The Transformation Digraph D^{++-}*

Juan Liu[1,2], Jixiang Meng[2]† Xindong Zhang[1],

1. College of Mathematics Sciences, Xinjiang Normal University, Urumqi, Xinjiang, 830054, P.R.China

2. College of Mathematics and System Sciences, Xinjiang University, Urumqi, Xinjiang, 830046, P.R.China

Abstract

Let D be a digraph with order at least two, the transformation digraph D^{++-} is the digraph with vertex set $V(D) \cup A(D)$ in which (x, y) is an arc of D^{++-} if one of the following conditions holds: (i) $x, y \in V(D)$, and (x, y) is an arc of D; (ii) $x, y \in A(D)$, and the head of x is the tail of y; (iii) $x \in V(D), y \in A(D)$, and x is not the tail of y; (iv) $x \in A(D), y \in V(D)$, and y is not the head of x. In this paper we determine the regularity and diameter of D^{++-}. Furthermore, we characterize maximally-arc-connected or super-arc-connected D^{++-}. We also give sufficient conditions for this kind of transformation digraph to be maximally-connected or super-connected.

Keywords: Transformation digraph; Super-arc-connected ; Super-connected.

1 Introduction

For graph-theoretical terminology and notation not defined here we follow Bondy and Murty [2]. We consider only strict digraph D(digraph contains no loops and no parallel arcs) with vertex set $V(D)$ and arc set $A(D)$.

Let D be a digraph, $X, Y \subseteq V(D)$. Set

*The research is supported by FSRPHEXJ (No.XJNU2010S30), China Postdoctoral Science Foundation (No.20100471000), Science Foundation for The Excellent Youth Scholars of Xinjiang Normal University (No.XJNU0920) and NSFXJ (No.2010211A06).

†Corresponding author. E-mail: liujuan1999@126.com.

$$E_D[X,Y] = \{e = (x,y) \in A(D) |\ x \in X, y \in Y\},$$
$$N_D^+(X) = \{x \in V(D)\backslash X | \exists\ y \in X \text{ such that } (y,x) \in A(D)\},$$
$$N_D^-(X) = \{x \in V(D)\backslash X | \exists\ y \in X \text{ such that } (x,y) \in A(D)\}.$$

If $X = \{x\}$, we write $N_D^+(x)$ and $N_D^-(x)$ instead of $N_D^+(\{x\})$ and $N_D^-(\{x\})$. Let $E_D^+(x) = E[\{x\}, V(D) \setminus \{x\}]$, $E_D^-(x) = E[V(D) \setminus \{x\}, \{x\}]$, $d_D^+(x) = |N_D^+(x)|$ and $d_D^-(x) = |N_D^-(x)|$. Then $d_D^+(x)$ and $d_D^-(x)$ are the outdegree and indegree of x, respectively. Set $\delta^+(D) = \min\{d_D^+(x) : x \in V(D)\}$, $\delta^-(D) = \min\{d_D^-(x) : x \in V(D)\}$, $\delta(D) = \min\{\delta^-(D), \delta^+(D)\}$, $\Delta^+(D) = \max\{d_D^+(x) : x \in V(D)\}$, $\Delta^-(D) = \max\{d_D^-(x) : x \in V(D)\}$, $\Delta(D) = \max\{\Delta^-(D), \Delta^+(D)\}$, which are called the minimum outdegree, the minimum indegree, the minimum degree, the maximum outdegree, the maximum indegree and the maximum degree of D, respectively.

An *arc-cut* of a strongly connected digraph D is a set of arcs whose remove makes D not strongly connected. The *arc-connectivity* $\lambda(D)$ is the minimum cardinality of an arc-cut over all arc-cuts of D. It is wellknown that $\lambda(D) \leq \delta(D)$. We call a digraph D *maximally-arc-connected*, for short, *max-*λ, if $\lambda(D) = \delta(D)$. The *connectivity* $\kappa(D)$ and *max-*κ can be similarly defined. A strongly connected digraph D is *super-arc-connected*, for short, *super-*λ, if every minimum arc-cut is either $E_D^+(v)$ or $E_D^-(v)$ for some vertex v. A digraph D is said to be *super-connected*, for short, *super-*κ, if every minimum vertex-cut is either $N_D^+(v)$ or $N_D^-(v)$ for some vertex v.

In [5], Wu and Meng introduced a kinds of transformation graphs and investigated some basic properties of them. Let G be a graph and x, y, z be three variables taking values $-$ or $+$. The *transformation graph* G^{xyz} is the graph with vertex set $V(G) \cup E(G)$, α and β are adjacent in G^{xyz} if and only if one of the following holds: (i) $\alpha, \beta \in V(G)$, α and β are adjacent in G if $x = +$ while α and β are not adjacent in G if $x = -$. (ii) $\alpha, \beta \in E(G)$, α and β are adjacent in G if $x = +$ while α and β are not adjacent in G if $x = -$. (iii) $\alpha \in V(G)$ and $\beta \in E(G)$, α and β are incident in G if $x = +$ while α and β are not incident in G if $x = -$. Clearly, G^{+++} is the wellknown total graph of G.

Now, we give the corresponding definitions of transformation digraphs.

Let $D = (V(D), A(D))$ be a digraph, where $|V(D)| = n$, $|A(D)| = m$ and $V(D) = \{v_1, v_2, \cdots, v_n\}$. The *line digraph* of D, denoted by $L(D)$, is the digraph with vertex set $V(L(D)) = \{a_{ij} | (v_i, v_j)$ is an arc in $D\}$, and a vertex a_{ij} is adjacent to a vertex a_{st} in $L(D)$ if and only if $v_j = v_s$ in D.

Definition 1.1. *Let $D = (V(D), A(D))$ be a digraph, x, y, z be three variables taking values $-$ or $+$. The transformation digraph of D, denoted by D^{xyz}, is a digraph with vertex set $V(D^{xyz}) = V(D) \cup A(D)$. For any vertex $a, b \in V(D^{xyz})$, $(a, b) \in A(D^{xyz})$ if and only if one of the following four cases holds:*

(i) If $a \in V(D)$ and $b \in V(D)$, then $(a, b) \in A(D)$ in D if $x = +$ and $(a, b) \notin A(D)$ in D if $x = -$.

(ii) If $a \in A(D)$ and $b \in A(D)$, then the head of arc a is the tail of arc b in D if $y = +$ and the head of arc a is not the tail of arc b in D if $y = -$.

(iii) If $a \in V(D)$ and $b \in A(D)$, then a is the tail of arc b in D if $z = +$ and a is not the tail of arc b in D if $z = -$.

(iv) If $a \in A(D)$ and $b \in V(D)$, then b is the head of arc a in D if $z = +$ and b is not the head of arc a in D if $z = -$.

Thus, as defined above, there are eight kinds of transformation digraphs, among which D^{+++} is usually known as the total digraph of D.

Wu and Meng [5] investigated some basic properties, including connectedness and diameters of G^{xyz}, Wu et al. [6] studied the connectivity, planarity, hamiltonity and isomorphism of G^{-++}. Chen [3] characterized the super-edge-connectivity of G^{xyz}. For transformation digraph, Liu and Meng [4] characterized super-arc-connected and super-connected total digraphs. In this paper we determine the regularity and diameter of D^{++-}. Furthermore, we characterize maximally-arc-connected or super-arc-connected D^{++-}. We also give sufficient conditions for this kind of transformation digraph to be maximally-connected or super-connected.

2 Regularity and Diameter of D^{++-}

By the definition of line digraph, we have $A(D) = V(L(D)) = \{a_{ij} | (v_i, v_j)$ is an arc in $D\}$. In fact, the digraph D^{++-} can be viewed as $V(D^{++-}) = V(D) \cup V(L(D))$ and $A(D^{++-}) = A(D) \cup A(L(D)) \cup A(D, L(D))$, where $A(D, L(D))$ denotes the arcs with one end in $V(D)$ and the other end in $V(L(D))$.

Proposition 2.1. *Let D be a digraph with n vertices and m arcs, then*

(i) $|V(D^{++-})| = m + n$*;*

(ii) $|A(D^{++-})| = 2mn - m + \sum_{x\in V(D)} d_D^+(x)d_D^-(x)$*;*

(iii) For $v \in V(D)$, $d_{D^{++-}}^+(v) = d_{D^{++-}}^-(v) = m$*;*

(iv) For $a_{ij} \in A(D)$, $d_{D^{++-}}^+(a_{ij}) = d_D^+(v_j)+n-1$, $d_{D^{++-}}^-(a_{ij}) = d_D^-(v_i)+n-1$.

Proof. (i) and (ii) can be obtained easily by the definition of D^{++-}. For $v \in V(D)$, since there are $d_D^+(v)$ out-arcs from v to vertices in $V(D)$ and $m - d_D^+(v)$ out-arcs from v to vertices in $V(L(D))$, thus $d_{D^{++-}}^+(v) = d_D^+(v) + (m - d_D^+(v)) = m$. For $a_{ij} \in V(L(D))$, since $d_{L(D)}^+(a_{ij}) = d_D^+(v_j)$ and there are $n-1$ out-arcs from a_{ij} to vertices in $V(D)$, thus $d_{D^{++-}}^+(a_{ij}) = d_D^+(v_j) + (|V(D)|-1) = d_D^+(v_j)+n-1$. Similarly, $d_{D^{++-}}^-(v) = d_D^-(v) + (|A(D)| - d_D^-(v)) = |A(D)| = m$, and $d_{D^{++-}}^-(a_{ij}) = d_D^-(v_i) + (|V(D)|-1) = d_D^-(v_i)+n-1$. □

By Proposition 2.1, we have $\delta(D^{++-}) = \min\{m, \delta(D)+n-1\}$ and $\Delta(D^{++-}) = \max\{m, \Delta(D)+n-1\}$.

Theorem 2.2. *Let D be a digraph with n vertices and m arcs, then D^{++-} is regular if and only if D is an $m-n+1$-regular digraph.*

Proof. By Proposition 2.1, for any vertex $v \in V(D)$, $d_{D^{++-}}^+(v) = d_{D^{++-}}^-(v) = m$, and for any arc $a_{ij} = (v_i, v_j) \in A(D)$, $d_{D^{++-}}^+(a_{ij}) = d_D^+(v_j)+n-1$, $d_{D^{++-}}^-(a_{ij}) = d_D^-(v_i)+n-1$. Therefore, if D^{++-} is regular, then $d_D^+(v) = d_D^-(v) = m-n+1$ for every vertex $v \in V(D)$, hence D is an $m-n+1$-regular digraph. On the other hand, if D is an $m-n+1$-regular digraph, then it is clear that D^{++-} is regular. □

In the following theorem, denote by K_1 an isolated vertex, $\overrightarrow{S_{n_1}}$ a digraph with n_1 vertices, where $V(\overrightarrow{S_{n_1}}) = \{v_1, v_2, \cdots, v_{n_1}\}$ and $A(\overrightarrow{S_{n_1}}) = \{(v_1, v_j) | \ \forall\ v_j \in V(\overrightarrow{S_{n_1}})\backslash\{v_1\}\}$. Denote by $\overleftarrow{S_{n_2}}$ a digraph with n_2 vertices, where $V(\overleftarrow{S_{n_2}}) = \{u_1, u_2, \cdots, u_{n_2}\}$ and $A(\overleftarrow{S_{n_2}}) = \{(u_j, u_1) | \ \forall\ u_j \in V(\overleftarrow{S_{n_2}})\backslash\{u_1\}\}$.

Theorem 2.3. *Let D be a digraph with at least one arc. Then $diam(D^{++-}) \le 3$, and the equality holds if and only if $D \cong m_1\overrightarrow{S_{n_1}} \cup m_2\overleftarrow{S_{n_2}} \cup m_3K_1$ with at least two non-negative integers of $\{m_1, m_2, m_3\}$ which are not* 0.

To prove our result, we first prove the following two claims:

Claim 1. Let D be a digraph, D^{++-} is strongly connected if and only if D has at least one arc.

Proof. If D contains no arc, then D^{++-} is not strongly connected. Therefore, if D^{++-} is strongly connected, D must have at least one arc.

On the other hand, Let $a_{ij} = (v_i, v_j)$ be an arc, then (v_i, v_j, a_{ij}, v_i) is a 3-cycle in D^{++-}.

For any $x, y \in V(L(D))$, if $(x, y) \in A(L(D))$, then $(x, y) \in A(D^{++-})$. Now we consider $(x, y) \notin A(L(D))$. Let $x = (v_s, v_t) \in A(D)$, $y = (u_s, u_t) \in A(D)$. If $v_s = u_s$ and $v_t \neq u_t$, then (x, u_t, y) is a path from x to y. If $v_s \neq u_s$, then (x, v_s, y) is a path from x to y.

For any $x \in V(D), y \in V(L(D))$, let $y = (u_s, u_t) \in A(D)$. If $x = u_s$, then (x, u_t, y) is a path from x to y. If $x \neq u_s$, then $(x, y) \in A(D^{++-})$. Furthermore, if $x = u_t$, then (y, u_s, x) is a path from y to x. If $x \neq u_t$, then $(y, x) \in A(D^{++-})$.

For any $x, y \in V(D)$, if $(x, y) \in A(D)$, then $(x, y) \in A(D^{++-})$. Now we consider $(x, y) \notin A(D)$. If $x = v_i$ and $y \neq v_j$, then (x, v_j, a_{ij}, y) is a path from x to y. If $x \neq v_i$ and $y = v_j$, then (x, a_{ij}, v_i, y) is a path from x to y. If $x \neq v_i$ and $y \neq v_j$, then (x, a_{ij}, y) is a path from x to y. Thus D^{++-} is strongly connected. □

Claim 2. Let D be a digraph with at least one arc. Then $diam(D^{++-}) = 3$ if and only if $D \cong m_1\overrightarrow{S_{n_1}} \cup m_2\overleftarrow{S_{n_2}} \cup m_3K_1$ with at least two non-negative integers of $\{m_1, m_2, m_3\}$ which are not 0.

Proof. If $D \cong m_1\overrightarrow{S_{n_1}} \cup m_2\overleftarrow{S_{n_2}} \cup m_3K_1$, then $diam(D^{++-}) = 3$. Conversely, if $diam(D^{++-}) = 3$, then there exist two vertices $x, y \in V(D^{++-})$ such that $d(x, y) = 3$. From the proof of Claim 1, we know that x, y must be in $V(D)$. Then $(x, y) \notin A(D^{++-})$, $N_D^+(x) \cap N_D^-(y) = \emptyset$ and there is no arc $a = (u_i, u_j) \in A(D)$ such that $x \neq u_i$, $y \neq u_j$. Hence each arc of D satisfies that either its tail is x or its head is y, and $N_D^+(x) \cap N_D^-(y) = \varnothing$, ie. $D \cong m_1\overrightarrow{S_{n_1}} \cup m_2\overleftarrow{S_{n_2}} \cup m_3K_1$ with at least two non-negative integers of $\{m_1, m_2, m_3\}$ which are not 0. □

Proof of Theorem 2.3. Since D has at least one arc, $diam(D^{++-})$ is well defined by Claim 1. By the proof of Claim 1, we know that $diam(D^{++-}) \leq 3$, by Claim 2 the equality holds if and only if $D \cong m_1\overrightarrow{S_{n_1}} \cup m_2\overleftarrow{S_{n_2}} \cup m_3K_1$ with at least two non-negative integers of $\{m_1, m_2, m_3\}$ which are not 0. □

Since there is no digraph D such that D^{++-} is a complete digraph, we have $diam(D^{++-}) \neq 1$ for any digraph D. We therefore deduce the following corollary.

Corollary 2.4. *Let D be a digraph with at least one arc. Then $diam(D^{++-}) = 2$ if and only if $D \not\cong m_1\overrightarrow{S_{n_1}} \cup m_2\overleftarrow{S_{n_2}} \cup m_3K_1$ with at least two non-negative integers of $\{m_1, m_2, m_3\}$ which are not* 0.

3 Super-arc-connected D^{++-}

In [3], Chen characterized super-edge-connected undirected transformation graph G^{++-}. For any given graph G with at least two edges, G^{++-} is super-λ if and only if $G \not\cong 2K_2 \cup mK_1$, $K_{1,2} \cup mK_1$, $K_3 \cup mK_1$, $2K_3$, $K_2 \cup K_3$, $K_2 \cup P_3$, P_4, where m is a non-negative integer. In the following, we will study super-arc-connected or maximally-arc-connected transformation digraphs D^{++-}.

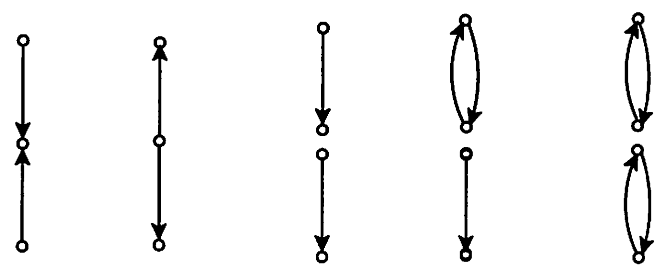

Figure 1

Theorem 3.1. *Let D be a digraph with at least one arc. Then D^{++-} is super-λ if and only if D is not isomorphic to $\overrightarrow{C_2} \cup mK_1$ $(m \geq 0)$ or digraphs shown in Fig.1, where $\overrightarrow{C_2}$ denotes the directed cycle of length* 2.

Proof. It is clear that if D is isomorphic to $\overrightarrow{C_2} \cup mK_1$ $(m \geq 0)$ or digraphs in Fig.1, then D^{++-} is not super-λ. On the other hand, if $n = 2$, then it is evident that D^{++-} is super-λ only if $D \not\cong \overrightarrow{C_2}$. If $n = 3$, it is easy to see that D^{++-} is super-λ only if D is not isomorphic to $\overrightarrow{C_2} \cup K_1$ or digraphs $D1, D2$ in Fig.1. Now we consider the case $n \geq 4$. In order to prove that D^{++-} is super-λ, it suffices to show that $\lambda(D^{++-}) \geq \delta(D^{++-})$ and every minimum arc-cut is either $E^+_D(v)$ or $E^-_D(v)$ for some vertex $v \in V(D^{++-})$. Let S be a minimum arc-cut of D^{++-}, then there exists a non-empty proper vertex subset $X \subseteq V(D^{++-})$ such that the subdigraph induced by X in D^{++-} is strongly connected and there are no arcs from X to $\overline{X}$ in $D^{++-} \setminus S$, where $\overline{X} = V(D^{++-}) \setminus X$.

We consider three cases.

Case 1. $X \subseteq V(D)$.

We claim that $|X| = 1$. In fact, if $2 \leq |X| \leq n-1$, since every vertex $a \in V(L(D))$ has at least $|X|-1$ in-neighbors in X, we have

$$|S| \geq |E[X, \overline{X} \cap V(D)]| + m(|X|-1) \geq m \geq min\{m, \delta(D)+n-1\} = \delta(D^{++-}).$$

The above equality holds if and only if the following conditions hold:

(1) $m \leq \delta(D) + n - 1$;

(2) $|X| = 2$, and the tails of all arcs of D are the vertices in X;

(3) $E[X, \overline{X} \cap V(D)] = \emptyset$.

It is evident that the above three conditions hold if and only if $D \cong \overrightarrow{C_2} \cup mK_1$ $(m \geq 2)$, a contradiction.

If $|X| = n$, then

$$|S| \geq m(n-1) > m \geq \delta(D^{++-}),$$

a contradiction.

It follows that $|X| = 1$. Let $X = \{x\}$, then $S = E^+_{D^{++-}}(x)$.

Case 2. $X \subseteq V(L(D))$.

If $2 \leq |X| \leq m-1$, since every vertex $a \in X$ has $n-1$ out-neighbors in $V(D)$, we have

$$|S| \geq |X|(n-1) + |E[X, \overline{X} \cap V(L(D))]| \geq 2(n-1) = n + (n-2) \geq n + \delta(D) - 1 \geq \delta(D^{++-}).$$

The above equality holds if and only if the following conditions hold:

(1) $|X| = 2$;

(2) $\delta(D) = n - 1$;

(3) $E[X, \overline{X} \cap V(L(D))] = \emptyset$.

It is evident that the above three conditions can not hold at the same time, a contradiction.

If $|X| = m$, then

$$|S| \geq m(n-1) > m \geq \delta(D^{++-}),$$

a contradiction.

Thus $|X| = 1$ and $S = E^+_{D^{++-}}(x)$, where $X = \{x\}$.

Case 3. $X \cap V(L(D)) \neq \varnothing$ and $X \cap V(D) \neq \varnothing$.

Let $|X \cap V(D)| = n_1$ and $|X \cap V(L(D))| = n_2$. If $V(D) \subseteq X$, then $\overline{X} \subseteq V(L(D))$. If $V(L(D)) \subseteq X$, then $\overline{X} \subseteq V(D)$, the result can be proved by a similar argument to Case 1 or Case 2. Thus we may suppose that $V(D) \nsubseteq X$ and $V(L(D)) \nsubseteq X$. Thus $1 \leq n_1 \leq n-1$ and $1 \leq n_2 \leq m-1$.

Subcase 3.1. $n_1 = 1$.

If $n_2 = 1$, let $X \cap V(D) = \{v\}$, $X \cap V(L(D)) = \{a\}$. Since a has at least $n-2$ out-neighbors in $\overline{X} \cap V(D)$ and v has at least $m - d^+_D(v) - 1$ out-neighbors in $\overline{X} \cap V(L(D))$, we have

$$\begin{aligned} |S| &\geq d^+_D(v) + d^+_{L(D)}(a) + n - 2 + m - d^+_D(v) - 1 \\ &= d^+_{L(D)}(a) + n - 3 + m \\ &> m \geq \delta(D^{++-}), \end{aligned}$$

a contradiction.

If $2 \leq n_2 \leq n-1$, since for any $a \in X \cap V(L(D))$, a has at least $n-2$ out-neighbors in $\overline{X} \cap V(D)$, we have

$$\begin{aligned} |S| &\geq d^+_D(v) + |E[X \cap V(L(D)), \overline{X} \cap V(L(D))]| + n_2(n-2) \\ &\geq \delta(D) + 2(n-2) \geq \delta(D) + n - 1 + (n-3) \\ &> \delta(D) + n - 1 \geq \delta(D^{++-}), \end{aligned}$$

a contradiction.

Subcase 3.2. $2 \leq n_1 \leq n-2$.

Since for any $a \in X \cap V(L(D))$, a has at least $n - n_1 - 1$ out-neighbors in $\overline{X} \cap V(D)$, and for any $b \in \overline{X} \cap V(L(D))$, b has at least $n_1 - 1$ in-neighbors in $X \cap V(D)$, we have

$$\begin{aligned} |S| &\geq |E[X \cap V(D), \overline{X} \cap V(D)]| + |E[X \cap V(L(D)), \overline{X} \cap V(L(D))]| \\ &\quad + n_2(n - n_1 - 1) + (n_1 - 1)(m - n_2) \\ &\geq n_2 + m - n_2 = m \geq \delta(D^{++-}), \end{aligned}$$

and the above equality holds if and only if the following conditions hold:

(1) $n_1 = 2, n = 4$;

(2) $E[X \cap V(D), \overline{X} \cap V(D)] = \emptyset$, and $E[X \cap V(L(D)), \overline{X} \cap V(L(D))] = \emptyset$;

(3) The heads of arcs corresponding to the vertices in $X \cap V(L(D))$ are the vertices in $\overline{X} \cap V(D)$. The tails of arcs corresponding to the vertices in $\overline{X} \cap V(L(D))$ are the vertices in $X \cap V(D)$.

From the above three cases we see that the heads and tails of arcs corresponding to the vertices in $X \cap V(L(D))$ are those in $\overline{X} \cap V(D)$, and the heads and tails of arcs corresponding to the vertices in $\overline{X} \cap V(L(D))$ are those in $X \cap V(D)$. Thus, it is evident that the above three cases hold if and only if D is isomorphic to $D3, D4, D5$ in Fig.1.

Subcase 3.3. $n_1 = n - 1$.

In this case, the result follows by applying the result of Subcase 3.1 to the reverse digraph of D.

We thus conclude that if $D \ncong \overrightarrow{C_2} \cup mK_1 (m \geq 2)$ with $|V(D)| \geq 4$, then D^{++-} is super-λ. Hence if D is not isomorphic to $\overrightarrow{C_2} \cup mK_1$ $(m \geq 0)$ or digraphs in Fig.1, then D^{++-} is super-λ. □

Corollary 3.2. *Let D be a digraph with order at least two, then D^{++-} is max-λ if and only if D has at least one arc.*

Proof. If D contains no arc, then D^{++-} is not max-λ. If D is isomorphic to $\overrightarrow{C_2} \cup mK_1$ $(m \geq 0)$ or digraphs in Fig.1, then D^{++-} is *max*-λ. Since super-arc-connected digraph is maximally arc-connected, hence if D is a digraph with at least one arc, then D^{++-} is max-λ.□

4 Super-connected D^{++-}

In this section, we will study super-connected transformation digraph D^{++-}.

Theorem 4.1. *Let D be a digraph with at least one arc. If $\lambda(D) = \delta(D) \geq 3$, then D^{++-} is super-κ.*

Proof. Let T be a minimum vertex-cut of D^{++-}. Then there exists a non-empty proper vertex subset $X \subseteq V(D^{++-})$ such that there is no arc from X to $\overline{X}$ in $D^{++-} \setminus T$ and $\overline{X} \neq \emptyset$, where $\overline{X} = V(D^{++-}) \setminus (X \cup T)$ and the subdigraph induced by X is strongly connected in D^{++-}.

Now we consider three cases.

Case 1. $X \subseteq V(D)$.

We claim that $|X| = 1$. In fact, if $2 \leq |X| \leq n-1$, note that either $N_D^+(X)$ is a vertex-cut of D or $N_D^+(X) = V(D) \setminus X$, since $\lambda(D) = \delta(D) \geq 3$, hence $N_D^+(X) \geq 1$, and for each vertex $a \in V(L(D))$, there is at least one in-neighbor of a in $V(D) \cap X$, we have

$$|T| \geq |N_D^+(X)| + m > m \geq min\{m, \delta(D) + n - 1\} \geq \delta(D^{++-}), \quad (1)$$

a contradiction.

If $|X| = n$, then for any vertex $a_{ij} \in V(L(D))$, there exists an arc $(v_j, a_{ij}) \in E[V(D), V(L(D))]$, so T must contain all vertices in $V(L(D))$, which is impossible.

It follows that $|X| = 1$. Let $X = \{x\}$, then $T = N_{D^{++-}}^+(x)$.

Case 2. $X \subseteq V(L(D))$.

If $2 \leq |X| \leq m - 1$, since the subdigraph induced by X is strongly connected, each vertex of D is out-neighbor of some vertex of X, then $V(D) \subseteq T$. If $N_{L(D)}^+(X) = V(L(D)) \setminus X$, then $\overline{X} = \emptyset$, a contradiction. If $N_{L(D)}^+(X) \neq V(L(D)) \setminus X$, then $|N_{L(D)}^+(X)| \geq \kappa(L(D))$, we have

$$\begin{aligned} |T| &\geq |N_{L(D)}^+(X)| + n \geq \kappa(L(D)) + n \geq \lambda(D) + n \\ &= \delta(D) + n > \delta(D) - 1 + n \geq \delta(D^{++-}), \end{aligned} \quad (2)$$

a contradiction.

If $|X| = m$. then for any vertex $v_i \in V(D)$, there exists an arc $(a_{ij}, v_i) \in E[V(L(D)), V(D)]$, so T must contain all vertices in $V(D)$, which is impossible.

Thus $|X| = 1$ and $T = N_{D^{++-}}^+(x)$, where $X = \{x\}$.

Case 3. $X \cap V(L(D)) \neq \emptyset$ and $X \cap V(D) \neq \emptyset$.

If $|X \cap V(D)| \geq 2$, each vertex in $V(L(D)) \setminus X$ is the out-neighbor of some vertex in $X \cap V(D)$, then $\overline{X} \subseteq V(D)$, the result can be proved by a similar argument to Case 1 . Now we consider $|X \cap V(D)| = 1$, let $X \cap V(D) = \{x\}$. Since $X \cap V(L(D)) \neq \emptyset$, there is at most one vertex in $V(D) \setminus \{x\}$ which is not out-neighbor of X. If $V(D) \setminus \{x\} \subseteq N_{D^{++-}}^+(X)$, then $\overline{X} \subseteq V(L(D))$, the result can be proved by a similar argument to Case 2. Otherwise there is only one vertex $y \in V(D) \setminus \{x\}$ such that $y \notin N_{D^{++-}}^+(X)$, all vertices in $X \cap V(L(D))$ are taking y as head. Thus $X \cap V(L(D))$ is an independent set. It is easy to see that $N_{L(D)}^+(X \cap V(L(D))) = E_D^+(y)$, for any $a \in X \cap V(L(D)), N_{L(D)}^+(a) = E_D^+(y) = d_D^+(y)$. Since D is a strict digraph, there is at most one arc b such that $b = (y, x) \in A(D)$, we have

$$
\begin{aligned}
|T| &\geq n-2+d_D^+(y)+d_D^-(x)-1 \geq n+\delta(D) \\
&> \delta(D)-1+n \geq \delta(D^{++-}),
\end{aligned} \tag{3}
$$

a contradiction. We thus conclude that if $\lambda(D)=\delta(D)\geq 3$, then D^{++-} is super-κ. □

By the proof of Theorem 4.1, we can obtain the following corollary.

Corollary 4.2. *Let D be a digraph with at least one arc. If $\lambda(D) \geq \delta(D)-1\geq 1$, then D^{++-} is max-κ.*

Proof. If $\lambda(D)\geq\delta(D)-1\geq 1$, then (1),(2),(3) will hold in the following forms,

$$
\begin{aligned}
|T| &\geq |N_D^+(X)|+m\geq m\geq min\{m,\delta(D)+n-1\}\geq\delta(D^{++-}); && (1') \\
|T| &\geq |N_{L(D)}^+(X)|+n\geq\kappa(L(D))+n\geq\lambda(D)+n\geq\delta(D)-1+n \\
&\geq \delta(D^{++-}); && (2') \\
|T| &\geq n-2+d_D^+(y)+d_D^-(x)-1\geq\delta(D)-1+n\geq\delta(D^{++-}). && (3')
\end{aligned}
$$

Thus $\kappa(D^{++-})\geq\delta(D^{++-})$ for each cases. Therefore, D^{++-} is max-κ.□

References

[1] J.Bang-Jensen, G. Gutin, Digraphs: Theory, Algorithms and Applications, Athenaum Press Ltd, 2001.

[2] J.A. Bondy, U.S.R. Murty, Graph Theory with Applications, Macmillan, London, Elsevier, New York, 1976.

[3] J. Chen, Super Edge-Connectivity of Two Class of Transformation Graphs, Ph. D.Thesis, Xinjiang University, 2006.

[4] J. Liu and J. Meng , Super-Arc-Connected and Super-Connected Total Digraphs, to appear.

[5] B. Wu and J. Meng, Basic Properties of Total Transformation Graphs, J. Math. Study, 34(2)(2001) 109-116.

[6] B. Wu and L. Zhang, Z. Zhang, the Transformation Graph G^{xyz} When $xyz=-++$, Discrete Mathematics, 296(2005) 263-270.