On the Magic Space of Locally Finite Graphs *

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Abstract

A *labelling* of a graph over a field $\mathbb{F}$, is a mapping of the edge set of the graph into $\mathbb{F}$. A labelling is called *magic* if for any vertex, the sum of the labels of all the edges incident to it is the same. The class of all such labellings forms a vector space over $\mathbb{F}$ and is called the *magic space* of the graph. For finite graphs, the dimensional structure of the magic space is well known. In this paper, we give the existence of magic labellings and discuss the dimensional structure of the magic space of *locally finite graphs*. In particular, for a class of locally finite graphs, we give an explicit basis of the magic space.

Keywords: Labellings, Magic Spaces, Locally Finite Graphs, Matroids.

1 Introduction

The concept of *magic graph* is due to [8, Sedláček]. He defined it to be a graph for which a real valued edge labelling exists satisfying

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1. the distinct edges have distinct non-negative labels, and
2. the sum of the labels of the edges incident to any vertex is the same for all vertices.

For finite graphs, the question of the precise structure of such labellings was investigated from several viewpoints. In [10], Stewart omitted the first condition and studied the properties of the real vector space of all magic labellings of finite graphs. Several researchers, (see *e.g.,* Stanley [9], Murty [7], Kotzig and Rosa [6], Guy [3] and Doob [2], *etc.*) studied the algebraic properties of graph labellings with different viewpoints. For finite graphs, Doob used the theory of matroids to study the dimensional structure of magic labellings over an integral domain.

In this paper, we investigate the structure of magic labellings of a class of locally finite graphs over a field $\mathbb{F}$. Section 2 gives definitions and known results for locally finite graphs. In Section 3, we define the notion of the constrained labelling of a locally finite graph and prove that a constrained labelling does exist for any locally finite graph. In Section 4, we prove that for a certain class of locally finite graphs, the dimension of the magic-space is always finite. In Section 5, we review a few results on matroid theory and use it to show that for a certain class of locally finite graphs, the dimension of the semi-magic space is exactly one more than the dimension of the zero-magic space.

Throughout this paper, $\mathbb{F}$ will denote a field, 0 will denote the zero element of the field, 1 will denote the multiplicative identity of the field and the word graph will mean an undirected connected graph, without loops or multi-edges. For all the graph-theoretic terms that have not been defined but are used in the paper, see Harary [4].

2 Preliminaries

Let S be a set. By $|S|$, we denote the number of elements in S. A graph $G = (V, E)$ is called a *finite graph* if $|V| < \infty$ (and so $|E| < \infty$), otherwise G is called an *infinite graph.* A subgraph of an infinite graph G is called a *singly infinite path* or a 1-*way ray* (see Figure 1) if it is isomorphic to the graph R with vertex set $V(R) = \{v_i : i = 0, 1, 2, \ldots\}$ and edge set $E(R) = \{e_i = (v_i, v_{i+1}) : i = 0, 1, 2, \ldots\}$ and is called a *doubly infinite path* or a 2-*way ray* (see Figure 2) if it is isomorphic to the graph R with vertex set $V(R) = \{v_i : i = 0, \pm 1, \pm 2, \ldots\}$ and edge set $E(R) = \{e_i = (v_i, v_{i+1}) : i = 0, \pm 1, \pm 2, \ldots\}$.

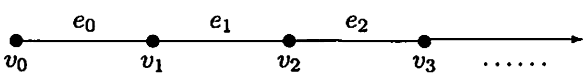

Figure 1: A 1-way Ray

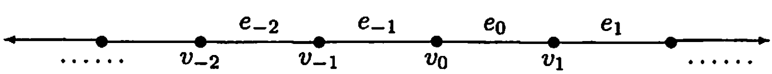

Figure 2: A 2-way Ray

For a vertex v of a graph $G = (V, E)$, let $d_G(v)$ (in short $d(v)$) denote the degree (or valency) of the vertex v in G. An infinite graph G is called *locally finite,* if $d(v) < \infty$, for all $v \in V$. A 1-way ray R in a locally finite graph G is called an *independent ray,* if all the vertices of R have degree 2 in G. An independent ray is called a *maximal independent ray,* if it is not properly contained in any other independent ray. With the above definitions, we now state three main results of this area which we need for our further use.

Theorem 2.1. *[5, König] Let G be a locally finite graph. Then the set of vertices as well as the set of edges of G are always countably infinite.*

Theorem 2.2. *[5, König] Let G be a locally finite graph. Then G always has a 1-way ray, where the initial vertex of this ray can be arbitrarily specified.*

Theorem 2.3. *[5,* **Infinity Lemma***] Let $\Pi_1, \Pi_2, \Pi_3, \ldots$ be a countably infinite sequence of finite, nonempty and pairwise disjoint sets of points. Let the points contained in these sets form the vertices of a graph G. If G has the property that every point of Π_{n+1} $(n = 1, 2, \ldots)$ is adjacent with some point of Π_n by an edge of G, then G has a singly infinite path R with $V(R) = \{v_n : n = 1, 2, \ldots\}$ and $E(R) = \{e_n = (v_n, v_{n+1}) : n = 1, 2, \ldots\}$, where $v_n \in \Pi_n$, for $n = 1, 2, \ldots$.*

3 Constrained labelling

Let $G = (V, E)$ be a locally finite graph with vertex set $V = \{v_i : i = 0, 1, \ldots\}$. A sequence $\mathbf{r} = \big(r(v)\big)_{v \in V} = \big(r(v_0), r(v_1), \ldots\big)$, where $r(v_i) \in \mathbb{F}$, for all $i \geq 0$, is called a constrained sequence of G over $\mathbb{F}$. Let f be a labelling of G, *i.e.*, a mapping $f : E \to \mathbb{F}$, where $f(e) \in \mathbb{F}$ is called the label of the edge e. If f is defined in such a way that the sum of the labels of the edges incident at v_i is $r(v_i)$, for $i = 0, 1, \ldots$ then f is said to be a constrained labelling of G with respect to the constrained sequence $\mathbf{r}$. In fact, f is a constrained labelling of G with respect to the constrained sequence $\mathbf{r}$ if

$$\sum_{e \in E(G)} \eta(v, e) f(e) = r(v), \text{ for all } v \in V(G),$$

where $\eta(v, e)$ is 1 if the vertex v is incident with the edge e and 0, otherwise.

In case of finite graphs, $\mathbf{r}$ is a finite sequence. For locally finite graphs, the above sum is always well defined. The class of all such labellings of G, corresponding to a constrained sequence $\mathbf{r}$ is denoted by $R(G, \mathbf{r})$.

A graph $G = (V, E)$ is called *bipartite* with a bipartition (S, U) if $V = S \cup U$, $S \cap U = \emptyset$ and every edge of G joins a vertex in S with a vertex in U. In a connected graph, the bipartition is always unique. For finite graphs, the following theorem is due to Doob [2], and will be useful in Section 4.

Theorem 3.1. *Let $G = (V, E)$ be a finite connected graph with a bipartition (S, U) and let $\mathbf{r} = \left(r(v)\right)_{v \in V}$ be a finite constrained sequence over $\mathbb{F}$. Then there exists a constrained labelling of G with respect to $\mathbf{r}$ if and only if $\sum_{v \in S} r(v) = \sum_{v \in U} r(v)$.*

We are now ready to state and prove our results related to constrained labelling of locally finite graphs.

Lemma 3.2. *Let $\mathbf{r}$ be a constrained sequence over $\mathbb{F}$. Then there always exists a constrained labelling of the 1-way ray, corresponding to the sequence $\mathbf{r}$.*

Proof : Let P be a 1-way ray with edge set $E(P) = \{e_i = (v_i, v_{i+1}) : i = 0, 1, 2, \ldots\}$. We exhibit a constrained labelling f of P, inductively as follows. Define

$$f(e_0) = r(v_0),\ f(e_1) = r(v_1) - r(v_0) \text{ and for } i \geq 1,\ f(e_{i+1}) = r(v_{i+1}) - f(e_i).$$

Then by the principle of mathematical induction, it can be easily verified that f is a constrained labelling of P. □

For a vertex v of a graph $G = (V, E)$, we define $N_i(v) = \{u \in V : d(u, v) \leq i\}$ for $i = 1, 2, \ldots$, and is called the i-th neighbourhood of v. As G is a locally finite graph, the set $N_i(v)$ is always finite for all $v \in V$ and for all $i = 1, 2, \ldots$. The set $B_i(v) = N_i(v) - N_{i-1}(v)$ is called the i-th boundary of v, $i = 1, 2, \ldots$, with the convention that $B_0(v) = \{v\}$. A pendant vertex of a graph G is a vertex of degree one.

Before stating and proving the main theorem of this section, we need to state a few results. The proof of these results are given in [1].

Lemma 3.3. *Let G be a locally finite connected graph and let $\mathbf{r} = \left(r(v)\right)_{v \in V(G)}$ be a constrained sequence over $\mathbb{F}$. If for some vertex v, any vertex of $B_i(v)$ is adjacent with some vertex in $B_{i+1}(v)$, $i = 1, 2, \ldots$, then there exists a constrained labelling of G with respect to the sequence* $\mathbf{r}$.

Before stating one more lemma and the main theorem, we need the following definitions and notations. We give the proof of these two results for the sake of completeness. Let G be a locally finite connected graph and $v \in V(G)$.Let us designate a finite path $P = v_0 v_1 v_2 \ldots v_n$, where $v_i v_{i+1}$ is an edge of G for $i = 0, 1, \ldots n-1$ in G, as an S-path, if the following conditions are satisfied :

(i) there exists i and j with $i \leq j$ such that $v_0 \in B_i(v)$ and $v_n \in B_j(v)$,

(ii) for any other intermediate vertex v_k, either $v_k, v_{k+1} \in B_j(v)$ or $v_k \in B_l(v)$ for some l with $i \leq l \leq j-1$ and in that case $v_{k+1} \in B_l(v) \cup B_{l+1}(v)$.

We call v_0 and v_n as the *first* and *last end vertices* respectively, of the S-path and other vertices as *interior vertices.*

Lemma 3.4. *Let G be a locally finite connected graph and $v \in V(G)$. Let W_i be the set of those vertices belonging to $B_i(v)$ that are not adjacent to any vertex of $B_{i+1}(v)$, $i = 1, 2, \ldots$ and let $W = \bigcup_{i=1}^{\infty} W_i$. Suppose W is an infinite set. Then there exists a collection $\{P_w : w \in W\}$ where P_w is a 1-way ray with initial vertex w, such that for any infinite subset $\overline{W}$ of W, we get $\bigcap_{w \in \overline{W}} E(P_w) = \emptyset$.*

Proof : Consider the sequence $\{B_i(v) : \ i = 1, 2, \ldots\}$. Clearly this sequence and so any tail *viz.* $X_j = \{B_{j+i}(v) : \ i = 0, 1, 2, \ldots\}$, $j = 1, 2, \ldots$ of this sequence satisfy the conditions of Infinity Lemma. Now we define

$R_0(v) =$ the class of all 1-way rays with initial vertex v ; and for $j = 1, 2, \ldots$.

$R_j(v) =$ the class of all 1-way rays having initial vertex from $B_j(v)$ but the rays does not contain any vertex from $B_{j-1}(v)$.

Then by Infinity Lemma, the sets $R_j(v)$ for $j = 0, 1, 2, \ldots$ are nonempty. Let $w \in W$. Then $w \in B_k(v)$, for some positive integer k. Let $\mathcal{S}_w$ be the class of all S-paths that starts at a vertex u and ends at w, where u is the initial vertex of some member of $R_j(v)$ for $0 \leq j \leq k$. Then the class $\mathcal{S}_w$ is nonempty and finite. Let $j = j(w)$ be the largest integer such that there is an S-path, say $S_w \in \mathcal{S}_w$ whose first end is the initial vertex of some 1-way ray in $R_j(v)$. Let us choose an element $P_j \in R_j(v)$ that starts at the first vertex of S_w and form the 1-way ray P_j^w by joining the initial vertex of P_j with the first end vertex of S_w. We now claim that the class $\mathcal{P} = \{P_j^w : w \in W\}$ has the desired property.

Suppose the claim is not true. This implies that there exists an edge $e = xy$ and an infinite set $\{w_1, w_2, \ldots\} \subseteq W$ such that $e \in \bigcap_{k=1}^{\infty} E(P_{j_k}^{w_k})$, where $j_k = j(w_k)$. In this case, there exists $l \geq 0$ such that $x, y \in B_l(v)$ or $x \in B_l(v)$ and $y \in B_{l+1}(v)$ or $y \in B_l(v)$ and $x \in B_{l+1}(v)$. As $e \in \bigcap_{k=1}^{\infty} E(P_{j_k}^{w_k})$ and j_k is the largest integer for which $P_{j_k}^{w_k}$ is defined, we have that $j_k \leq l$, for all $k = 1, 2, \ldots$.

Let H be the subgraph of G having vertex set $\bigcup_{k=1}^{\infty} V(S_{w_k}) - N_{l-1}(v)$ and edge set $\bigcup_{k=1}^{\infty} E(S_{w_k}) - E(\langle N_l(v) \rangle)$, where $\langle N_l(v) \rangle$ is the subgraph induced by $N_l(v)$. Note that H is an infinite graph with at most finitely many isolated vertices.

As H is an infinite graph, we need to consider the following two cases:

Case I: H has finitely many connected components.

In this case, one component, say H_1, must be infinite. By the definition of $\mathcal{S}_{w_k}$, no interior vertex of S_{w_k} which is also a vertex of H can be the initial vertex of some 1-way ray. Therefore Theorem 2.2 implies that the infinite component H_1 is not locally finite, a contradition to the fact that G is locally finite.

Case II: H has infinitely many connected components.

Becasue $B_l(v)$ is a finite set, H cannot have infinitely many components. Therefore this case is not possible.

Hence the class $\mathcal{P}$ has the desired property. □

Theorem 3.5. *Let $G = (V, E)$ be a locally finite graph and let $\mathbf{r} = (r(v))_{v \in V}$ be a constrained sequence over $\mathbb{F}$. Then there exists a constrained labelling of G corresponding to $\mathbf{r}$.*

Proof : Let W be as in Lemma 3.4. Let us attach a new 1-way ray to each vertex in W. Now apply Lemma 3.3 to the obtained graph and remove all those rays again. As the new edges of the 1-way rays are not incident with vertices in $V - W$, the resulting labelling, say f of G satisfies the constrained conditions for all the vertices of $V - W$. Let us put

$$s(w_j) = r(w_j) - \sum_{e \in E(G)} \eta(w_j, e) f(e), \quad \text{for } j = 1, 2, \dots.$$

Case I. W is an infinite set :

Let $W = \{w_1, w_2, \dots\}$. Now for each $j = 1, 2, \dots,$ choose a 1-way ray P_j with initial vertex w_j such that any edge e of G belong to finitely many P_j's (existence of such P_j's is guaranteed by Lemma 3.4).

Let $E(P_j) = \{e_{ji} \mid e_{ji} \text{ is incident with } e_{j,i+1},\ i = 1, 2, \dots\}$. Let f_j be the labelling of G such that

$$f_j(e_{ji}) = \begin{cases} s(w_j), & \text{if } i \text{ is odd}, \\ -s(w_j), & \text{if } i \text{ is even}, \end{cases}$$

$$\text{and} \quad f_j(e) = 0, \text{if} \quad e \notin E(P_j).$$

Now it is easy to observe that the sum $f + \sum_{j=1}^{\infty} f_j$ is well defined, as any edge belong to finitely many P_j's. Also $f + \sum_{j=1}^{\infty} f_j$ is a labelling of G which satisfies the constrained conditions at each vertex.

Case II. W is a finite set :

Let $W = \{w_1, w_2, \ldots, w_k\}$. Now for $1 \leq j \leq k$, we can choose any 1-way ray P_j with initial vertex w_j and proceed exactly as in Case I to obtain a constrained labelling of G with respect to the sequence $\mathbf{r}$. □

Corollary 3.6. *Let $G = (V, E)$ be a locally finite graph and let $\mathbf{r} = \left(r(v)\right)_{v \in V}$ be a constrained sequence over $\mathbb{F}$. Then there exists a constrained labelling of G corresponding to $\mathbf{r}$ such that the set of edges of G with non-zero label is acyclic.*

Proof : By Zorn's Lemma, G has a spanning tree, say T. By Theorem 3.5, there exists a labelling of T corresponding to the constrained sequence $\mathbf{r}$. This labelling is extended to the graph G by labelling all the edges of $G - T$ with zero. Clearly, this extension has the required property. □

Remark 3.7. *The above results are not always true for finite graphs. In case of finite graphs, we get necessary and sufficient condition on the constrained sequence which depends on whether the graph considered is bipartite or not. See [2] for details.*

Remark 3.8. *The above results are also true if we consider an Abelian group in place of the field $\mathbb{F}$.*

4 The Zero Magic Space

Let $G = (V, E)$ be a graph and consider the set $A(G)$, of all functions with the edge set as domain and the field $\mathbb{F}$ as the co-domain. Then $A(G)$ is a vector space over $\mathbb{F}$, where the vector addition is defined by

$$(f_1 + f_2)(e) = f_1(e) + f_2(e), \quad \text{for all } f_1, f_2 \in A(G) \text{ and } e \in E,$$

and the scalar multiplication is defined by

$$(xf)(e) = x\, f(e), \quad \text{for all } f \in A(G),\ x \in \mathbb{F} \text{ and } e \in E.$$

This vector space is called the *edge space* of G. It is easy to see that if G is a finite graph then $\dim(A(G)) = |E|$. In fact, if $E = \{e_1, e_2, \ldots, e_n\}$, then for $1 \leq i \leq n$, the set of functions $\{f_1, f_2, \ldots, f_n\}$, defined by $f_i(e_j) = \delta_{ij}$, where δ_{ij} is the Kronecker delta function, forms a basis of $A(G)$. Using a similar idea, it follows that if G is an infinite graph then $\dim(A(G)) \geq |E|$.

We now look at a subset of $A(G)$ defined by

$$Z(G) = \{f \in A(G) \mid \sum_{e \in E} \eta(v,e) f(e) = 0, \text{ for all } v \in V\}.$$

It is clear that $Z(G)$ is a vector subspace of $A(G)$ over $\mathbb{F}$. The elements of $Z(G)$ are called zero magic labellings. In fact, $Z(G) = R(G, \mathbf{0})$, where $\mathbf{0} = (0, 0, 0, \ldots)$ is the zero constrained sequence. Before coming to the main result of this section, we need the following results and notations.

Lemma 4.1. *[2] If T is a finite tree, then $Z(T) = \{\mathbf{0}\}$.*

Proof : It is immediate by induction on the number of edges, as every finite tree has a vertex of degree one. □

Proposition 4.2. *[2, Doob] Let G be a graph and $\mathbf{r}$ be a constrained sequence over $\mathbb{F}$. Then the set $R(G, \mathbf{r})$ is a translation of $Z(G)$.*

Proof : Let f be a constrained labelling in $R(G, \mathbf{r})$. If g is any other element in $R(G, \mathbf{r})$, then it is clear that $g - f \in Z(G)$. So, $R(G, \mathbf{r}) \subseteq f + Z(G)$. The other inclusion is also obvious. Hence, $R(G, \mathbf{r})$ is a translation of $Z(G)$. □

Lemma 4.3. *Let T be a locally finite tree having only one maximal independent ray, then $Z(T) = \{\mathbf{0}\}$.*

Proof : As T has only one maximal independent ray, for any vertex v in $V(T)$, we can find a positive integer i, such that the induced subgraph $\langle T - N_i(v)\rangle$ is a 1-way ray. As T is a tree, the induced subgraph $\langle N_i(v)\rangle$ is also a tree. So by Lemma 4.1, $Z(\langle N_i(v)\rangle) = \{0\}$. Let e be the edge joining a vertex in $N_i(v)$ with a vertex in $N_{i+1}(v)$. Now note that a tree is a bipartite graph and therefore by Theorem 3.1, for any $f \in Z(T)$, the restriction of f to $\langle N_{i+1}(v)\rangle$ gives $f(e) = 0$. That is, f restricted to $\langle N_{i+1}(v)\rangle$ is the zero function. Also $f(e) = 0$ implies that f is zero on the edges of $\langle T - N_{i+1}(v)\rangle$. Hence f is identically zero *i.e.* $Z(T) = \{0\}$. □

Now for a certain class of locally finite graphs, our goal is to find an explicit basis of $Z(G)$. For this purpose we need three special zero magic labellings and a few definitions. Suppose -1 denotes the additive inverse of the unity 1 in $\mathbb{F}$, while 2 stands for $1 + 1$.

1. Let R be a double ray. Then we have exactly one linearly independent element in $Z(R)$, say $\mathbf{x}$, given as follows:
choose an edge $e \in E(R)$, and let $\mathbf{x}(e) = 1$, and label the other edges as in Figure 3. This particular labelling will be denoted by $\mathbf{x}_e$.

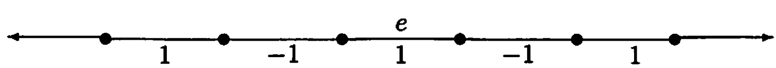

Figure 3: A zero magic labelling of a 2-way Ray.

2. Let C_{2n} be an even cycle and let e be an edge of C_{2n}. Then a labelling of C_{2n} is given in Figure 4, and is denoted by $\mathbf{y}_e$.

3. Recall that a *kite* is a graph, which contains a cycle (called the head) and a path (called the tail) that is attached to a vertex of the cycle. A kite is called odd or even according as the cycle is odd or even and is called an infinite kite if the path attached to it is a 1-way ray.

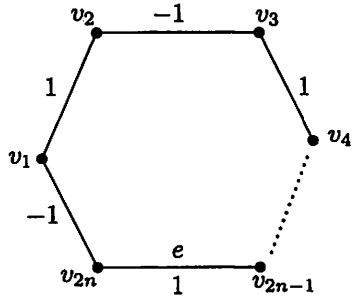

Figure 4: A zero magic labelling of an even cycle.

Let K be an infinite odd kite. Then the labelling given in Figure 5, denoted by $\mathbf{z}_e$ is an element of $Z(K)$.

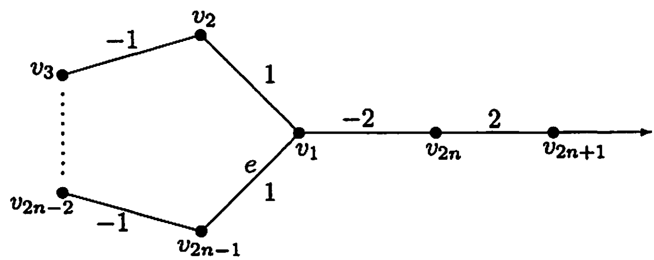

Figure 5: A zero magic labelling of an infinite odd kite.

With the labellings as defined above, we state and prove the main theorem of this section.

Theorem 4.4. *Let G be a locally finite graph having finitely many cycles and finitely many maximal independent 1-way rays. Then the dimension of $Z(G)$ is always finite.*

Proof : Let T be a spanning tree of $G = (V, E)$. Suppose G has $(n+1)$ maximal independent rays and finitely many finite cycles. Let $v \in V$. It is evident that there exists a positive integer m, such that, $G - N_m(v)$ contains $(n+1)$ distinct components, where each component is an independent ray. Let us denote the components of $G - N_m(v)$ by P_i, $1 \leq i \leq n+1$. For any two distinct 1-way rays P_i and P_j, it is obvious that there is a unique finite path in T which joins P_i and P_j and therefore forms a double ray, which we denote by P_{ij}. From each P_i, $i \geq 2$, choose one $e_i \in P_i$ and form the following sets:

$$\begin{aligned} E_1 &= \{e_i : 2 \leq i \leq n+1\}, \\ E_2 &= \{e \in E : \{e\} \cup E(T) \text{ contains an even cycle }\} \text{ and} \\ E_3 &= \{e \in E : \{e\} \cup E(T) \text{ contains an odd cycle }\}. \end{aligned}$$

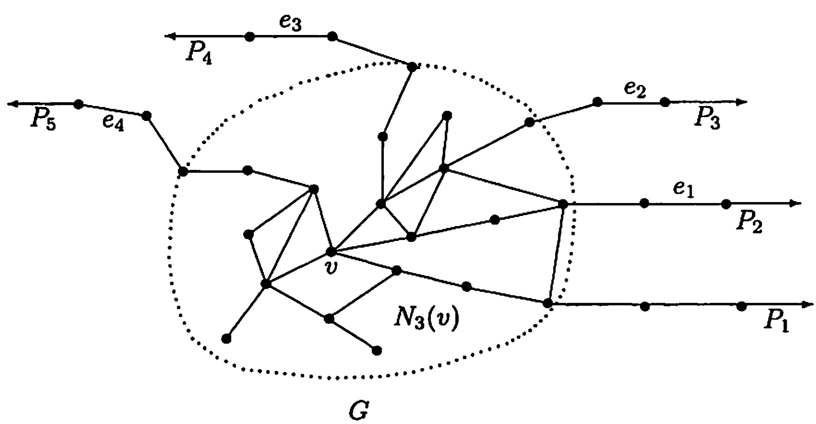

Figure 6: An example for clarity.

For $2 \leq i \leq n+1$, consider the labelling $\mathbf{x}_{e_i}$ of P_{1i}. This labelling can be extended to an element of $Z(G)$, by labelling all other edges by 0, the zero element of $\mathbb{F}$. We denote this extension also by $\mathbf{x}_{e_i}$. Similarly for each $e \in E_2$, we get an element $\mathbf{y}_e$ of $Z(G)$.

So far, we have used the sets E_1 and E_2 to get distinct elements of $Z(G)$. We will now use the set E_3 to get another set of elements of $Z(G)$. Note that, each $e \in E_3$ give rise to an odd cycle. Since T is a spanning tree, this cycle is connected to P_1 by a unique path, say Q. Then this odd cycle along with Q and P_1 forms an infinite kite, say C_{1e}. For each of the

C_{1e}'s, we get the labelling $\mathbf{z}_e$. This labelling is also extended to the whole of G, by simply labelling all other edges by 0 and the new labelling is also denoted by $\mathbf{z}_e$.

Note that for any $e \in E_1 \cup E_2 \cup E_3$, there is exactly one element of

$$\{\mathbf{x}_e : e \in E_1\} \cup \{\mathbf{y}_e : e \in E_2\} \cup \{\mathbf{z}_e : e \in E_3\},$$

which is non zero at the edge e. Hence we conclude that this set is linearly independent. We now wish to show that

$$Z(G) = \left\{ \sum_{e \in E_1} \alpha_e \, \mathbf{x}_e + \sum_{e \in E_2} \alpha_e \, \mathbf{y}_e + \sum_{e \in E_3} \alpha_e \, \mathbf{z}_e : \alpha_e \in \mathbb{F} \right\}.$$

Here, we only need to show that any element of $Z(G)$ can be expressed as the above sum.

Suppose f is any element of $Z(G)$ and consider the labelling

$$g = f - \left[\sum_{e \in E_1} f(e) \, \mathbf{x}_e + \sum_{e \in E_2} f(e) \, \mathbf{y}_e + \sum_{e \in E_3} f(e) \, \mathbf{z}_e \right] \in Z(G)$$

Let $\widetilde{T}$ be the sub-tree of T, obtained by removing all the 1-way rays having initial edges from E_1. Then $\widetilde{T}$ is a tree having exactly one 1-way ray. Hence by Lemma 4.3, $Z(\widetilde{T}) = \{0\}$. It is clear that $g(e) = 0$, for any edge $e \in E_1 \cup E_2 \cup E_3$ and hence $g(e) = 0$, for the edges in P_i, $2 \leq i \leq n+1$. Hence the restriction of g to $\widetilde{T}$ is in $Z(\widetilde{T}) = \{0\}$. Thus

$$g = 0 \Rightarrow f = \sum_{e \in E_1} f(e) \, \mathbf{x}_e + \sum_{e \in E_2} f(e) \, \mathbf{y}_e + \sum_{e \in E_3} f(e) \, \mathbf{z}_e.$$

Thus the proof of the theorem is complete. □

We now define the graph operation of joining two graphs. This operation is called *amalgamation*, and is defined as follows: let G_1 and G_2 be any two graphs. Without loss of generality, we can assume that G_1 and G_2 have no common vertices or edges (for if they are not disjoint, we replace G_2 by an isomorphic copy G_3 that is disjoint from G_1 and form the amalgamation of G_1 and G_3). Select a vertex v_1 of G_1 and a vertex v_2 of G_2. Then the amalgamation of G_1 and G_2 is formed by taking the disjoint union of G_1 and G_2 and then identifying v_1 with v_2. With the above definition, we have the following result:

Corollary 4.5. *Let G be a locally finite graph. Let the graph $\widetilde{G}$ be obtained by amalgamating some finite trees with some vertices of G. Then $Z(G)$ is isomorphic to $Z(\widetilde{G})$.*

Proof : Using the proof technique of Lemma 4.3, it is easy to observe that for any $f(\neq 0) \in Z(\widetilde{G})$, $f(e) = 0$ if e is an edge of a finite tree that are amalgamated with G. Hence $Z(G) \cong Z(\widetilde{G})$. □

Remark 4.6. *From Theorem 4.4, it follows that for locally finite graphs, the structure of $Z(G)$ is independent of the characteristic of the field $\mathbb{F}$. But it is not the case for finite graphs. See [2] for further details.*

5 The Semi-Magic Space

If we consider the constrained sequence $\mathbf{r} = (r(v))_{v \in V}$ to be any constant sequence, then $\bigcup_{\mathbf{r}} R(G, \mathbf{r})$ is called the semi-magic space and is denoted by $S(G)$. That is, we define

$$S(G) = \bigcup_{r \in \mathbf{F}} \left\{ f \in A(G) \mid \sum_{e \in E} \eta(v,e) f(e) = r, \text{ for all } v \in V \right\}.$$

It is easy to see that $S(G)$ is a vector subspace of $A(G)$. For a given $f \in S(G)$, the constant vertex sum r of f is sometimes called the *index* of f.

Now we wish to use matroid theory to investigate the dimensional structure of $S(G)$. To do this, we start with a few definitions related with matroid theory. For more details on matroid theory, see [11].

A *matroid* M is a pair $M = (E, C)$, where E is a non-empty finite set, usually called the set of edges and C is a certain class of subsets of E, usually called the set of circuits (sometimes these are called atoms) such that the following two conditions hold:

(i) no circuit properly contains another circuit.

(ii) if C_1 and C_2 are circuits, $e_1 \in C_1 \cap C_2$ and $e_2 \in C_1 - C_2$, then there is a circuit $C_3 \subset C_1 \cup C_2$, such that $C_3 \cap \{e_1, e_2\} = \{e_2\}$.

A *dendroid* D in a matroid M is a set of edges that has non-empty intersection with any circuit and is minimal with this property. Any two dendroids in a matroid M have the same cardinality. This value is called the *rank* of the matroid and is denoted by $r(M)$.

We now give some more definitions which will help us to relate some results on matroid theory with graph theory. A *chain group* on a finite set E over an integral domain $\mathbb{D}$ is a set of maps $f : E \to \mathbb{D}$, called *chains*, which is closed under point wise addition and scalar multiplication. The *support* of f, denoted $\mathsf{s}(f)$, is defined as $\mathsf{s}(f) = \{e \in E : f(e) \neq 0\}$. If $\mathsf{s}(f)$ is empty, then f corresponds to the zero chain. A chain f is called *elementary* if its support is nonempty and there is no non-zero chain g such that $\mathsf{s}(g) \subsetneq \mathsf{s}(f)$. That is, for any chain g, if $\mathsf{s}(g) \subseteq \mathsf{s}(f)$ and if f is elementary then either $\mathsf{s}(g) = \mathsf{s}(f)$ or $\mathsf{s}(g) = \emptyset$.

Theorem 5.1. *[11, Tutte] If N is a chain group on a nonempty set E over an integral domain $\mathbb{D}$, then the class M of the supports of all elementary chains of N is a matroid on E.*

If D is a dendroid of the matroid of a chain group, then for any $e \in D$, we can find a chain f_e such that $f_e(e) \neq 0$. Then the product $w = \prod_{e \in D} f_e(e)$ is called the weight of the chains $\{f_e : e \in D\}$.

Theorem 5.2. *[11, Tutte] Given a dendroid D and a set of chains $\{f_e : e \in D\}$ of weight w; for any other chain g; there exists $a_e \in \mathbb{D}$, for each $e \in D$; such that $wg = \sum_{e \in D} a_e f_e$.*

Now it is easy to see that if $G = (V, E)$ is a finite graph, then we can consider the labellings as chains in the straight forward manner and that $Z(G)$ and $S(G)$ form chain groups. But if G is an infinite graph, we cannot directly consider the labellings as chains, as in this case the edge set is an infinite set. That is, in this case we can not give a matroid structure on $E(G)$ directly. But for a locally finite graph G, we observe that if there are only finitely many maximal independent rays and finitely many cycles in G, then it is nothing but a finite graph $\bar{G}$ with finitely many 1-way rays attached to some vertices. Therefore, for a labelling of index r, as soon as the label of the edges which joins a 1-way ray with $\bar{G}$ is known, the label of all other edges of this ray is determined.

So, for a locally finite graph G, we consider a new finite graph formed by replacing all the maximal independent rays of G by pendent edges (a path of length one) such that restriction of having constant vertex sum at the newly formed pendent vertices are omitted. The space of all labellings of a finite graph having constant vertex sum at all but the newly formed pendent vertices will be called the *semi-restricted magic space*. We shall give an isomorphism of $S(G)$ $\big($or $Z(G)\big)$ to the semi-restricted magic space and use matroid theory to find out the dimension of this new semi-restricted magic space. We start with the following:

Let $G = (V, E)$ be a finite graph and let $U \subseteq V$. Define

$$S_U(G) = \bigcup_{r \in \mathbb{F}} \left\{ f \in A(G) \middle| \sum_{e \in E(G)} \eta(v, e) f(e) = r, \text{ for all } v \in V - U \right\}.$$

Note that if $U = \emptyset$, then $S_U(G) = S(G)$, the usual semi magic space for finite graphs. So in what follows, the subset U of V is non-empty.

It is easy to observe that $S_U(G)$ is also a vector subspace of $A(G)$. In a similar way, we define $Z_U(G)$ as well. Note that for $Z_U(G)$, $r = 0$. If $U = \{u\}$ a singleton set, we denote $S_U(G)$ and $Z_U(G)$ by $S_u(G)$ and $Z_u(G)$, respectively. It is worthwhile to note that if U is properly contained in V, then for any $f \in S_U(G)$, there is some vertex where the vertex sum is restricted. In such a case, for an $f \in S_U(G)$, the fixed vertex sum r corresponding to the vertices of $V - U$ is called the *index* of f. If $U = V$, an $f \in S_V(G)$ is called a labelling without an index. Before proceeding further, we need the following lemmas and notations.

Lemma 5.3. *Let $G = (V, E)$ be a finite graph, and U be a non-empty proper subset of V. Then for any $r \in \mathbb{F}$, there is an element of $S_U(G)$ with index r*

Proof : Let T be a spanning tree of G and let $u \in U$. Since a tree is bipartite, by demanding the vertex sum r at all vertices but u and a suitable vertex sum at u, Theorem 3.1 gives us a labelling as desired. □

Lemma 5.4. *Let T be a finite tree and let u be a pendent vertex of T. Then $Z_u(T) = \{0\}$.*

Proof : It is clear by Theorem 3.1. □

Let R be a 1-way ray with edge set $E(R) = \{(v_i, v_{i+1}) : i \geq 1\}$. Then the edges (v_{2i-1}, v_{2i}) and (v_{2i}, v_{2i+1}) for $i \geq 1$ are called *odd* and *even* edges of R respectively.

Proposition 5.5. *Let $G = (V, E)$ be a finite graph and $U = \{u_1, \ldots, u_k\}$ be a non-empty finite subset of V and let $\widetilde{G}$ be the locally finite graph obtained by amalgamating the pendent vertices of the 1-way rays $R_1, R_2, \ldots, R_k$ at the vertices $u_1, u_2, \ldots, u_k$, respectively. Then the following holds:*

1. *$Z_U(G) \cong Z(\widetilde{G})$. In particular, if $U = V$ then $A(G) = S_V(G) = Z_V(G) \cong Z(\widetilde{G})$.*

2. If U is a proper subset of V then $S_U(G) \cong S(\widetilde{G})$.

3. If $U = V$ then $dim(S_V(G)) = dim(S(\widetilde{G})) - 1$.

Proof : For any $f \in Z_U(G)$ and $1 \leq i \leq k$ let $s_f(u_i) = \sum_{e \in E} \eta(u_i, e) f(e)$. We use the $s_f(u_i)$'s to define the isomorphisms and obtain the above mentioned results.

Part 1. Define $T_1 : Z_U(G) \longrightarrow Z(\widetilde{G})$ by $T_1(f) = \tilde{f}$ where $\tilde{f}$ is a labelling on $\widetilde{G}$ defined by

$$\tilde{f}(e) = \begin{cases} f(e), & \text{if } e \in E(G) \\ -s_f(u_i), & \text{if } e \in E(R_i) \text{ is an odd edge, } 1 \leq i \leq k, \\ s_f(u_i), & \text{if } e \in E(R_i) \text{ is an even edge, } 1 \leq i \leq k. \end{cases}$$

It is easy to check that T_1 is an isomorphism. In particular, if $U = V$ the result follows from definition.

Part 2. Let U be a proper subset of V. For any $f \in S_U(G)$, let r_f be the index of f. Define $T_2 : S_U(G) \longrightarrow S(\widetilde{G})$, by $T_2(f) = \tilde{f}$ where $\tilde{f}$ is a labelling on $\widetilde{G}$ defined by

$$\tilde{f}(e) = \begin{cases} f(e), & \text{if } e \in E(G) \\ r_f - s_f(u_i), & \text{if } e \in E(R_i) \text{ is an odd edge, } 1 \leq i \leq k, \\ s_f(u_i), & \text{if } e \in E(R_i) \text{ is an even edge, } 1 \leq i \leq k. \end{cases}$$

One can easily check that T_2 is an isomorphism.

Part 3. We need to show $dim(S_V(G)) = dim(S(\widetilde{G})) - 1$. To do so, it is sufficient to establish that $S(\widetilde{G}) \cong Z(\widetilde{G}) \oplus W$, where W is the 1-dimensional subspace of $S_V(G)$ spanned by $\tilde{f}$ and

$$\tilde{f}(e) = \begin{cases} 1, & \text{if } e \in E(R_i) \text{ is an odd edge,} \\ 0, & \text{if } e \in E(G) \text{ or } e \in E(R_i) \text{ is an even edge.} \end{cases}$$

Clearly $Z(\widetilde{G}) \cap W = \{0\}$. Define $T_3 : Z(\widetilde{G}) \oplus W \longrightarrow S(\widetilde{G})$ by

$$T_3(f \oplus g)(e) = f(e) + g(e), \text{ whenever } f \in Z(\widetilde{G}) \text{ and } g \in W.$$

It can be easily verified that T_3 is an isomorphism. Thus the proof of the proposition is complete. □

Proposition 5.5 gives us relations between the magic spaces of a class of locally finite graphs and the semi-restricted magic spaces of some finite graphs. We use this and the theory of matroids to find an explicit basis of $S(G)$. To do so, we define three special semi-restricted zero magic labellings (as defined in Section 4) and use these labellings as the building blocks for our required basis.

(i) Let P be a finite path. If both the pendent vertices u and v of P are not restricted, then $Z_{\{u,v\}}(P)$ has exactly one linearly independent element. This element is defined as follows: choose any non-pendent edge $e \in E(P)$, and put $\mathbf{x}(e) = 1$ and label the other edges as in Figure 7. We denote this particular labelling by $\mathbf{x}_e$.

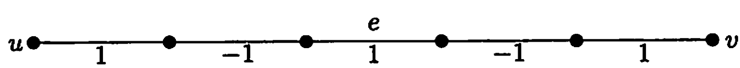

Figure 7: A semi-restricted zero magic labelling of a Path.

(ii) For any even cycle we already know the labelling $\mathbf{y}_e$ from Section 4.

(iii) For a kite K with an odd cycle (an odd kite) in which the pendent vertex v has no restriction, choose an edge e from the cycle and form the labelling as in Figure 8. We denote this labelling by $\mathbf{z}_e$.

With the help of the above defined labellings, we now have the proof of the main theorem of this section.

Theorem 5.6. *Let $G = (V, E)$ be a finite graph with at least one pendent vertex. Let U be a non-empty subset of the set of all pendent vertices of G. Then $dim(S_U(G)) = 1 + dim(Z_U(G))$.*

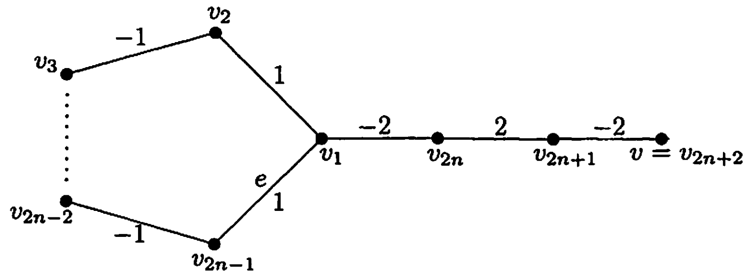

Figure 8: A semi-restricted zero magic labelling of an odd kite.

Proof : We prove the theorem by constructing an explicit basis of $Z_U(G)$ and $S_U(G)$.

It is clear that $S_U(G)$ and $Z_U(G)$ carry a matroid structure from the concerned chain groups. We denote these matroids by $S_U(G)$ and $Z_U(G)$ itself. Let $\{e_1, e_2, \ldots e_k\}$ be the set of pendent edges whose end vertices belong to U and let $v_i \in U$ be the pendent vertex incident with e_i, $1 \leq i \leq k$. Let T be a spanning tree of $G - \{v_2, v_3, \ldots v_k\}$. Put $D = E(G) - E(T)$. We first show that D is a dendroid of $Z_U(G)$. Note that for any $e \in D$, $E(T) \cup \{e\}$ contains either a path having e_1 and e_i $(2 \leq i \leq k)$ as pendent edges or an even cycle or an odd kite with pendent edge e_1. Let $e \in D$ be such that $E(T) \cup \{e\}$ contains a path, say P, having e_1 and some e_i $(2 \leq i \leq k)$ as pendent edges. Let f_e be the labelling of G where $f_e = \mathbf{x}_e$ on the edges of P and $f_e = 0$, elsewhere. Similarly in the other cases, f_e be the extension of $\mathbf{y}_e$ or $\mathbf{z}_e$ respectively. We observe the following :

(i) $D \cap \mathsf{s}(f_e) = \{e\}$, if $e \in D$ and $E(T) \cup \{e\}$ contains a path having e_1 and e_i $(2 \leq i \leq k)$ as pendent edges.

(ii) $D \cap \mathsf{s}(f_e) = \{e\}$, if $e \in D$ and $E(T) \cup \{e\}$ contains an even cycle.

(iii) $D \cap \mathsf{s}(f_e) = \{e\}$, if $e \in D$ and $E(T) \cup \{e\}$ contains an odd kite with e_1 as a pendent edge.

By Lemma 5.4, $Z_{v_1}(T) = \{0\}$. Hence by Theorem 5.1, no circuit is contained in $E(T)$ and so we can conclude that D has non empty intersection with every circuit. Again, if we remove an edge e from D, it is clear that there would be a circuit of the form $\mathsf{s}(f_e)$ which is disjoint from D. Hence D is minimal and so it is a dendroid of $Z_U(G)$. Note that $\{f_e : e \in D\}$ is a set of chains (labellings) of weight 1. Therefore by Theorem 5.2, we ensure that all these chains span $Z_U(G)$. Also $\mathsf{s}(f_e) \cap D = \{e\}$ implies that all these chains are linearly independent and hence form a basis of $Z_U(G)$.

Now we proceed to find out the rank of the matroid induced by $S_U(G)$. By Lemma 5.3, we get an element of $S_U(T)$ with index $r \neq 0$, which can be extended to $f \in S_U(G)$ by defining it as 0 on all edges in $E(G) \setminus E(T)$. Clearly $\mathsf{s}(f) \subseteq E(T)$. Let us choose an $e \in E(T)$ such that $f(e) \neq 0$, and put $\widetilde{D} = D \cup \{e\}$. We wish to show that $\widetilde{D}$ is a dendroid of $S_U(G)$.

Suppose that g $(\neq 0) \in S_U(G)$ and $\mathsf{s}(g) \cap D = \emptyset$. Since D is a dendroid of $Z_U(G)$, $g \notin Z_U(G)$ and so g has index $s \neq 0$. Therefore $sf - rg \in Z_U(G)$. Also the relations $\mathsf{s}(f) \subseteq E(T)$ and $D \cap E(T) = \emptyset$ imply that $\mathsf{s}(f) \cap D = \emptyset$. As $\mathsf{s}(g) \cap D = \emptyset$, $sf - rg$ is zero on every edge in D and $\mathsf{s}(sf - rg) \cap D = \emptyset$. Thus, using the fact that $sf - rg \in Z_U(G)$ and D is a dendroid of $Z_U(G)$, we have $sf = rg$ and $g(e) \neq 0$. Hence, $\widetilde{D}$ has non empty intersection with every circuit in $S_U(G)$.

Also from $Z_U(G) \subseteq S_U(G)$ and $\mathsf{s}(g) \cap D = \emptyset$, we observe that $\widetilde{D}$ is minimal. Hence $\widetilde{D}$ is a dendroid of $S_U(G)$. Clearly $\{f\} \cup \{f_e : e \in D\}$ is a basis for the vector space $S_U(G)$ over $\mathbb{F}$. Hence $dim(S_U(G)) = dim(Z_U(G)) + 1$. □

In the statement of Theorem 5.6, the set of unrestricted vertices were taken as a subset of the pendent vertices. The next corollary shows that this assumption is not necessary.

Corollary 5.7. *Let $G = (V, E)$ be a finite graph and U be a non-empty* *subset of* V. *Then* $dim(S_U(G)) = dim(Z_U(G)) + 1$.

Proof : If each vertex of U has degree one, the result follows from Theorem 5.6. Otherwise, let $\bar{U} = \{u_1, u_2, \ldots, u_k\}$ be the set of non pendent vertices of U. In this case, form a new graph $\widetilde{G}$ by amalgamating an edge $e_i = (u_i, v_i)$ at each of the vertices u_i, $1 \leq i \leq k$ and observe that $S_U(G) \cong S_{\widetilde{U}}(\widetilde{G})$ and $Z_U(G) \cong Z_{\widetilde{U}}(\widetilde{G})$, where $\widetilde{U} = (U - \bar{U}) \cup \{v_1, v_2, \ldots, v_k\}$. Now the use of Theorem 5.6 gives the desired result. □

Theorem 5.8. *Let* G *be a locally finite graph having finitely many maximal independent* 1*-way rays and finitely many cycles. Then dimension of* $S(G)$ *is exactly one more than that of* $Z(G)$.

Proof : The proof follows from Proposition 5.5 and Corollary 5.7. □

Remark 5.9. *From Proposition 5.5 and Theorem 5.6, it is clear that given any locally finite graph* G *having finitely many cycles and finitely many maximal independent rays, we can always construct an explicit basis of* $S(G)$.

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