The generalized connectivity of complete bipartite graphs*

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Abstract

Let G be a nontrivial connected graph of order n, and k an integer with $2 \leq k \leq n$. For a set S of k vertices of G, let $\kappa(S)$ denote the maximum number ℓ of edge-disjoint trees $T_1, T_2, \ldots, T_\ell$ in G such that $V(T_i) \cap V(T_j) = S$ for every pair i, j of distinct integers with $1 \leq i, j \leq \ell$. Chartrand et al. generalized the concept of connectivity as follows: The *k-connectivity*, denoted by $\kappa_k(G)$, of G is defined by $\kappa_k(G)$ =min$\{\kappa(S)\}$, where the minimum is taken over all k-subsets S of $V(G)$. Thus $\kappa_2(G) = \kappa(G)$, where $\kappa(G)$ is the connectivity of G. Moreover, $\kappa_n(G)$ is the maximum number of edge-disjoint spanning trees of G.

This paper mainly focus on the k-connectivity of complete bipartite graphs $K_{a,b}$, where $1 \leq a \leq b$. First, we obtain the number of edge-disjoint spanning trees of $K_{a,b}$, which is $\lfloor \frac{ab}{a+b-1} \rfloor$, and specifically give the $\lfloor \frac{ab}{a+b-1} \rfloor$ edge-disjoint spanning trees. Then, based on this result, we get the k-connectivity of $K_{a,b}$ for all $2 \leq k \leq a+b$. Namely, if $k > b-a+2$ and $a-b+k$ is odd then $\kappa_k(K_{a,b}) = \frac{a+b-k+1}{2} + \lfloor \frac{(a-b+k-1)(b-a+k-1)}{4(k-1)} \rfloor$, if $k > b-a+2$ and $a-b+k$ is even then $\kappa_k(K_{a,b}) = \frac{a+b-k}{2} + \lfloor \frac{(a-b+k)(b-a+k)}{4(k-1)} \rfloor$, and if $k \leq b-a+2$ then $\kappa_k(K_{a,b}) = a$.

Keywords: k-connectivity, complete bipartite graph, edge-disjoint spanning trees

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1 Introduction

We follow the terminology and notation of [1]. As usual, denote by $K_{a,b}$ the complete bipartite graph with bipartition of sizes a and b. The *connectivity* $\kappa(G)$ of a graph G is defined as the minimum cardinality of a set Q of vertices of G such that $G - Q$ is disconnected or trivial. A well-known theorem of Whitney [4] provides an equivalent definition of the connectivity. For each 2-subset $S = \{u, v\}$ of vertices of G, let $\kappa(S)$ denote the maximum number of internally disjoint uv-paths in G. Then $\kappa(G)$ =min$\{\kappa(S)\}$, where the minimum is taken over all 2-subsets S of $V(G)$.

In [2], the authors generalized the concept of connectivity. Let G be a nontrivial connected graph of order n, and k an integer with $2 \leq k \leq n$. For a set S of k vertices of G, let $\kappa(S)$ denote the maximum number ℓ of edge-disjoint trees $T_1, T_2, \ldots, T_\ell$ in G such that $V(T_i) \cap V(T_j) = S$ for every pair i, j of distinct integers with $1 \leq i, j \leq \ell$ (Note that the trees are vertex-disjoint in $G \backslash S$). A collection $\{T_1, T_2, \ldots, T_\ell\}$ of trees in G with this property is called an *internally disjoint set of trees connecting* S. The *k-connectivity*, denoted by $\kappa_k(G)$, of G is then defined as $\kappa_k(G)$ =min$\{\kappa(S)\}$, where the minimum is taken over all k-subsets S of $V(G)$. Thus, $\kappa_2(G) = \kappa(G)$ and $\kappa_n(G)$ is the maximum number of edge-disjoint spanning trees of G.

In [3], the authors focused on the investigation of $\kappa_3(G)$ and mainly studied the relationship between the 2-connectivity and the 3-connectivity of a graph. They gave sharp upper and lower bounds for $\kappa_3(G)$ for general graphs G, and showed that if G is a connected planar graph, then $\kappa(G) - 1 \leq \kappa_3(G) \leq \kappa(G)$. Moreover, they studied the algorithmic aspects for $\kappa_3(G)$ and gave an algorithm to determine $\kappa_3(G)$ for a general graph G.

Chartrand et al. in [2] proved that if G is the complete 3-partite graph $K_{3,4,5}$, then $\kappa_3(G) = 6$. They also gave a general result for the complete graph K_n:

Theorem 1.1. *For every two integers n and k with $2 \leq k \leq n$,*

$$\kappa_k(K_n) = n - \lceil k/2 \rceil.$$

Okamoto and Zhang in [5] investigated the generalized connectivity for regular complete bipartite graphs $K_{a,a}$. In this paper, we consider this connectivity for general complete bipartite graphs $K_{a,b}$. First, we give the number of edge-disjoint spanning trees of $K_{a,b}$, namely $\kappa_{a+b}(K_{a,b})$.

Theorem 1.2. *For any two integers a and b,*

$$\kappa_{a+b}(K_{a,b}) = \lfloor \frac{ab}{a+b-1} \rfloor.$$

Actually, we specifically give the $\lfloor \frac{ab}{a+b-1} \rfloor$ edge-disjoint spanning trees of $K_{a,b}$. Then based on Theorem 1.2, we obtain the k-connectivity of $K_{a,b}$ for all $2 \leq k \leq a+b$.

2 Proof of Theorem 1.2

Without loss of generality, we may assume that $a \leq b$. Since $K_{a,b}$ contains ab edges and a spanning tree needs $a+b-1$ edges, the number of edge-disjoint spanning trees of $K_{a,b}$ is at most $\lfloor \frac{ab}{a+b-1} \rfloor$, namely, $\kappa_{a+b}(K_{a,b}) \leq \lfloor \frac{ab}{a+b-1} \rfloor$. Thus, it suffices to prove that $\kappa_{a+b}(K_{a,b}) \geq \lfloor \frac{ab}{a+b-1} \rfloor$. To this end, we want to find out all the $\lfloor \frac{ab}{a+b-1} \rfloor$ edge-disjoint spanning trees. $K_{1,b}$ is a star which has exactly $\lfloor \frac{ab}{a+b-1} \rfloor = 1$ spanning tree. So we can restrict our attention to $K_{a,b}$ for $a \geq 2$. Hence, $\lfloor \frac{ab}{a+b-1} \rfloor < a$. Let $X = \{x_1, x_2, \ldots, x_a\}$ and $Y = \{y_1, y_2, \ldots, y_b\}$ be the bipartition of $K_{a,b}$.

We can describe a spanning tree in $K_{a,b}$ by giving the set of neighbors of x_j for $1 \leq j \leq a$. Now we give the first spanning tree T_1 we find:

vertex	neighbors	degree
x_1	$y_1,\ y_2,\ \ldots,\ y_{d_1}$	d_1
x_2	$y_{d_1},\ y_{d_1+1},\ \ldots,\ y_{d_1+d_2-1}$	d_2
x_3	$y_{d_1+d_2-1},\ y_{d_1+d_2},\ \ldots,\ y_{d_1+d_2+d_3-2}$	d_3
$\ldots$	$\ldots$	$\ldots$
x_j	$y_{d_1+d_2+\cdots+d_{j-1}-(j-2)},\ \ldots,\ y_{d_1+d_2+\cdots+d_j-(j-1)}$	d_j
$\ldots$	$\ldots$	$\ldots$
x_a	$y_{d_1+d_2+\cdots+d_{a-1}-(a-2)},\ \ldots,\ y_{d_1+d_2+\cdots+d_a-(a-1)}$	d_a

where d_j denotes the degree of x_j in T_1, and $d_1 + d_2 + \cdots + d_a = a + b - 1$.

To simplify the subscript, we denote $i_0 = 1$, $i_1 = d_1$, $i_2 = d_1 + d_2 - 1$, ..., $i_j = d_1 + d_2 + \cdots + d_j - (j-1)$, ..., $i_a = d_1 + d_2 + \cdots + d_a - (a-1) = b$. Note that, $i_j - i_{j-1} = d_j - 1$. So in T_1, the set of neighbors of x_j is $\{y_{i_{j-1}}, y_{i_{j-1}+1}, \cdots, y_{i_j}\}$ for $1 \leq j \leq a$.

Here and in what follows, the subscript j of $y_j \in Y$ is expressed modulo b as one of $1, 2, \ldots, b$. The subscript $j \neq 0$ of i_j is expressed modulo a as one of $1, 2, \ldots, a$. And the subscript j of d_j is expressed modulo a as one of $1, 2, \ldots, a$.

Then we can describe the second spanning tree T_2 we find. In T_2, the set of neighbors of x_j is $\{y_{i_j+1}, y_{i_j+2}, \cdots, y_{i_{j+1}+1}\}$ for $1 \leq j \leq a-1$ and the set of neighbors of x_a is $\{y_{i_a+1},\ y_{i_a+2},\ \ldots,\ y_{i_{a+1}}\}$. Note that $y_{i_a+1} = y_1$. Therefore $d_{T_2}(x_j) = i_{j+1} - i_j + 1 = d_{j+1}$ for $1 \leq j \leq a-1$ and $d_{T_2}(x_a) = i_{a+1} - 1 + 1 = d_1$.

We can see that T_2 and T_1 are edge-disjoint, if and only if for every vertex x_j, $d_j + d_{j+1} \leq b$. If T_2 and T_1 are edge-disjoint, then we continue to find T_3. In T_3, the set of neighbors of x_j is $\{y_{i_{j+1}+2},\ y_{i_{j+1}+3},\ \ldots,\ y_{i_{j+2}+2}\}$ for $1 \leq j \leq a-2$, the set of neighbors of x_{a-1} is $\{y_{i_a+2},\ y_{i_a+3},\ \ldots,\ y_{i_{a+1}+1}\}$

and the set of neighbors of x_a is $\{y_{i_{a+1}+1}, y_{i_{a+1}+2}, \dots, y_{i_{a+2}+1}\}$. Note that $y_{i_a+2} = y_2$. Therefore $d_{T_3}(x_j) = i_{j+2} - i_{j+1} + 1 = d_{j+2}$ for $1 \le j \le a-2$, $d_{T_3}(x_{a-1}) = i_{a+1} + 1 - 2 + 1 = d_1$ and $d_{T_3}(x_a) = i_{a+2} - i_{a+1} + 1 = i_2 - i_1 + 1 = d_2$.

We can see that T_3 and T_1, T_2 are edge-disjoint, if and only if for every vertex x_j, $d_j + d_{j+1} + d_{j+2} \le b$. If T_3 and T_1, T_2 are edge-disjoint, then we continue to find T_4. Continuing the procedure, our goal is to find the maximum l, such that T_l and $T_1, T_2, \dots, T_{l-1}$ are edge-disjoint. In T_l, the set of neighbors of x_j is $\{y_{i_{j+l-2}+(l-1)}, y_{i_{j+l-2}+l}, \dots, y_{i_{j+l-1}+(l-1)}\}$ for $1 \le j \le a-l+1$, the set of neighbors of x_{a-l+2} is $\{y_{i_a+(l-1)}, y_{i_a+l}, \dots, y_{i_{a+1}+(l-2)}\}$ and the set of neighbors of x_j is $\{y_{i_{j+l-2}+(l-2)}, y_{i_{j+l-2}+(l-1)}, \dots, y_{i_{j+l-1}+(l-2)}\}$ for $a-l+3 \le j \le a$. Note that $y_{i_a+(l-1)} = y_{l-1}$. Therefore $d_{T_l}(x_j) = i_{j+l-1} - i_{j+l-2} + 1 = d_{j+l-1}$ for $1 \le j \le a-l+1$, $d_{T_l}(x_{a-l+2}) = i_{a+1} + (l-2) - (l-1) + 1 = d_1$ and $d_{T_l}(x_j) = i_{j+l-1} - i_{j+l-2} + 1 = i_{j+l-1-a} - i_{j+l-2-a} + 1 = d_{j+l-1-a}$, for $a-l+3 \le j \le a$. That is, we want to find the maximum l, such that $d_j + d_{j+1} + \cdots + d_{j+l-1} \le b$ for any $1 \le j \le a$.

Let $D_j^t = d_j + d_{j+1} + \cdots + d_{j+t-1}$. It can be observed that $D_j^t = D_{j+1}^t$ if and only if $d_j = d_{j+t}$. We will show that for any fixed integer t, $1 \le t < a$, by assigning appropriate values to d_j, we can make $| D_i^t - D_j^t | \le 1$ for any integers $1 \le i, j \le a$. We describe the method for assigning values to d_j and prove its validity for two cases. Consider the numbers $1, t+1, 2t+1, \dots, (a-1)t+1$, where addition is performed modulo a.

Case 1. $1, t+1, 2t+1, \dots, (a-1)t+1$ are pairwise distinct.

Then we can assign the values to d_j as follows: Let $a+b-1 = ka+c$, where k, c are integers, and $0 \le c \le a-1$. Then $a+b-1 = (k+1)c + k(a-c)$. If $c = 0$, let $d_j = k$ for all $1 \le j \le a$. If $c > 0$, let $d_{(i-1)t+1} = k+1$ for all $1 \le i \le c$, and let the other $d_j = k$.

If $c = 0$, $d_j = k$ for all $1 \le j \le a$. Then $D_i^t = D_j^t$ for any integers $1 \le i, j \le a$.

If $c > 0$, we construct a weighted cycle: $C = x_1 x_{t+1} x_{2t+1} \dots x_{(a-1)t+1} x_1$ and $w(x_{(i-1)t+1}) = d_{(i-1)t+1}$ for $1 \le i \le a$. According to the assignment, we have $w(x_1) = w(x_{t+1}) = \cdots = w(x_{(c-1)t+1}) = k+1$ and $w(x_{ct+1}) = w(x_{(c+1)t+1}) = \cdots = w(x_{(a-1)t+1}) = k$.

Since $D_i^t = D_{i+1}^t$ if and only if $d_i = d_{i+t}$, then $D_{(i-1)t+1}^t = D_{(i-1)t+1+1}^t$ if and only if $w(x_{(i-1)t+1}) = w(x_{it+1})$. Similarly, $D_{(i-1)t+1}^t = D_{(i-1)t+1+1}^t + 1$ if and only if $w(x_{(i-1)t+1}) = w(x_{it+1}) + 1$, and $D_{(i-1)t+1}^t = D_{(i-1)t+1+1}^t - 1$ if and only if $w(x_{(i-1)t+1}) = w(x_{it+1}) - 1$. We know that $w(x_{(c-1)t+1}) = w(x_{ct+1}) + 1$, $w(x_{(a-1)t+1}) = w(x_1) - 1$, and $w(x_{(i-1)t+1}) = w(x_{it+1})$ for $1 \le i \le a-1$ and $i \ne c$. For simplicity, let $(c-1)t+1 = \alpha\ (mod\ a)$, $(a-1)t+1 = \beta\ (mod\ a)$. Therefore we can get $D_\alpha^t = D_{\alpha+1}^t + 1$, $D_\beta^t = D_{\beta+1}^t - 1$ and $D_{(i-1)t+1} = D_{(i-1)t+1+1}$, for $1 \le i \le a-1$ and $i \ne c$, namely, if $\alpha < \beta$, then $D_1^t = D_2^t = \cdots = D_\alpha^t = D_{\alpha+1}^t + 1 = D_{\alpha+2}^t + 1 = \cdots = D_\beta^t + 1 = D_{\beta+1}^t = D_{\beta+2}^t = \cdots = D_a^t$; if $\alpha > \beta$, then $D_1^t = D_2^t = \cdots = D_\beta^t = D_{\beta+1}^t - 1 = D_{\beta+2}^t - 1 = \cdots = D_\alpha^t - 1 = D_{\alpha+1}^t = D_{\alpha+2}^t = \cdots = D_a^t$.

We have $| D_i^t - D_j^t | \leq 1$ for any integers $1 \leq i, j \leq a$.

Case 2. Some of the numbers $1, t+1, 2t+1, \ldots, (a-1)t+1$ are equal.

Suppose that $it + 1 = jt + 1 (mod\ a)$ such that $0 \leq i < j \leq a-1$ and $1, t+1, 2t+1, \ldots, (j-1)t+1$ are pairwise distinct integers (in $\mathbb{Z}_a$). We claim that $i = 0$. Otherwise $(j-i)t + 1 = 1 (mod\ a)$ and $0 < j - i \leq j - 1$, a contradiction. Then $1 \leq j \leq a - 1$.

Claim 1. $it + 1 \neq 2\ (mod\ a)$ for any integer i.

If $it + 1 = 2\ (mod\ a)$, then we have $it = 1\ (mod\ a)$. Thus $\lambda it + 1 = \lambda + 1\ (mod\ a)$ for any integer λ. So $jit + 1 = j + 1\ (mod\ a)$. Since $1 \leq j \leq a-1$, $2 \leq j + 1 \leq a$. On the other hand $jt + 1 = 1\ (mod\ a)$, namely $jit + 1 = 1\ (mod\ a)$, a contradiction. Thus, $it + 1 \neq 2\ (mod\ a)$ for any integer i.

Claim 2. $2, t+2, 2t+2, \ldots, (j-1)t+2$ are pairwise distinct.

If $j_1 t + 2 = j_2 t + 2\ (mod\ a)$, where $0 \leq j_1 < j_2 \leq j-1$, then $j_1 t + 1 = j_2 t + 1\ (mod\ a)$. But $1, t+1, 2t+1, \ldots, (j-1)t+1$ are pairwise distinct, a contradiction.

Claim 3. $\{1, t+1, 2t+1, \ldots, (j-1)t+1\} \cap \{2, t+2, 2t+2, \ldots, (j-1)t+2\} = \emptyset$.

If $i_1 t + 1 = i_2 t + 2\ (mod\ a)$, then $(i_1 - i_2)t + 1 = 2\ (mod\ a)$. But $it + 1 \neq 2\ (mod\ a)$ for any integer i, a contradiction by Claim 1. Thus, $1, t+1, 2t+1, \ldots, (j-1)t+1, 2, t+2, 2t+2, \ldots, (j-1)t+2$ are pairwise distinct.

Now, if $2 = \frac{a}{j}$, then we order $1, \ldots, a$ by $1, t+1, 2t+1, \ldots, (j-1)t+1, 2, t+2, 2t+2, \ldots, (j-1)t+2$. If $2 < \frac{a}{j}$, we will prove that $1 + it \neq 3\ (mod\ a)$ and $2 + it \neq 3\ (mod\ a)$ for any integer i.

Claim 4. If $2 < \frac{a}{j}$, then $1 + it \neq 3\ (mod\ a)$ and $2 + it \neq 3\ (mod\ a)$ for any integer i.

If $2 + it = 3\ (mod\ a)$, then $1 + it = 2\ (mod\ a)$, a contradiction by Claim 1. If $1 + it = 3\ (mod\ a)$, then we have $it = 2\ (mod\ a)$. Thus $\lambda it + 1 = 2\lambda + 1\ (mod\ a)$ for any integer λ. So $jit + 1 = 2j + 1\ (mod\ a)$. Since $2 \leq 2j < a$, $3 \leq 2j + 1 \leq a$. On the other hand $jt + 1 = 1\ (mod\ a)$, namely $jit + 1 = 1\ (mod\ a)$, a contradiction. Hence, if $2 < \frac{a}{j}$, then $1 + it \neq 3\ (mod\ a)$ and $2 + it \neq 3\ (mod\ a)$ for any integer i.

If $3 = \frac{a}{j}$, then we order $1, \ldots, a$ by $1, t+1, 2t+1, \ldots, (j-1)t+1, 2, t+2, 2t+2, \ldots, (j-1)t+2, 3, t+3, 2t+3, \ldots, (j-1)t+3$. If $3 < \frac{a}{j}$, then continue the similar discussion until we reach some integer $s = \frac{a}{j}$. Similarly, we can prove that $p + it \neq q\ (mod\ a)$ for $1 \leq p < q \leq s$. Thus we can get the following claim:

Claim 5. $1, t+1, 2t+1, \ldots, (j-1)t+1, 2, t+2, 2t+2, \ldots, (j-1)t+$

$2,\ldots,s,t+s,2t+s,\ldots,(j-1)t+s$ are pairwise distinct. And hence $\{1,t+1,2t+1,\ldots,(j-1)t+1\}\cup\{2,t+2,2t+2,\ldots,(j-1)t+2\}\cup\cdots\cup\{\frac{a}{j},t+\frac{a}{j},2t+\frac{a}{j},\ldots,(j-1)t+\frac{a}{j}\}=\{1,2,\ldots,a\}$.

The proof is similar to those of Claims 2, 3 and 4. Then we order $1,2,\ldots,a$ by $1,t+1,2t+1,\ldots,(j-1)t+1,2,t+2,2t+2,\ldots,(j-1)t+2,\ldots,s,t+s,2t+s,\ldots,(j-1)t+s$. Now, we can assign the values of d_j as follows:

Let $a+b-1=ka+c$, where k,c are integers, and $0\le c\le a-1$. Then $a+b-1=(k+1)c+k(a-c)$. In the case that $c=0$, let $d_j=k$ for all $1\le j\le a$. In the case that $c>0$ for the first c numbers of our ordering, if d_j uses one of them as subscript, then $d_j=k+1$, else $d_j=k$.

Next, we will show that $|D_i^t-D_j^t|\le 1$ for any integers $1\le i,j\le a$.

If $c=0$, $d_j=k$ for all $1\le j\le a$. Then $D_i^t=D_j^t$ for any integers $1\le i,j\le a$.

If $c>0$, we construct s weighted cycles: $C_i=x_ix_{t+i}\ldots x_{(j-1)t+i}x_i$, $1\le i\le s$, and $w(x_{(p-1)t+i})=d_{(p-1)t+i}$, $1\le p\le j$. Since $D_i^t=D_{i+1}^t$ if and only if $d_i=d_{i+t}$, then $D_{(p-1)t+i}^t=D_{(p-1)t+i+1}^t$ if and only if $w(x_{(p-1)t+i})=w(x_{pt+i})$. By the assignment, there is at most one cycle in which the vertices have two distinct weights. If such cycle does not exist, clearly, we have $D_{(p-1)t+i}^t=D_{(p-1)t+i+1}^t$ for all $1\le i\le s$ and $1\le p\le j$, namely, $D_1^t=D_2^t=\cdots=D_a^t$. So we may assume that for some cycle C_r, $w(x_{(\gamma-1)t+r})=w(x_{\gamma t+r})+1$ and $w(x_{(j-1)t+r})=w(x_r)-1$. Similar to the proof of Case 1, we can get that $|D_i^t-D_j^t|\le 1$ for any integers $1\le i,j\le a$.

Then, we can show that, with the assignment we can get $l\ge\lfloor\frac{ab}{a+b-1}\rfloor$.

Let $t'=\lfloor\frac{ab}{a+b-1}\rfloor<a$. We have $D_1^{t'}+D_2^{t'}+\cdots+D_a^{t'}=(d_1+d_2+\cdots+d_{t'})+(d_2+d_3+\cdots+d_{t'+1})+\cdots+(d_a+d_1+\cdots+d_{t'-1})=t'(d_1+d_2+\cdots+d_a)=t'(a+b-1)$.

Since for fixed $t'=\lfloor\frac{ab}{a+b-1}\rfloor$, $|D_i^{t'}-D_j^{t'}|\le 1$ for any integers $1\le i,j\le a$,

$$D_j^{t'}\le\lceil\frac{t'(a+b-1)}{a}\rceil<\frac{t'(a+b-1)}{a}+1\le\frac{ab}{a+b-1}\frac{a+b-1}{a}+1=b+1.$$

The third inequality holds since $t'=\lfloor\frac{ab}{a+b-1}\rfloor\le\frac{ab}{a+b-1}$. Since $D_j^{t'}$ is an integer, we have $D_j^{t'}\le b$ for all $1\le j\le a$. Since l is the maximum integer such that $D_j^l=d_j+d_{j+1}+\cdots+d_{j+l-1}\le b$ for any $1\le j\le a$, then $l\ge t'=\lfloor\frac{ab}{a+b-1}\rfloor$. So we can find at least $\lfloor\frac{ab}{a+b-1}\rfloor$ edge-disjoint spanning trees of $K_{a,b}$. And hence $\kappa_{a+b}(K_{a,b})\ge\lfloor\frac{ab}{a+b-1}\rfloor$. So we have proved that $\kappa_{a+b}(K_{a,b})=\lfloor\frac{ab}{a+b-1}\rfloor$. ∎

3 Main result

Next, we will calculate $\kappa_k(K_{a,b})$ for $2\le k\le a+b$.

Recall that $\kappa_k(G) = \min\{\kappa(S)\}$, where the minimum is taken over all k-element subsets S of $V(G)$. $X = \{x_1, x_2, \ldots, x_a\}$ and $Y = \{y_1, y_2, \ldots, y_b\}$ be the bipartition of $K_{a,b}$. Actually, all vertices in X are equivalent and all vertices in Y are equivalent. So instead of considering all k-element subsets S of $V(G)$, we can restrict our attention to the k-element subsets $S_i = \{x_1, x_2, \ldots, x_i, y_1, y_2, \ldots, y_{k-i}\}$ for $0 \le i \le k$. Notice that, if $i > a$ or $k - i > b$, then S_i does not exist. So, we need only to consider S_i for $\max\{0,\ k-b\} \le i \le \min\{a,\ k\}$.

Now, let A be a maximum set of internally disjoint trees connecting S_i. Let $\mathfrak{A}_0$ be the set of trees connecting S_i whose vertex set is S_i, let $\mathfrak{A}_1$ be the set of trees connecting S_i whose vertex set is $S_i \cup \{u\}$, where $u \notin S_i$ and let $\mathfrak{A}_2$ be the set of trees connecting S_i whose vertex set is $S_i \cup \{u, v\}$, where $u, v \notin S_i$ and they belong to distinct partitions.

Lemma 3.1. *Let A be a maximum set of internally disjoint trees connecting S_i. Then we can always find a set A' of internally disjoint trees connecting S_i, such that $|A| = |A'|$ and $A' \subset \mathfrak{A}_0 \cup \mathfrak{A}_1 \cup \mathfrak{A}_2$.*

Proof. Let $A = \{T_1, T_2, \ldots, T_p\}$. If for some tree T_j in A, $T_j \notin \mathfrak{A}_0 \cup \mathfrak{A}_1 \cup \mathfrak{A}_2$, then let $V(T_j) = S_i \cup U \cup V$, where $(U \cup V) \cap S_i = \emptyset$, $U \subseteq X$ and $V \subseteq Y$. One of U and V can be empty but not both. If U and V are not empty, let $u_1 \in U$ and $v_1 \in V$. The tree T_j' with vertex set $V(T_j') = S_i \cup \{u_1, v_1\}$ and edge set $E(T_j') = \{u_1y_1, \ldots, u_1y_{k-i}, v_1x_1, \ldots, v_1x_i, u_1v_1\}$ is a tree in $\mathfrak{A}_0 \cup \mathfrak{A}_1 \cup \mathfrak{A}_2$. Since $V(T_j) \cap V(T_k) = S_i$ and $E(T_j) \cap E(T_k) = \emptyset$ for every tree $T_k \in A$, where $k \ne j$, T_k will not contain u_1, v_1 nor the edges incident with u_1, v_1. Therefore, $V(T_j') \cap V(T_k) = S_i$ and $E(T_j') \cap E(T_k) = \emptyset$ for $1 \le k \le p, k \ne j$. If one of U and V is empty, say V, let $U = \{u_1, u_2, \ldots, u_q\}$. Then we connect all neighbors of $u_2, \ldots, u_q$ to u_1 by some new edges and delete $u_2, \ldots, u_q$ and any resulting multiple edges. Obviously, the new graph we obtain is a tree $T_j' \in \mathfrak{A}_0 \cup \mathfrak{A}_1 \cup \mathfrak{A}_2$ that connects S_i. For every tree $T_k \in A$, where $k \ne j$, T_k will not contain u_1 nor the edges incident with u_1. Therefore, $V(T_j') \cap V(T_k) = S_i$ and $E(T_j') \cap E(T_k) = \emptyset$ for $1 \le k \le p, k \ne j$. Replacing each $T_j \notin \mathfrak{A}_0 \cup \mathfrak{A}_1 \cup \mathfrak{A}_2$ by T_j', we finally get the set $A' \subset \mathfrak{A}_0 \cup \mathfrak{A}_1 \cup \mathfrak{A}_2$ which has the same cardinality as A. ∎

So, we can assume that the maximum set A of internally disjoint trees connecting S_i is contained in $\mathfrak{A}_0 \cup \mathfrak{A}_1 \cup \mathfrak{A}_2$.

Next, we will define the standard structure of trees in $\mathfrak{A}_0$, $\mathfrak{A}_1$ and $\mathfrak{A}_2$, respectively.

Every tree in $\mathfrak{A}_0$ is of standard structure. A tree T in $\mathfrak{A}_1$ with vertex set $V(T) = S_i \cup \{u\}$, where $u \in X \setminus S_i$, is of standard structure, if u is adjacent to every vertex in $S_i \cap Y$. Since $|E(T)| = |V(T)| - 1 = k$ and $d_T(u) = |S_i \cap Y| = k - i$, there remains i edges incident with $S_i \cap X$. We know that $|S_i \cap X| = i$ and each vertex must have degree at least 1 in T.

So every vertex in $S_i \cap X$ has degree 1. A tree T in $\mathfrak{A}_1$ with vertex set $V(T) = S_i \cup \{v\}$, where $v \in Y \setminus S_i$, is of standard structure, if v is adjacent to every vertex in $S_i \cap X$. Similarly, every vertex in $S_i \cap Y$ has degree 1. A tree T in $\mathfrak{A}_2$ with vertex set $V(T) = S_i \cup \{u, v\}$, where $u \in X \setminus S_i$ and $v \in Y \setminus S_i$, is of standard structure, if u is adjacent to every vertex in $S_i \cap Y$, v is adjacent to every vertex in $S_i \cap X$, and u is adjacent to v. We then denote the resulting tree T by $T_{u,v}$. Denote the set of trees in $\mathfrak{A}_0$, $\mathfrak{A}_1$ and $\mathfrak{A}_2$ with the standard structure by $\mathcal{A}_0$, $\mathcal{A}_1$ and $\mathcal{A}_2$, respectively. Clearly, $\mathcal{A}_0 = \mathfrak{A}_0$.

Lemma 3.2. *Let A be a maximum set of internally disjoint trees connecting S_i, $A \subset \mathfrak{A}_0 \cup \mathfrak{A}_1 \cup \mathfrak{A}_2$. Then we can always find a set A'' of internally disjoint trees connecting S_i, such that $| A |=| A'' |$ and $A'' \subset \mathcal{A}_0 \cup \mathcal{A}_1 \cup \mathcal{A}_2$.*

Proof. Let $A = \{T_1, T_2, \ldots, T_p\}$. Suppose that there is a tree T_j in A such that $T_j \in \mathfrak{A}_1$, but $T_j \notin \mathcal{A}_1$. Let $V(T_j) = S_i \cup \{u\}$, where $u \in X \setminus S_i$. Note that the case $u \in Y \setminus S_i$ is similar. Since $T_j \notin \mathcal{A}_1$, there are some vertices in $S_i \cap Y$, say $y_{i_1}, \ldots, y_{i_t}$, not adjacent to u. Then we can connect y_{i_1} to u by a new edge. It will produce a unique cycle. Delete the other edge incident with y_{i_1} on the cycle. The graph remains a tree. Do the same operation to $y_{i_2}, \ldots, y_{i_t}$ in turn. Finally we get a tree T_j' whose vertex set is $S_i \cup \{u\}$ and u is adjacent to every vertex in $S_i \cap Y$, that is, T is of standard structure. For each tree $T_n \in A \setminus \{T_j\}$, clearly T_n does not contain u nor the edges incident with u. So $V(T_j') \cap V(T_n) = S_i$ and $E(T_j') \cap E(T_n) = \emptyset$. Suppose that there is a tree T_j in A such that $T_j \in \mathfrak{A}_2$, but $T_j \notin \mathcal{A}_2$. Let $V(T_j) = S_i \cup \{u, v\}$, where $u \in X \setminus S_i$ and $v \in Y \setminus S_i$. Then $T_j' = T_{u,v}$ is the tree in $\mathcal{A}_2$ whose vertex set is $S_i \cup \{u, v\}$. For each tree $T_n \in A \setminus \{T_j\}$, $V(T_j') \cap V(T_n) = S_i$ and $E(T_j') \cap E(T_n) = \emptyset$. Replacing each $T_j \notin \mathcal{A}_0 \cup \mathcal{A}_1 \cup \mathcal{A}_2$ by T_j', we finally get the set $A'' \subset \mathcal{A}_0 \cup \mathcal{A}_1 \cup \mathcal{A}_2$ which has the same cardinality as A. ∎

So, we can assume that the maximum set A of internally disjoint trees connecting S_i is contained in $\mathcal{A}_0 \cup \mathcal{A}_1 \cup \mathcal{A}_2$. Namely, all trees in A are of standard structure.

For simplicity, we denote the union of the vertex sets of all trees in set A by $V(A)$ and the union of the edge sets of all trees in set A by $E(A)$. Let $A_0 := A \cap \mathcal{A}_0$, $A_1 := A \cap \mathcal{A}_1$ and $A_2 := A \cap \mathcal{A}_2$. Then $A = A_0 \cup A_1 \cup A_2$.

Lemma 3.3. *Let $A \subset \mathcal{A}_0 \cup \mathcal{A}_1 \cup \mathcal{A}_2$ be a maximum set of internally disjoint trees connecting S_i. Then either $X \subseteq V(A)$ or $Y \subseteq V(A)$.*

Proof. If $X \not\subseteq V(A)$ and $Y \not\subseteq V(A)$, let $x \in X \setminus V(A)$ and $y \in Y \setminus V(A)$. Then the tree $T_{x,\,y} \in \mathcal{A}_2$ with vertex set $S_i \cup \{x, y\}$ is a tree that connects S_i. Moreover, $V(T_{x,\,y}) \cap V(A) = S_i$ and since all edges of $T_{x,\,y}$ are incident with x or y, so $T_{x,\,y}$ and T are edge-disjoint for any tree $T \in A$. So, $A \cup \{T_{x,\,y}\}$

is also a set of internally disjoint trees connecting S_i, contradicting to the maximality of A. ∎

So we conclude that if A is a maximum set of internally disjoint trees connecting S_i, then $X \subseteq V(A)$ or $Y \subseteq V(A)$.

Lemma 3.4. *Let $A \subset \mathcal{A}_0 \cup \mathcal{A}_1 \cup \mathcal{A}_2$ be a maximum set of internally disjoint trees connecting S_i, and $A = A_0 \cup A_1 \cup A_2$. If there is a vertex $x \in X \setminus V(A)$ and a tree $T \in A_1$ with vertex set $S_i \cup \{y\}$, where $y \in Y \setminus S_i$, then we can find a maximum set $A' = A'_0 \cup A'_1 \cup A'_2$ of internally disjoint trees connecting S_i, such that $A'_0 = A_0$, $|A'_1| = |A_1| - 1$, and $|A'_2| = |A_2| + 1$.*

Proof. Let $T_{x,\,y}$ be the tree in $\mathcal{A}_2$ whose vertex set is $S_i \cup \{x, y\}$. Then $A' = A \setminus T \cup \{T_{x,\,y}\}$ is just the set we want. ∎

The case that there is a vertex $y \in Y \setminus V(A)$ and a tree $T \in A_1$ with vertex set $S_i \cup \{x\}$, where $x \in X \setminus S_i$, is similar.

Next, we will show that we can always find a maximum set A of internally disjoint trees connecting S_i, such that all vertices in $V(A_1) \setminus S_i$ belong to the same partition. To show this, we need the following lemma.

Lemma 3.5. *Let p, q be two nonnegative integers. If $p(k-1)+qi \leq i(k-i)$, and there are q vertices $u_1, u_2, \ldots, u_q \in X \setminus S_i$, then we can always find p trees $T_1, T_2, \ldots, T_p$ in $\mathcal{A}_0$ and q trees $T_{p+1}, T_{p+2}, \ldots, T_{p+q}$ in $\mathcal{A}_1$, such that $V(T_j) = S_i$ for $1 \leq j \leq p$, $V(T_{p+m}) = S_i \cup \{u_m\}$ for $1 \leq m \leq q$, and T_r and T_s are edge-disjoint for $1 \leq r < s \leq p+q$. Similarly, if $p(k-1)+q(k-i) \leq i(k-i)$, and there are q vertices $v_1, v_2, \ldots, v_q \in Y \setminus S_i$, then we can always find p trees $T_1, T_2, \ldots, T_p$ in $\mathcal{A}_0$ and q trees $T_{p+1}, T_{p+2}, \ldots, T_{p+q}$ in $\mathcal{A}_1$, such that $V(T_j) = S_i$ for $1 \leq j \leq p$, $V(T_{p+m}) = S_i \cup \{v_m\}$ for $1 \leq m \leq q$, and T_r and T_s are edge-disjoint for $1 \leq r < s \leq p+q$.*

Proof. If $p(k-1) + qi \leq i(k-i)$, then $p(k-1) \leq i(k-i)$, namely $p \leq \lfloor \frac{i(k-i)}{k-1} \rfloor$. Then with the method which we used to find edge-disjoint spanning trees in the proof of Theorem 1.2, we can find p edge-disjoint trees $T_1, T_2, \ldots, T_p$ in $\mathcal{A}_0$, just by taking $a = i$, $b = k - i$ and $t = p$. Moreover, let D_s^p denote the number of edges incident with x_s in all of the p trees. Then according to the method, $|D_s^p - D_t^p| \leq 1$ for $1 \leq s, t \leq i$. Now, denote by B_s^p the number of edges incident with x_s which we have not used in the p trees. Then $|B_s^p - B_t^p| \leq 1$ for $1 \leq s, t \leq i$. Since $B_1^p + B_2^p + \cdots + B_i^p = i(k-i) - p(k-1) \geq qi$, $B_s^p \geq q$. Because for each tree in $\mathcal{A}_1$ with vertex set $S_i \cup \{u\}$, where $u \in X \setminus S_i$, the vertices in $S_i \cap X$ all have degree 1, we can find q edge-disjoint trees $T_{p+1}, T_{p+2}, \ldots, T_{p+q}$ in $\mathcal{A}_1$. Since the edges in $T_{p+1}, T_{p+2}, \ldots, T_{p+q}$ are not used in $T_1, T_2, \ldots, T_p$ for $1 \leq r < s \leq p+q$, T_r and T_s are edge-disjoint. The proof of the second part of the lemma is similar. ∎

Lemma 3.6. *Let $A \subset \mathcal{A}_0 \cup \mathcal{A}_1 \cup \mathcal{A}_2$ be a maximum set of internally disjoint trees connecting S_i, and $A = A_0 \cup A_1 \cup A_2$. If there are s trees $T_1, T_2, \dots, T_s \in A_1$ with vertex set $S_i \cup \{u_1\}$, $S_i \cup \{u_2\}$, ..., $S_i \cup \{u_s\}$ respectively, where $u_j \in X \setminus S_i$ for $1 \leq j \leq s$, and t trees $T_{s+1}, T_{s+2}, \dots, T_{s+t} \in A_1$ with vertex set $S_i \cup \{v_1\}$, $S_i \cup \{v_2\}$, ..., $S_i \cup \{v_t\}$ respectively, where $v_j \in Y \setminus S_i$ for $1 \leq j \leq t$. Then we can find a set $A' = A_0' \cup A_1' \cup A_2'$ of internally disjoint trees connecting S_i, such that $|A| = |A'|$ and all vertices in $V(A_1') \setminus S_i$ belong to the same partition.*

Proof. Let $|A_0| = p$. Since A is a set of internally disjoint trees connecting S_i, we have $p(k-1) + si + t(k-i) \leq i(k-i)$, where si denote the si edges incident with $x_1, \dots, x_i$ in $T_1, T_2, \dots, T_s$, and $t(k-i)$ denote the $t(k-i)$ edges incident with $y_1, \dots, y_{k-i}$ in $T_{s+1}, T_{s+2}, \dots, T_{s+t}$. If $s \leq t$, then $p(k-1) + si + s(k-i) + (t-s)(k-i) \leq i(k-i)$, and hence $(p+s)(k-1) + (t-s)(k-i) \leq i(k-i)$. Obviously, there are $t-s$ vertices $v_{s+1}, v_{s+2}, \dots, v_t \in Y \setminus S_i$, and therefore by Lemma 3.5, we can find $p+s$ trees in $\mathcal{A}_0$ and $t-s$ trees in $\mathcal{A}_1$, such that all these trees are internally disjoint trees connecting S_i. Now let A_0' be the set of the $p+s$ trees in $\mathcal{A}_0$, A_1' be the set of the $t-s$ trees in $\mathcal{A}_1$ and $A_2' := A_2 \cup \{T_{u_j,v_j}, 1 \leq j \leq s\}$. Then $A' = A_0' \cup A_1' \cup A_2'$ is just the set we want. The case that $s > t$ is similar. ∎

From Lemmas 3.4 and 3.6, we can see that, if A' is a set of internally disjoint trees connecting S_i which we find currently, $X \setminus V(A) \neq \emptyset$ and $Y \setminus V(A) \neq \emptyset$, then no matter how many edges there are in $E(K_{a,b}[S_i]) \setminus E(A')$, we always add to A' the trees in $\mathcal{A}_2$ rather than the trees in $\mathcal{A}_1$ to form a larger set of internally disjoint trees connecting S_i.

Lemma 3.7. *Let $A \subset \mathcal{A}_0 \cup \mathcal{A}_1 \cup \mathcal{A}_2$ be a maximum set of internally disjoint trees connecting S_i, and $A = A_0 \cup A_1 \cup A_2$. If $V(A) \subset V(G)$ and $A_0 \neq \emptyset$, then we can find a maximum set $A' = A_0' \cup A_1' \cup A_2'$ of internally disjoint trees connecting S_i, such that $|A_0'| = |A_0| - 1$, $|A_1'| = |A_1| + 1$, and $A_2' = A_2$.*

Proof. Let $u \in V(G) \setminus V(A)$ and $T \in A_0$. Without loss of generality, suppose $u \in X$. Then we can add the edge uy_1 to T and get a tree $T' \in \mathfrak{A}_1$. Using the method in Lemma 3.2, we can transform T' into a tree T'' of standard structure. Then $T'' \in \mathcal{A}_1$. Let $A_0' := A_0 \setminus T$, $A_1' := A_1 \cup \{T''\}$ and $A_2' = A_2$. It is easy to see that $A' = A_0' \cup A_1' \cup A_2'$ is a set of internally disjoint trees connecting S_i. Since $|A_0'| = |A_0| - 1$, $|A_1'| = |A_1| + 1$, and $A_2' = A_2$, A' is a maximum set of internally disjoint trees connecting S_i. ∎

So, we can assume that for the maximum set A of internally disjoint trees connecting S_i, either $V(A) = V(G)$ or $A_0 = \emptyset$. Moreover, if A' is a set of internally disjoint trees connecting S_i which we find currently, $V(A') \subset V(G)$ and the edges in $E(K_{a,b}[S_i]) \setminus E(A')$ can form a tree T in

$\mathcal{A}_0$, then we will add to A' the tree T'' in Lemma 3.7 rather than the tree T to form a larger set of internally disjoint trees connecting S_i.

Next, let us state and prove our main result.

Theorem 3.1. *Given any two positive integers $a \le b$, let $K_{a,b}$ denote a complete bipartite graph with a bipartition of sizes a and b, respectively. Then we have the following results: if $k > b-a+2$ and $a-b+k$ is odd, then*

$$\kappa_k(K_{a,b}) = \frac{a+b-k+1}{2} + \lfloor \frac{(a-b+k-1)(b-a+k-1)}{4(k-1)} \rfloor;$$

if $k > b-a+2$ and $a-b+k$ is even, then

$$\kappa_k(K_{a,b}) = \frac{a+b-k}{2} + \lfloor \frac{(a-b+k)(b-a+k)}{4(k-1)} \rfloor;$$

and if $k \le b-a+2$, then

$$\kappa_k(K_{a,b}) = a.$$

Proof. Let $X = \{x_1, x_2, \ldots, x_a\}$ and $Y = \{y_1, y_2, \ldots, y_b\}$ be the bipartition of $K_{a,b}$. As we have mentioned, we can restrict our attention to the k-element subsets $S_i = \{x_1, x_2, \ldots, x_i, y_1, y_2, \ldots, y_{k-i}\}$ for $\max\{0,\ k-b\} \le i \le \min\{a,\ k\}$.

From the above lemmas, we can decide our principle to find the maximum set of internally disjoint trees connecting S_i. Namely, first we find as many trees in $\mathcal{A}_2$ as possible, next we find as many trees in $\mathcal{A}_1$ as possible, and finally we find as many trees in $\mathcal{A}_0$ as possible. Let A be the maximum set of internally disjoint trees connecting S_i we finally find. We now compute $|A|$.

Case 1. $k \le b-a+2$.

Obviously, $\kappa(S_0) = a$. For S_1, since $k \le b-a+2$, then $b-(k-1) = b-k+1 \ge a-2+1 = a-1$. So, $|A_2| = a-1$. If $b-k+1 = a-1$, then $|A_1| = 0$ and $|A_0| = 1$. If $b-k+1 > a-1$, then $|A_1| = 1$ and $|A_0| = 0$. No matter which case happens, we have $\kappa(S_1) = |A_2| + |A_1| + |A_0| = a$.

For S_i, $i \ge 2$, since $k \le b-a+2$, then $b-(k-i) = b-k+i \ge a-2+i > a-i$. So, $|A_2| = a-i$. Since $b-k+i-(a-i) = b-a-k+2i \ge -2+2i \ge i$, then $|A_1| = i$ and $|A_0| = 0$. Thus $\kappa(S_i) = |A_2| + |A_1| + |A_0| = a$.

In summary, if $k \le b-a+2$, then $\kappa_k(G) = a$.

Case 2. $k > b-a+2$.

First, let us compare $\kappa(S_i)$ with $\kappa(S_{k-i})$, for $0 \le i \le \lfloor \frac{k}{2} \rfloor$. If $a = b$, clearly, $\kappa(S_i) = \kappa(S_{k-i})$. So we may assume that $a < b$.

For $i = 0$, $\kappa(S_0) = a < b = \kappa(S_k)$.

For $1 \le i \le \lfloor \frac{k}{2} \rfloor$, we will give the expressions of $\kappa(S_i)$ and $\kappa(S_{k-i})$.

First for S_i, since every pair of vertices $u \in X \setminus S_i$ and $v \in Y \setminus S_i$ can form a tree $T_{u,v}$, then $|A_2| = \min\{a-i, b-(k-i)\}$. Namely,

$$|A_2| = \begin{cases} a-i & \text{if } i \geq \frac{a-b+k}{2} ; \\ b-k+i & \text{if } i < \frac{a-b+k}{2} . \end{cases}$$

Next, since every tree T in A_1 has a vertex in $V \setminus (S_i \cup V(A_2))$, we have

$$|A_1| \leq \begin{cases} b-k+i-(a-i) & \text{if } i \geq \frac{a-b+k}{2} ; \\ a-i-(b-k+i) & \text{if } i < \frac{a-b+k}{2} . \end{cases}$$

On the other hand, if the tree T has vertex set $S_i \cup \{u\}$, where $u \in X \setminus S_i$, then every vertex in $S_i \cap X$ is incident with one edge in $E(S_i)$, where $E(S_i)$ denotes the set of edges whose ends are both in S_i. And if the tree T has vertex set $S_i \cup \{v\}$, where $v \in Y \setminus S_i$, then every vertex in $S_i \cap Y$ is incident with one edge in $E(S_i)$. Since every vertex in $S_i \cap X$ is incident with $k-i$ edges in $E(S_i)$ and every vertex in $S_i \cap Y$ is incident with i edges in $E(S_i)$, we have

$$|A_1| \leq \begin{cases} i & \text{if } i \geq \frac{a-b+k}{2} ; \\ k-i & \text{if } i < \frac{a-b+k}{2} . \end{cases}$$

Combining the two inequalities, we get

$$|A_1| = \begin{cases} \min\{b-a-k+2i, i\} & \text{if } i \geq \frac{a-b+k}{2} ; \\ \min\{a-b+k-2i, k-i\} & \text{if } i < \frac{a-b+k}{2} . \end{cases}$$

Thus

$$|A_1| = \begin{cases} i & \text{if } i \geq a-b+k ; \\ b-a-k+2i & \text{if } \frac{a-b+k}{2} \leq i < a-b+k ; \\ a-b+k-2i & \text{if } i < \frac{a-b+k}{2} . \end{cases}$$

Finally, by Lemma 3.5 we have

$$|A_0| = \begin{cases} \lfloor \frac{i(k-i)-|A_1|(k-i)}{k-1} \rfloor & \text{if } i \geq \frac{a-b+k}{2} ; \\ \lfloor \frac{i(k-i)-|A_1|i}{k-1} \rfloor & \text{if } i < \frac{a-b+k}{2} . \end{cases}$$

Thus

$$|A_0| = \begin{cases} 0 & \text{if } i \geq a-b+k ; \\ \lfloor \frac{[i-(b-a-k+2i)](k-i)}{k-1} \rfloor & \text{if } \frac{a-b+k}{2} \leq i < a-b+k ; \\ \lfloor \frac{[k-i-(a-b+k-2i)]i}{k-1} \rfloor & \text{if } i < \frac{a-b+k}{2} . \end{cases}$$

Hence

$$\kappa(S_i) = \begin{cases} a & \text{if } i \geq a-b+k ; \\ b-k+i+\lfloor \frac{[i-(b-a-k+2i)](k-i)}{k-1} \rfloor & \text{if } \frac{a-b+k}{2} \leq i < a-b+k ; \\ a-i+\lfloor \frac{[k-i-(a-b+k-2i)]i}{k-1} \rfloor & \text{if } i < \frac{a-b+k}{2} . \end{cases}$$

Notice that $i \geq 1$, and hence $k - i \leq k - 1$.

If $\frac{a-b+k}{2} \leq i < a-b+k$, then $\lfloor\frac{[i-(b-a-k+2i)](k-i)}{k-1}\rfloor \leq i-(b-a-k+2i) = a-b+k-i$. So, $\kappa(S_i) \leq b-k+i+a-b+k-i = a$.

If $i < \frac{a-b+k}{2}$, then $a-b+k-2i > 0$, $k-i-(a-b+k-2i) < k-i \leq k-1$, and hence $\lfloor\frac{[k-i-(a-b+k-2i)]i}{k-1}\rfloor \leq i$. So, $\kappa(S_i) \leq a-i+i = a$

Thus $\kappa(S_i) \leq a$ for $i \geq 1$.

Next, considering S_{k-i}, similarly, we have $|A_2| = \min\{a-(k-i), b-i\}$.

Since $a < b$ and $i \leq \lfloor\frac{k}{2}\rfloor \leq \lceil\frac{k}{2}\rceil \leq k-i$, then $b-i > a-(k-i)$. So $|A_2| = a-k+i$ and $|A_1| = \min\{b-i-(a-k+i), k-i\}$. Hence

$$|A_1| = \begin{cases} k-i & \text{if } i \leq b-a\,; \\ b-a+k-2i & \text{if } i > b-a\,. \end{cases}$$

Moreover,

$$|A_0| = \begin{cases} 0 & \text{if } i \leq b-a\,; \\ \lfloor\frac{[k-i-(b-a+k-2i)]i}{k-1}\rfloor & \text{if } i > b-a\,. \end{cases}$$

So,

$$\kappa(S_{k-i}) = \begin{cases} a & \text{if } i \leq b-a\,; \\ b-i+\lfloor\frac{[k-i-(b-a+k-2i)]i}{k-1}\rfloor & \text{if } i > b-a\,. \end{cases}$$

Now, we can compare $\kappa(S_i)$ with $\kappa(S_{k-i})$. For $i \leq b-a$, $\kappa(S_{k-i}) = a \geq \kappa(S_i)$. For $i > b-a$, there must be $b-a < k-i$, that is, $i < a-b+k$. Note that for any two real numbers s,t, $\lfloor s+t\rfloor \geq \lfloor s\rfloor + \lfloor t\rfloor$.

If $\frac{a-b+k}{2} \leq i < a-b+k$, then

$$\begin{aligned} \kappa(S_{k-i}) - \kappa(S_i) &= b-i+\lfloor\frac{[k-i-(b-a+k-2i)]i}{k-1}\rfloor \\ &\quad -\{b-k+i+\lfloor\frac{[i-(b-a-k+2i)](k-i)}{k-1}\rfloor\} \\ &\geq (k-2i)+\lfloor\frac{(k-2i)(b-a-k)}{k-1}\rfloor \\ &\geq (k-2i)+\lfloor\frac{(k-2i)(1-k)}{k-1}\rfloor = 0. \end{aligned}$$

So, $\kappa(S_{k-i}) \geq \kappa(S_i)$. If $i < \frac{a-b+k}{2}$, then

$$\begin{aligned} \kappa(S_{k-i}) - \kappa(S_i) &= b-i+\lfloor\frac{[k-i-(b-a+k-2i)]i}{k-1}\rfloor \\ &\quad -\{a-i+\lfloor\frac{[k-i-(a-b+k-2i)]i}{k-1}\rfloor\} \\ &\geq (b-a)+\lfloor\frac{(2i)(a-b)}{k-1}\rfloor. \end{aligned}$$

Since $i < \frac{a-b+k}{2}$, then $2i \le k-1$, and hence $\frac{(2i)(a-b)}{k-1} \ge a-b$. So, $\kappa(S_{k-i}) - \kappa(S_i) \ge b-a+a-b = 0$. Thus, $\kappa(S_{k-i}) \ge \kappa(S_i)$.

In summary, $\kappa(S_{k-i}) \ge \kappa(S_i)$ for $0 \le i \le \lfloor \frac{k}{2} \rfloor$. So, in order to get $\kappa_k(G)$, it is enough to consider $\kappa(S_i)$ for $0 \le i \le \lfloor \frac{k}{2} \rfloor$.

Next, let us compare $\kappa(S_i)$ with $\kappa(S_{i+1})$, for $0 \le i \le \lfloor \frac{k}{2} \rfloor - 1$. For $i=0$, $\kappa(S_i) = a \ge \kappa(S_{i+1})$. For $1 \le i \le \lfloor \frac{k}{2} \rfloor - 1$,

$$\kappa(S_i) = \begin{cases} a & \text{if } i \ge a-b+k \text{ ;} \\ b-k+i+\lfloor \frac{[i-(b-a-k+2i)](k-i)}{k-1} \rfloor & \text{if } \frac{a-b+k}{2} \le i < a-b+k \text{ ;} \\ a-i+\lfloor \frac{[k-i-(a-b+k-2i)]i}{k-1} \rfloor & \text{if } i < \frac{a-b+k}{2} \text{ .} \end{cases}$$

and

$$\kappa(S_{i+1}) = \begin{cases} a & \text{if } i \ge a-b+k-1 \text{ ;} \\ b-k+i+1 & \text{if } \frac{a-b+k}{2} - 1 \le i \\ +\lfloor \frac{[i+1-(b-a-k+2i+2)](k-i-1)}{k-1} \rfloor & \text{and } i < a-b+k-1 \text{ ;} \\ a-i-1 & \text{if } i < \frac{a-b+k}{2} - 1 \text{ .} \\ +\lfloor \frac{[k-i-1-(a-b+k-2i-2)](i+1)}{k-1} \rfloor & \end{cases}$$

So, $\kappa(S_{a-b+k}) = \kappa(S_{a-b+k+1}) = \cdots = \kappa(S_{\min\{a,k\}}) = a$.

If $i < \frac{a-b+k}{2} - 1$, then

$$\begin{aligned} \kappa(S_i) - \kappa(S_{i+1}) &= a-i+\lfloor \frac{[k-i-(a-b+k-2i)]i}{k-1} \rfloor - (a-i-1) \\ &- \lfloor \frac{[k-i-1-(a-b+k-2i-2)](i+1)}{k-1} \rfloor \\ &\ge 1+\lfloor \frac{(a-b-2i-1)}{k-1} \rfloor \ge 1+\lfloor \frac{1-k}{k-1} \rfloor = 0. \end{aligned}$$

So, $\kappa(S_i) \ge \kappa(S_{i+1})$. Namely, if $a-b+k$ is odd, we have $\kappa(S_0) \ge \kappa(S_1) \ge \cdots \ge \kappa(S_{\frac{a-b+k-3}{2}}) \ge \kappa(S_{\frac{a-b+k-1}{2}})$; and if $a-b+k$ is even, we have $\kappa(S_0) \ge \kappa(S_1) \ge \cdots \ge \kappa(S_{\frac{a-b+k-4}{2}}) \ge \kappa(S_{\frac{a-b+k-2}{2}})$.

If $a-b+k$ is even, then $\kappa(S_{\frac{a-b+k}{2}-1}) = \frac{a+b-k}{2}+1+\lfloor \frac{(b-a+k-2)(a-b+k-2)}{4(k-1)} \rfloor$ and $\kappa(S_{\frac{a-b+k}{2}}) = \frac{a+b-k}{2} + \lfloor \frac{(b-a+k)(a-b+k)}{4(k-1)} \rfloor$. Since $(a-b+k)(b-a+k) - (b-a+k-2)(a-b+k-2) = (a-b+k)(b-a+k) - [(a-b+k)(b-a+k) - 2(b-a+k) - 2(a-b+k-2)] = 4(k-1)$, we have $\kappa(S_{\frac{a-b+k}{2}-1}) = \kappa(S_{\frac{a-b+k}{2}})$. If $a-b+k$ is odd, then $\kappa(S_{\frac{a-b+k-1}{2}}) = \frac{a+b-k+1}{2} + \lfloor \frac{(b-a+k-1)(a-b+k-1)}{4(k-1)} \rfloor$ $= \kappa(S_{\frac{a-b+k+1}{2}})$.

If $\frac{a-b+k}{2} \leq i < a-b+k-1$, then

$$\begin{aligned}\kappa(S_{i+1}) - \kappa(S_i) &= b-k+i+1 \\ &\quad + \lfloor \frac{[i+1-(b-a-k+2i+2)](k-i-1)}{k-1} \rfloor \\ &\quad - \{b-k+i+\lfloor \frac{[i-(b-a-k+2i)](k-i)}{k-1} \rfloor\} \\ &\geq 1+\lfloor \frac{(b-a-2k+2i+1)}{k-1} \rfloor \geq 1+\lfloor \frac{1-k}{k-1} \rfloor = 0.\end{aligned}$$

So, $\kappa(S_{i+1}) \geq \kappa(S_i)$. Namely, if $a-b+k$ is odd, we have $\kappa(S_{\frac{a-b+k+1}{2}}) \leq \kappa(S_{\frac{a-b+k+3}{2}}) \leq \cdots \leq \kappa(S_{a-b+k-1}) \leq a = \kappa(S_{a-b+k})$, and if $a-b+k$ is even, we have $\kappa(S_{\frac{a-b+k}{2}}) \leq \kappa(S_{\frac{a-b+k+2}{2}}) \leq \cdots \leq \kappa(S_{a-b+k-1}) \leq a = \kappa(S_{a-b+k})$.

Thus, if $k > b-a+2$ and $a-b+k$ is odd,

$\kappa_k(K_{a,b}) = \kappa(S_{\frac{a-b+k-1}{2}}) = \frac{a+b-k+1}{2} + \lfloor \frac{(a-b+k-1)(b-a+k-1)}{4(k-1)} \rfloor$,

and if $k > b-a+2$ and $a-b+k$ is even,

$\kappa_k(K_{a,b}) = \kappa(S_{\frac{a-b+k}{2}}) = \frac{a+b-k}{2} + \lfloor \frac{(a-b+k)(b-a+k)}{4(k-1)} \rfloor$.

The proof is complete. ∎

Notice that, when $k = a+b$, the result coincides with Theorem 1.2.

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