Spectral norm, Eigenvalues and Determinant of Circulant Matrix involving the Generalized k-Horadam numbers

Y.Yazlik, N.Taskara *
Selcuk University, Science Faculty,
Department of Mathematics,
42075, Campus, Konya, Turkey

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Abstract

In this paper, firstly, we define the generalized k-Horadam sequence and investigate some its properties. In addition, by also defining circulant matrix $C_n(H)$ whose entries are the generalized k-Horadam numbers, we compute the spectral norm, eigenvalues and the determinant of the this matrix.

Keywords : Generalized k-Horadam sequence, circulant matrix, spectral norm, eigenvalue, determinant.

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1 Introduction

There are so many studies in the literature that concern about the special second order sequences such as generalized k-Fibonacci and k-Lucas, k-Fibonacci, Horadam, Fibonacci, Lucas, Pell, Jacobsthal and Jacobsthal Lucas sequences (see, for instance, [1-3,8,10]). Recently, it has been studied the norms of some special matrices with entries these sequences. For instance, Solak [9] defined the $n \times n$ circulant matrices $A = [a_{ij}]$ and $B = [b_{ij}]$, where $a_{ij} \equiv F_{(\text{mod}(j-i,n))}$ and $b_{ij} \equiv L_{(\text{mod}(j-i,n))}$. Additionally, he investigated the upper and lower bounds of the matrices A and B. In [7], it has been studied the norms, eigenvalues and determinants of some matrices related to different numbers. In [5], Ipek obtained the spectral norms

*email: yyazlik@selcuk.edu.tr ntaskara@selcuk.edu.tr

of circulant matrices $A = [a_{ij}]$ and $B = [b_{ij}]$, where $a_{ij} \equiv F_{(\text{mod}(j-i,n))}$ and $b_{ij} \equiv L_{(\text{mod}(j-i,n))}$.

In the light of the above material, in here, to give a new direction of these numbers, we define the generalized k-Horadam sequence which is generalized the special all second order sequences in the literature. Moreover, we present some proporties of the generalized k-Horadam sequence as Binet formula, sum. Additionally, as the main goal of this paper, we find spectral norm, eigenvalues, determinant of circulant matrix involving the generalized k-Horadam numbers.

2 The Generalized k-Horadam Sequence

Since main results of this paper concern about the spectral norm, eigenvalues and determinant of the circulant matrix entried by the generalized k-Horadam numbers, we need to introduce and investigate some proporties of this sequence. We note that most of the following preliminary material are actually defined the first time.

Definition 1 *Let k be any positive real number and $f(k)$, $g(k)$ are scaler-value polynomials. For $n \geq 0$ and $f^2(k) + 4g(k) > 0$, the generalized k-Horadam sequence $\{H_{k,n}\}_{n\in N}$ are defined by*

$$H_{k,n+2} = f(k)\, H_{k,n+1} + g(k)\, H_{k,n}, \tag{1}$$

with initial conditions $H_{k,0} = a,\ H_{k,1} = b$.

Particular cases of the previous definition are

- If $f(k) = k,\ g(k) = 1$, the generalized k-Fibonacci and k-Lucas sequence is obtained

$$G_{k,n+2} = kG_{k,n+1} + G_{k,n},\ G_{k,0} = a,\ G_{k,1} = b.$$

- If $f(k) = k,\ g(k) = 1$, $a = 0$ and $b = 1$, the k-Fibonacci sequence is obtained

$$F_{k,n+2} = kF_{k,n+1} + F_{k,n},\ F_{k,0} = 0,\ F_{k,1} = 1.$$

- If $f(k) = p,\ g(k) = q$, the Horadam sequence is obtained

$$H_{n+2} = pH_{n+1} + qH_n,\ H_0 = a,\ H_1 = b.$$

- If $f(k) = 1,\ g(k) = 1$, $a = 0$ and $b = 1$, the Fibonacci sequence is obtained

$$F_{n+2} = F_{n+1} + F_n,\ F_0 = 0,\ F_1 = 1.$$

- If $f(k) = 1$, $g(k) = 1$, $a = 2$ and $b = 1$, the Lucas sequence is obtained

$$L_{n+2} = L_{n+1} + L_n,\ L_0 = 2,\ L_1 = 1.$$

- If $f(k) = 2$, $g(k) = 1$, $a = 0$ and $b = 1$, the Pell sequence is obtained

$$P_{n+2} = 2P_{n+1} + P_n,\ P_0 = 0,\ P_1 = 1.$$

- If $f(k) = 1$, $g(k) = 2$, $a = 0$ and $b = 1$, the Jacobsthal sequence is obtained

$$J_{n+2} = J_{n+1} + 2J_n,\ J_0 = 0,\ J_1 = 1.$$

- If $f(k) = 1$, $g(k) = 2$, $a = 2$ and $b = 1$, the Jacobsthal Lucas sequence is obtained

$$j_{n+2} = j_{n+1} + 2j_n,\ j_0 = 2,\ j_1 = 1.$$

We can find the more information associated with these sequence in [1-3,8,10].

Binet Formula enables us to state the generalized k-Horadam number. It can be clearly obtained from the roots r_1 and r_2 of characteristic equation of (1) as the form $x^2 = f(k)x + g(k)$, where $r_1 = \frac{f(k)+\sqrt{f^2(k)+4g(k)}}{2}$, $r_2 = \frac{f(k)-\sqrt{f^2(k)+4g(k)}}{2}$.

Theorem 2 *For every $n \in \mathbb{N}$, we can write the Binet formula*

$$H_{k,n} = \frac{Xr_1^n - Yr_2^n}{r_1 - r_2},$$

where $X = b - ar_2$ and $Y = b - ar_1$.

Proof. Let us use the principle of mathematical induction on n. For $n = 0$,

$$H_{k,0} = \frac{(b - ar_2) - (b - ar_1)}{r_1 - r_2} = a.$$

It is easy to see that, for $n = 1$, we have

$$H_{k,1} = \frac{(b - ar_2)\, r_1 - (b - ar_1)\, r_2}{r_1 - r_2} = b.$$

As the usual next step of inductions, let us assume that it is true for all positive integers m. That is,

$$H_{k,m} = \frac{Xr_1^m - Yr_2^m}{r_1 - r_2}. \quad (2)$$

Therefore, we have to show that it is true for $m+1$. In other words, we need to check

$$H_{k,m+1} = \frac{Xr_1^{m+1} - Yr_2^{m+1}}{r_1 - r_2}.$$

By using (1) and (2), we have

$$\begin{aligned} H_{k,m+1} &= f(k)H_{k,m} + g(k)H_{k,m-1} \\ &= \frac{f(k)}{r_1 - r_2}\left[Xr_1^m - Yr_2^m\right] + \frac{g(k)}{r_1 - r_2}\left[Xr_1^{m-1} - Yr_2^{m-1}\right] \\ &= \frac{Xr_1^{m-1}\left[f(k)r_1 + g(k)\right] - Yr_2^{m-1}\left[f(k)r_2 + g(k)\right]}{r_1 - r_2} \\ &= \frac{Xr_1^{m+1} - Yr_2^{m+1}}{r_1 - r_2}. \end{aligned}$$

which ends up the induction. ■

Lemma 3 *For $n \geq 1$, we have*

$$\sum_{i=0}^{n-1} H_{k,i} = \frac{H_{k,n} + g(k)\, H_{k,n-1} - H_{k,1} + H_{k,0}(f(k) - 1)}{f(k) + g(k) - 1}.$$

Proof. Using the Binet formula in Theorem 2, the proof is clear. ■

3 Main Results and Their Proofs

At this point we first remind that the circulant matrix $C = [c_{ij}] \in M_{n,n}(\mathbb{C})$ is defined by the form

$$c_{ij} = \begin{cases} c_{j-i}, & j \geq i \\ c_{n+j-i}, & j < i \end{cases}.$$

We further remind that, for a matrix $X = [x_{i,j}] \in M_{m,n}(\mathbb{C})$, the spectral norm of X is given by

$$\|X\|_2 = \sqrt{\max_{1 \leq i \leq n} \lambda_i (X^* X)}$$

where X^* is the conjugate transpose of matrix X.

In this section, we formulate eigenvalues, the spectral norm and determinant of the circulant matrix with the generalized k-Horadam numbers entries. In order to do that, we can also define *the circulant matrix* as follows.

Definition 4 *An $(n \times n)$ circulant matrix with generalized k-Horadam numbers entries is defined by*

$$C_n(H) = \begin{pmatrix} H_{k,0} & H_{k,1} & H_{k,2} & \dots & H_{k,n-1} \\ H_{k,n-1} & H_{k,0} & H_{k,1} & \dots & H_{k,n-2} \\ H_{k,n-2} & H_{k,n-1} & H_{k,0} & \dots & H_{k,n-3} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ H_{k,1} & H_{k,2} & H_{k,3} & \dots & H_{k,0} \end{pmatrix}. \quad (3)$$

Lemma 5 *[6] Let $C(x) \in M_{n\times n}(\mathbb{C})$ be a circulant matrix. Then the diagonalization is written as the form $C = F^* diag(\lambda_0(x), \lambda_1(x), \dots, \lambda_{n-1}(x))F$. In here $\lambda_j(x) = \sum_{k=0}^{n-1} u_k w^{-jk}$, where $w = e^{\frac{2\pi i}{n}}$ for $i = \sqrt{-1}$, $j = 0, 1, \dots n-1$ and F is a Fourier matrix.*

Lemma 6 *[4]Let A be an $n \times n$ matrix with eigenvalues $\lambda_1, \lambda_2, \dots, \lambda_n$. Then, A is a normal matrix if and only if the eigenvalues of AA^* are $|\lambda_1|^2, |\lambda_2|^2, \dots, |\lambda_n|^2$, where A^* is the conjugate transpose of the matrix A.*

The following Theorem gives us the eigenvalues of the matrix in (3).

Theorem 7 *Let $C_n(H) = circ(H_{k,0}, H_{k,1}, \dots H_{k,n-1})$ be circulant matrix as in (3). Then the eigenvalues of $C_n(H)$ are*

$$\lambda_j(C_n(H)) = \frac{H_{k,n} - H_{k,0} + (g(k) H_{k,n-1} - H_{k,1} + f(k)H_{k,0}) w^{-j}}{g(k) w^{-2j} + f(k) w^{-j} - 1},$$

where $H_{k,n}$ is the nth generalized k-Horadam number and $w = e^{\frac{2\pi i}{n}}$, $i = \sqrt{-1}$, $j = 0, 1, \dots n-1$.

Proof. By Lemma 5, we have $\lambda_j(C_n(H)) = \sum_{k=0}^{n-1} H_{k,i} w^{-jk}$. Moreover, by Theorem 2, $\sum_{k=0}^{n-1} H_{k,i} w^{-jk} = \sum_{i=0}^{n-1} \left(\frac{X r_1^i - Y r_2^i}{r_1 - r_2}\right) \left(w^{-j}\right)^i$. Then, we obtain

$$\begin{aligned} \sum_{k=0}^{n-1} H_{k,i} w^{-jk} &= \frac{1}{r_1 - r_2} \left[X \sum_{i=0}^{n-1} \left(r_1 w^{-j}\right)^i - Y \sum_{i=0}^{n-1} \left(r_2 w^{-j}\right)^i \right] \\ &= \frac{1}{r_1 - r_2} \left[\frac{X r_1^n - X}{r_1 w^{-j} - 1} - \frac{Y r_2^n - Y}{r_2 w^{-j} - 1} \right] \\ &= \frac{H_{k,n} - H_{k,0} + (g(k) H_{k,n-1} - H_{k,1} + f(k)H_{k,0}) w^{-j}}{g(k) w^{-2j} + f(k) w^{-j} - 1}. \end{aligned}$$

Hence the result. ■

We have the following theorems that deals with the spectral norm and the determinant of $C_n(H)$.

Theorem 8 *Let $C_n(H)$ be an $n \times n$ circulant matrix with the generalized k-Horadam numbers entries. Then the spectral norms of $C_n(H)$ is*

$$\|C_n(H)\|_2 = \sum_{i=0}^{n-1} H_{k,i}$$

where $H_{k,n}$ is nth generalized k-Horadam number.

Proof. By Lemma 6 , we have

$$\begin{aligned}\|C_n(H)\|_2 &= \sqrt{\max_{0\le j\le n-1} \lambda_j \left(C_n(H)\, C_n^*(H)\right)} \\ &= \sqrt{\max_{0\le j\le n-1} |\lambda_j (C_n(H))|^2}\end{aligned}$$

In this last equality, if we take $j = 0$, then λ_0 becomes the maximum eigenvalue. Thus $\|C_n(H)\|_2 = |\lambda_0(C_n(H))|$. In addition, by Lemma 3 and Theorem 7, we clearly obtain

$$\begin{aligned}\|C_n(H)\|_2 &= \frac{H_{k,n} + g(k) H_{k,n-1} - H_{k,1} + H_{k,0}(f(k)-1)}{f(k) + g(k) - 1} \\ &= \sum_{i=0}^{n-1} H_{k,i},\end{aligned}$$

as required. ■

Theorem 9 *The determinant of $C_n(H) = circ(H_{k,0}, H_{k,1}, \ldots H_{k,n-1})$ is*

$$\det(C_n(H)) = \frac{(-H_{k,n} + H_{k,0})^n - (g(k) H_{k,n-1} - H_{k,1} + f(k) H_{k,0})^n}{(-g(k))^n - r_1^n - r_2^n + 1}.$$

Proof. The fundamental fact about determinants tells us that $\det(C_n(H)) = \prod_{j=0}^{n-1} \lambda_j(C_n(H))$. By Theorem 7, we then get

$$\prod_{j=0}^{n-1} \frac{(-H_{k,n} + H_{k,0}) - (g(k) H_{k,n-1} - H_{k,1} + f(k) H_{k,0})\, w^{-j}}{(r_1 w^{-j} - 1)(r_2 w^{-j} - 1)}.$$

Also, by considering the well-known identity $\prod_{k=0}^{n-1}\left(x-yw^{k}\right)=x^{n}-y^{n}$, we have

$$\begin{aligned}\det(C_n(H)) &= \frac{(-H_{k,n}+H_{k,0})^n-(g(k)H_{k,n-1}-H_{k,1}+f(k)H_{k,0})^n}{(1-r_1^n)(1-r_2^n)} \\ &= \frac{(-H_{k,n}+H_{k,0})^n-(g(k)H_{k,n-1}-H_{k,1}+f(k)H_{k,0})^n}{(-g(k))^n-r_1^n-r_2^n+1},\end{aligned}$$

Hence the results. ∎

Conclusion 10 *We should note that choosing suitable values on $f(k), g(k)$, a and b in Definition 4, it is actually obtained the spectral norm, eigenvalues and the determinant for the special all second order sequences. For example, by taking $f(k)=g(k)=1$, $a=0$ and $b=1$ in Theorem 8, then we turn into $\|C_n(H)\|_2=F_{n+1}-1$ in [5], where F_n is the nth classical Fibonacci number.*

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