Multi-level distance labelings for helm graphs *

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Abstract

For a graph G and any two vertices u and v in G, let $d_G(u,v)$ denote the distance between them and let diam(G) be the diameter of G. A multi-level distance labeling (or radio labeling) for G is a function f that assigns to each vertex of G a positive integer such that for any two distinct vertices u and v, $d_G(u,v) + |f(u) - f(v)| \geq$ diam$(G) + 1$. The largest positive integer in the range of f is called the span of f. The radio number of G, denoted $rn(G)$, is the minimum span of a multi-level distance labeling for G.
A helm graph H_n is obtained from the wheel W_n by attaching a vertex of degree one to each of the n vertices of the cycle of the wheel. In this paper the radio number of the helm graph is determined for every $n \geq 3$: $rn(H_3) = 13, rn(H_4) = 21$ and $rn(H_n) = 4n + 2$ for every $n \geq 5$. Also, a lower bound of $rn(G)$ related to the length of a maximum hamiltonian path in the graph of distances of G is proposed.

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1 Introduction

Multi-level distance labelings of graphs are motivated by restrictions inherent in assigning channel frequencies for radio transmitters [2] and they can be regarded as an extension of distance two labeling.
For a set of given cities (or stations), the task is to assign to each city a channel, which is a positive integer, so that the interference is prohibited, and the span of the channels assigned is minimized. To avoid interference, to transmitters that are geographically close must be assigned channels with large frequency differences, transmitters that are further apart may receive channels with relatively close frequencies. This situation can be physically modeled by considering the transmitters to be the vertices of a graph. Positive integers are assigned to the vertices of the graph with restriction between them and our goal is to minimize the largest integer used.
Let G be a connected graph. For any two vertices u and v of G, $d(u,v)$ represents the distance between them and $\mathrm{diam}(G)$ the diameter of G. A *multi-level distance labeling* of G [1] is a one-to-one mapping $f : V(G) \rightarrow \mathbb{Z}^+$ satisfying the condition

$$d(u,v) + |f(u) - f(v)| \geq \mathrm{diam}(G) + 1 \qquad (1)$$

for every two distinct vertices $u, v \in V(G)$. The *span* of a multi-level distance labeling f is the maximum integer in the range of f. The *radio number* of G, denoted $rn(G)$, is the lowest span in all multi-level distance labelings of G. The inequality (1) will be referred as the multi-level distance labeling condition (or the radio condition). Note that from this condition it follows that all vertices must receive distinct labels, hence $rn(G) \geq |V(G)|$ for all graphs G. For a graph G and a multi-level distance labeling f of G denote by $S(G,f)$ the set of consecutive integers $\{m, m+1, \ldots, M\}$, where $m = \min_{u \in V(G)} f(u)$ and $M = \max_{u \in V(G)} f(u)$ is the span of f. It is clear that $rn(G) = \min_f \max S(G,f)$ and since (1) contains only the difference of the labels, a multi-level distance labeling realizing $rn(G)$ must have $m = 1$. In [4] the radio number of paths and cycles was determined. We shall use the terminology of monographs [5] and [6]. A complete survey on multi-level distance labelings of graphs can be found in [3].
As $\mathrm{diam}(K_n)$=1, it is easy to see that the radio condition is satisfied if the vertices of K_n are labeled with consecutive integers $1, \ldots, n$, hence the radio number of the complete graph on n vertices is $rn(K_n) = n$ [1]. The star graph S_n $(n \geq 2)$ is the tree on $n+1$ vertices $K_{1,n}$.

Lemma 1.1 [1] $rn(S_n) = n + 2$ *for* $n \geq 2$.

The wheel graph W_n ($n \geq 3$) consists of an n-cycle together with a center vertex that is adjacent to all n vertices of the cycle. As $W_3 = K_4$ we have $rn(W_3) = 4$. Also, it is an easy exercise to show that $rn(W_4) = 7$. For larger wheels the radio number is given by the following lemma.

Lemma 1.2 [1] $rn(W_n) = n + 2$ *for* $n \geq 5$.

If G is a connected graph of order n, let DG represent the weighted complete graph K_n having $V(K_n) = V(G)$ and the length of an edge ij defined by $l(ij) = d_G(i,j)$. If $hp_{\max}(DG)$ denotes the maximum length of a hamiltonian path in DG, the following result holds.

Theorem 1.3 $rn(G) \geq (n-1)(\text{diam}(G)+1) - hp_{\max}(DG) + 1$.

Proof: Let f be a multi-level distance labeling of G. Since f is injective we can associate with f the hamiltonian path in DG, denoted by $hp(f)$: $x_{i_1}, x_{i_2}, \ldots, x_{i_n}$, where $x_{i_1}x_{i_2}\ldots x_{i_n}$ is a permutation of $V(G)$ such that $f(x_{i_1}) < f(x_{i_2}) < \ldots < f(x_{i_n})$. Applying (1) we can write

$$f(x_{i_{k+1}}) - f(x_{i_k}) \geq \text{diam}(G) + 1 - d(x_{i_k}, x_{i_{k+1}})$$

for every $k = 1, \ldots, n-1$. Adding up these inequalities we get

$$f(x_{i_n}) \geq (n-1)(\text{diam}(G)+1) - l(hp(f))) + f(x_{i_1})$$

$$\geq (n-1)(\text{diam}(G)+1) - hp_{\max}(DG) + 1.$$

□

Although the determination of $hp_{\max}(DG)$ is an NP-hard problem, this lower bound may be useful at least for small graphs, as we shall see in the next section.

2 Multi-level distance labelings and radio number of helm graphs

Helm graphs are obtained from wheels by attaching a pendant edge to each vertex of the n-cycle. It follows that the helm graph denoted H_n has $2n + 1$ vertices (n vertices of degree 4, n vertices of degree one and one vertex of degree n) and $3n$ edges. We have diam(H_3)=3 and diam(H_n)=4 for $n \geq 4$. We shall denote the central vertex by z; $v_1, \ldots, v_n$ and $u_1, \ldots, u_n$ are the vertices of degrees four and one, respectively, such that $u_iu_{i+1}, v_iv_{i+1}, u_iv_i \in E(H_n)$ for every $1 \leq i \leq n-1$. Also $u_nu_1, v_nv_1, u_nv_n \in E(H_n)$.

Theorem 2.1 *For $n \geq 4$, $rn(H_n) \geq 4n + 2$.*

Proof: Assume $n \geq 4$. Because diam(H_n)=4, any multi-level distance labeling f of H_n must satisfy the radio condition

$$d(u, v) + |f(u) - f(v)| \geq 5 \tag{2}$$

for all distinct $u, v \in V(H_n)$. We shall count the minimum number of forbidden values in $S(G, f)$. For example, if we label the central vertex z with label a, then as $d(z, v) \leq 2$ for all vertices $v \neq z$, we cannot assign any label from the set $\{a - 2, a - 1, a + 1, a + 2\}$ to any other vertex v since this contradicts condition (2) for the pair $\{z, v\}$. Hence in this case the integers $a - 2, a - 1, a + 1, a + 2$ are forbidden to be used as labels. Also as $d(v_i, r) \leq 3$ for all $r \neq v_i$, one value above and one value below $f(v_i)$ is forbidden to be used as a label for any vertex r. It follows that for the center z we have at least two forbidden labels in $S(G, f)$ and z has exactly two forbidden labels only if $f(z)$ is the lowest or the highest label. Similarly, for any vertex from $\{v_1, \ldots, v_n\}$ there exists at least one forbidden label: we have exactly one forbidden label only if its label is the lowest or the highest label, otherwise there exist two forbidden labels in $S(G, f)$. Also note that any two forbidden labels in $S(G, f)$ cannot coincide. If α is a forbidden label for z and β is a forbidden label for a vertex v_i then $\alpha \in \{a-2, a-1, a+1, a+2\}$ and $\beta \in \{b-1, b+1\}$, where $f(z) = a$ and $f(v_i) = b$. If $\alpha = \beta$ this would imply that $d(z, v_i) + |f(z) - f(v_i)| \leq 4$, which contradicts the radio condition (2). Similarly, if β_i and β_j are two forbidden labels in $S(G, f)$ corresponding to vertices v_i and v_j $(i \neq j)$ then $\beta_i = \beta_j$ would imply that $|f(v_i) - f(v_j)| \leq 2$. Because $d(v_i, v_j) \leq 2$ we deduce $d(v_i, v_j) + |f(v_i) - f(v_j)| \leq 4$, which again contradicts (2).
It follows that in $S(G, f)$ there exist at least $2 + 1 + 2(n - 1) = 2n + 1$ forbidden labels, which implies $|S(G, f)| \geq 4n + 2$. Since $S(G, f)$ is a set of consecutive positive integers it follows that $\max S(G, f) \geq 4n+2$, hence $rn(H_n) \geq 4n + 2$.

□

Theorem 2.2 $rn(H_3) = 13$.

Proof: Since diam(H_3)=3 the condition (1) becomes

$$d(u, v) + |(f(u) - f(v)| \geq 4 \tag{3}$$

for every two distinct vertices $u, v \in V(H_3)$. The multi-level distance labeling of H_3 illustrated in Fig. 1 shows that $rn(H_3) \leq 13$. If the central vertex z has label a, since $d(z, v) \leq 2$ for all vertices $v \neq z$, if any positive value from the set $\{a - 1, a + 1\}$ is assigned to any other vertex say v, then the condition (3) for the pair $\{z, v\}$ is not satisfied.

Also since $d(v_i, r) \leq 2$ for all v_i and for any $r \neq v_i$, if $f(v_i) = b$ then $b-1$ and $b+1$ cannot be assigned to any other vertex of H_3. We deduce that the center z and any vertex from $\{v_1, v_2, v_3\}$ have at least one forbidden label in

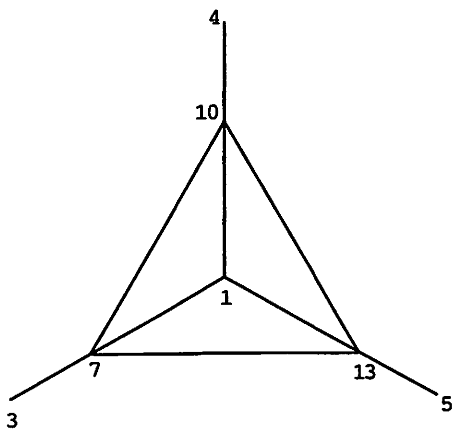

Figure 1: A multi-level distance labeling f of H_3 with span(f)=13.

$S(H_3, f)$ and this label is unique if and only if its label is the lowest or the highest label. Otherwise, it has two forbidden labels in $S(H_3, f)$. As in the proof of Theorem 2.1 one can deduce that any two forbidden labels cannot coincide. It follows that in $S(H_3, f)$ there exist at least $1 + 2 \cdot 2 + 1 = 6$ forbidden labels, which implies $|S(H_3, f)| \geq 6 + |V(H_3)| = 13$ for any multi-level distance labeling f, hence $rn(H_3) \geq 13$. We conclude that $rn(H_3) = 13$.

□

Theorem 2.3 $rn(H_4) = 21$.

Proof: We have diam(H_4)=4 and any multi-level distance labeling f of H_4 must satisfy condition (2). The multi-level distance labeling f of H_4 represented in Fig. 2a) shows that $rn(H_4) \leq 21$.

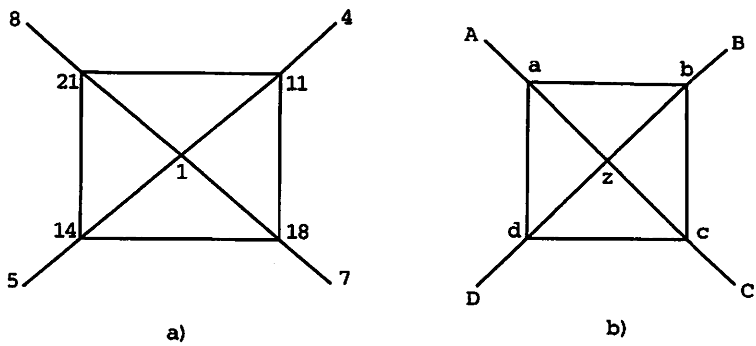

Figure 2: A multi-level distance labeling f of H_4 with span(f)=21.

The bound $rn(H_4) \geq 18$ provided by Theorem 2.1 is not sufficient to prove the reverse inequality. We will find some more arguments to prove that in fact $rn(H_4) \geq 21$. In order to simplify the notation we shall refer to the vertices of H_4 denoted as in Fig. 2b). Let f be a multi-level distance labeling of H_4. Firstly, we shall prove that there exist some new forbidden labels associated with the vertices A, B, C, D, which are different from the forbidden labels assigned to the vertices a, b, c, d and z as in Theorem 2.1. Suppose that $f(B) = x$ and $f(z) < x$. It follows that the forbidden labels of z are less than or equal to $x - 1$. We shall consider two cases: α) $f(D) = x + 1$ and β) $f(D) \neq x + 1$.

α) In this case condition (2) implies that $f(a), f(b), f(c), f(d), f(A), f(C), f(z) \neq x + 2$ and $x + 2$ is not a forbidden label for a, b, c, d, z (determined as in Theorem 2.1). Indeed, if $x+2$ would be a forbidden label for a, then $f(a) \in \{x+1, x+3\}$. Since $f(D) = x+1$ we deduce that $f(a) = x+3$. In this case $|f(a) - f(D)| + d(a, D) = 4 < 5$, a contradiction. Similarly, $x+2$ is not a forbidden label for b, c, d nor for z since $f(z) < x$. In this case the forbidden label relatively to the diametral pair $\{B, D\}$ is defined to be $x + 2$.

β) If $f(D) \neq x + 1$ we shall also consider two subcases: β1) $f(d) \neq x + 2$ and β2) $f(d) = x + 2$.

β1) In this subcase $x + 1$ is defined to be a forbidden label relatively to $\{B, D\}$, which is different from all forbidden labels of a, b, c, d, z (determined as in Theorem 2.1). If $x + 1$ would be a forbidden label for a, then $f(a) = x + 2$ and $|f(a) - f(B)| + d(a, B) = 4 < 5$. A similar situation holds for c. Also $x + 1$ cannot be a forbidden label for d since $f(d) \neq x, x + 2$.

β2) In this subcase we define $x+4$ to be new forbidden label having required properties. This cannot be the label of any vertex of H_4 and is different from forbidden labels of a, b, c, d, z since otherwise $x + 3$ or $x + 5$ are the labels of some vertices in the set $\{a, b, c\}$ and this contradicts (2). The same construction can be performed relatively to D and the same diametral pair $\{B, D\}$.

If $f(z) > x$ we shall consider, by a similar reasoning, $x-2, x-1$ or $x-4$, respectively to be the new forbidden label of the pair $\{B, D\}$. A similar construction can be made for the pair $\{A, C\}$ of diametral vertices; if $f(A) = y$ then the forbidden label of this pair will be by definition $y+2, y+1$ or $y+4$ (if $f(z) < y$) and $y-2, y-1$ or $y-4$ (if $f(z) > y$), respectively.

It is necessary to show that in all cases these forbidden labels of the pairs $\{A, C\}$ and $\{B, D\}$ are different.

If $f(z) < \min(x, y)$, since $|x-y| \geq 2$ by (2) we can have equality between some labels in the sets $\{x+1, x+2, x+4\}$ and $\{y+1, y+2, y+4\}$ only when: $x+1 = y+4$ or equivalently, $x = y+3$ (in this case we have $f(B) = x = y+3, f(c) = y+2$ and (2) is not satisfied); $x+2 = y+4$ or equivalently, $x = y+2$ (when $f(B) = f(c)$, which contradicts the injectivity of f); the remaining two subcases are obtained by interchanging x and y between them and may be solved in the same manner.

If $f(z) > \max(x, y)$ then the sets of labels are $\{x-1, x-2, x-4\}$ and $\{y-1, y-2, y-4\}$ and this case may be settled in a similar way.

Suppose that $x < f(z) < y$; we get $\{x-1, x-2, x-4\} \cap \{y+1, y+2, y+4\} = \emptyset$.

If $f(z) \neq \min(\max)S(H_4, f)$ and $f(a), f(b), f(c), f(d) \neq \max(\min)S(H_4, f)$ then three new forbidden values appear in $S(H_4, f)$ relatively to those considered in the proof of Theorem 2.1, hence in this case $|S(H_4, f)| \geq 21$ and the result is proved.

Otherwise, we shall consider the following cases: γ) $f(z) \neq \min(\max)S(H_4, f)$ and one of $f(a), \ldots, f(d)$ equals $\max(\min)S(H_4, f)$; $\delta1$) $f(z) = \min(\max)S(H_4, f)$ and one of $f(a), \ldots, f(d)$ equals $\max(\min)S(H_4, f)$; $\delta2$) $f(z) = \min(\max)S(H_4, f)$ and none of $f(a), \ldots, f(d)$ is equal to $\max(\min)S(H_4, f)$.

γ) Without loss of generality let $f(z) \neq \min S(H_4, f)$ and $f(b) = \max S(H_4, f)$. Condition (2) implies $f(B) \leq f(b) - 4$. If $f(z) < f(B)$ then a forbidden label relatively to the diametral pair $\{B, D\}$ belongs to the set $\{f(B)+1, f(B)+2, f(B)+4\} \subset S(H_4, f)$. In this case vertex z has two new forbidden labels since $\min S(H_4, f) < f(z) < \max S(H_4, f)$. We have a total of three new forbidden labels, which implies $rn(H_4) \geq 21$.

If $f(z) > f(B)$, then we have seen that the new forbidden label relatively to $\{B, D\}$ lies in the set $\{f(B)-1, f(B)-2, f(B)-4\}$ and it may be possible that it does not belong to $S(H_4, f)$. If $f(z) < f(A)$, then the forbidden value relatively to the diametral pair $\{A, C\}$ belongs to the set $\{f(A)+1, f(A)+2, f(A)+4\}$. Condition (2) implies $f(A) \leq f(b) - 3$ and the forbidden label relatively to $\{A, C\}$ does not belong to $S(H_4, f)$ when

it is equal to $f(A) + 4$ and $f(A) = f(b) - 3$. This forbidden label may appear when $f(c) = f(A)+2 = f(b)-1$, which contradicts $f(c) \leq f(b)-4$ deduced from (2).
If $f(z) > f(A)$, then the forbidden label relatively to $\{A, C\}$ belongs to $\{f(A) - 1, f(A) - 2, f(A) - 4\}$ and it may happen that it does not belong to $S(H_4, f)$. Condition (2) also provides $|f(A) - f(B)| \geq 2$.
If $f(A) < f(B)$ then $f(A) \leq f(B) - 2$. Both forbidden labels relatively to $\{B, D\}$ and $\{A, C\}$ do not belong to $S(H_4, f)$ only if the forbidden label relatively to $\{B, D\}$ equals $f(B) - 4$ and $f(A) \in \{f(B) - 2, f(B) - 3\}$. But in this case $f(d) = f(B) - 2$, which implies $f(A) = f(B) - 3$. We get $|f(A) - f(d)| = 1$, which contradicts (2). A similar conclusion holds if $f(A) > f(B)$, when $f(B) = f(A) - 3, f(c) = f(A) - 2$ and this contradicts (2).
Hence in case γ) three new forbidden labels appear, thus implying $rn(H_4) \geq 21$.

$\delta 1$) Without loss of generality suppose that $f(z) = 1$ and $f(a) = \max S(H_4, f)$. As in the proof of Theorem 1.3 consider the complete graph K_9 having vertex set $V(K_9) = \{z, a, b, c, d, A, B, C, D\}$, where the length in K_9 of the edge uv is defined by $l_{K_9}(u, v) = d_{H_4}(u, v)$ (defined above as DH_4). Each multi-level distance labeling f of H_4 induces a hamiltonian path in DH_4 having extremities z and a by ordering its vertices $z, x_1, x_2, \ldots, x_7, a$ such that $1 = f(z) < f(x_1) < \ldots < f(x_7) < f(a)$.

For example, the multi-level distance labeling of H_4 illustrated in Fig. 2 induces the hamiltonian path $z, B, D, C, A, b, d, c, a$. As in the proof of Theorem 1.3 we get $f(a) \geq 41 - l(z, x_1, \ldots, x_7, a)$. Hence we can obtain a lower bound for $rn(H_4)$ by considering a hamiltonian path of maximum length having extremities z and a in DH_4. Such a hamiltonian path is exactly $z, B, D, C, A, b, d, c, a$ of length equal to 20, hence $rn(H_4) = \min f(a) \geq 41 - 20 = 21$.

$\delta 2$) In this case suppose that $f(z) = 1$ and $\max S(H_4, f) = f(B)$. A hamiltonian path of maximum length in DH_4 defined as above is $z, D, C, A, c, a, b, d, B$ of length equal to 20, hence $f(B) \geq 41 - 20 = 21$ and the proof is complete.
Note that this method cannot be applied to the case γ since a hamiltonian path in DH_4 having one extremity in a and other in c is $a, C, A, D, B, d, b, z, c$ of length equal to 21, hence we can deduce only $rn(H_4) = \min f(c) \geq 41 - 21 = 20$.

□

Another optimal multi-level distance labeling of H_4 is represented in Fig. 3 and corresponds to the case when $f(a) = 1$ and $f(z) = \max S(H_4, f) = 21$.

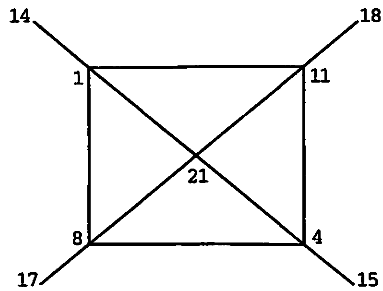

Figure 3: Another multi-level distance labeling of H_4 with span(f)=21.

Theorem 2.4 $rn(H_n) = 4n + 2$ *for every* $n \geq 5$.

Proof: We shall propose a multi-level distance labeling of H_n with span $4n + 2$, which implies $rn(H_n) \leq 4n + 2$. By Theorem 2.1 the opposite inequality is also valid, which will prove the equality.
Let $n \geq 5$. The multi-level distance labeling $f : V(H_n) \rightarrow \mathbb{Z}^+$ is defined as follows:
A. If n is odd: $f(z) = 1, f(u_{2i-1}) = 3 + i$ for $1 \leq i \leq (n+1)/2$; $f(u_{2i}) = 3 + i + (n+1)/2$ for $1 \leq i \leq (n-1)/2$; $f(v_{2i-1}) = n + 2 + 3i$ for $1 \leq i \leq (n+1)/2$, $f(v_{2i}) = n + 2 + 3(n+1)/2 + 3i$ for $1 \leq i \leq (n-1)/2$.
B. If n is even: $f(z) = 1, f(u_{2i-1}) = 3 + i$ for $1 \leq i \leq n/2$; $f(u_{2i}) = 3 + n/2 + i$ for $1 \leq i \leq n/2 - 2$, $f(u_{n-2}) = n + 3$, $f(u_n) = n+2$; $f(v_{2i-1}) = n+2+3i$ for $1 \leq i \leq n/2$; $f(v_{2i}) = n+2+3n/2+3i$ for $1 \leq i \leq n/2$.

In both cases the span of f is equal to $4n+2$ and it is reached for $f(v_{n-1})$ or $f(v_n)$ if n is odd or even, respectively.
For $n = 5$ and $n = 6$ the multi-level distance labeling of H_n described above is illustrated in figures 4 and 5, respectively.

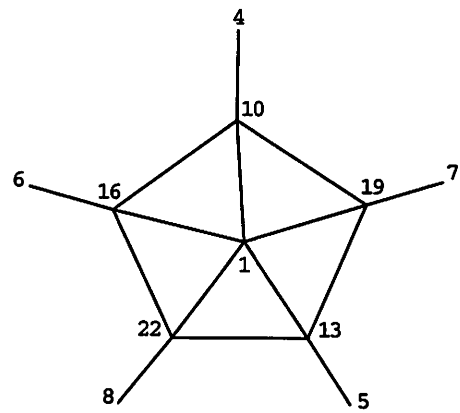

Figure 4: An optimal multi-level distance labeling of H_5.

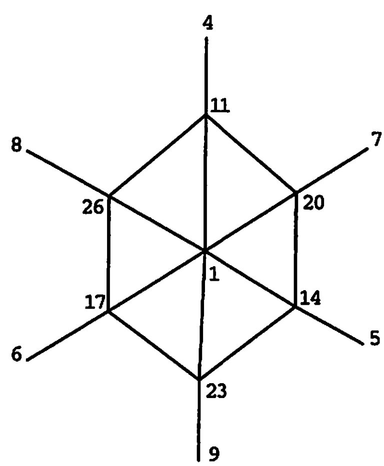

Figure 5: An optimal multi-level distance labeling of H_6.

It is clear that (2) is satisfied if: $u = z$; $\{u, v\}$ = a) $\{u_i, u_j\}$ since if v_i and v_j are consecutive on the cycle then $|f(u_i) - f(u_j)| \geq 2$ and their distance is equal to three and if v_i and v_j are non-consecutive their distance is equal to four; b) $\{v_i, v_j\}$ since if v_i, v_j are consecutive then $d(v_i, v_j) = 1$ and $|f(v_i) - f(v_j)| \geq 6$; otherwise $d(v_i, v_j) = 2$ and $|f(v_i) - f(v_j)| \geq 3$; c) $\{u_i, v_j\}$ since the set of labels of the vertices u_i and v_j $(1 \leq i, j \leq n)$ are equal to $\{4,5,\ldots,n+3\}$ and $\{n+5,n+8,\ldots,4n+2\}$, respectively. We have $|d(u_i)-d(v_j)| \geq 4$ unless $f(u_i)=n+3$, $f(v_j)=n+5$ or $f(u_i)=n+2$ and $f(v_j)=n+5$. The corresponding pairs of vertices in these cases are $\{u_{n-1},v_1\},\{u_{n-3},v_1\}$ for n odd and $\{u_{n-2},v_1\},\{u_n,v_1\}$ for n even. The condition (2) is verified in all these cases.

□

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