The full metamorphosis of λ-fold block designs with block size four into λ-fold 4-cycle systems

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Abstract

Let (X, B) be a λ-fold block design with block size 4. If a pair of disjoint edges are removed from each block of B the resulting collection of 4-cycles C is a partial λ-fold 4-cycle system (X, C). If the deleted edges can be arranged into a collection of 4-cycles D, then $(X, C \cup D)$ is a λ-fold 4-cycle system [10]. Now for each block $b \in B$ specify a 1-factorization of b as $\{F_1(b), F_2(b), F_3(b)\}$ and define for *each* $i = 1, 2, 3$, sets C_i and D_i as follows: for each $b \in B$, put the 4-cycle $b \setminus F_i(b)$ in C_i and the 2 edges belonging to $F_i(b)$ in D_i. If the edges in D_i can be arranged into a collection of 4-cycles D_i^* then $M_i = (X, C_i \cup D_i^*)$ is a λ-fold 4-cycle system, called the ith

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metamorphosis of (X, B). The *full metamorphosis* is the set of three metamorphoses $\{M_1, M_2, M_3\}$. We give a complete solution of the following problem: for which n and λ does there exist a λ-fold block design with block size 4 having a full metamorphosis into a λ-fold 4-cycle system?

1 Introduction

A λ-fold block design of order n and block size 4 is a pair (X, B), where B is a collection of edge disjoint copies of K_4 which partitions the edge set of λK_n with vertex set X. The copies of K_4 are called blocks. A λ-fold 4-cycle system of order n is a pair (X, C), where C is a a collection of edge disjoint 4-cycles which partitions the edge set of λK_n with vertex set X. The spectrum for both λ-fold block designs and λ-fold 4-cycle systems, for all λ is known and can be found in [1].

Let (X, B) be a λ-fold block design with block size 4. If a pair of disjoint edges are removed from each block of B the resulting collection of 4-cycles C is a partial λ-fold 4-cycle system (X, C). If the deleted edges can be rearranged into a collection of 4-cycles D, then $(X, C \cup D)$ is a λ-fold 4-cycle system, called a metamorphosis of the λ-fold block design (X, B) into a λ-fold 4-cycle system. In [10] a complete solution is given to the following problem: For which n and λ does there exist a λ-fold block design with block size 4 having a metamorphosis into λ-fold 4-cycle system?

Now label the elements of each block b with b_1, b_2, b_3, and b_4 (in any manner). For each block $b = [b_1, b_2, b_3, b_4]$ belonging to B let $\{F_1, F_2, F_3\}$ be the 1-factorization of b defined by $F_1 = \{\{b_1, b_2\}, \{b_3, b_4\}\}$, $F_2 = \{\{b_1, b_3\}, \{b_2, b_4\}\}$, and $F_3 = \{\{b_1, b_4\}, \{b_2, b_3\}\}$. For each $i = 1, 2, 3$ define a set of 4-cycles C_i and a set of deleted edges D_i as follows: for each 1-factor F_i of b, place the 4-cycle $b \setminus F_i$ in C_i and F_i in D_i. Then (X, C_i) is a partial

λ-fold 4-cycle system. Now if the edges belonging to D_i can be rearranged into a collection of 4-cycles D_i^*, then $M_i = (X, C_i \cup D_i^*)$ is a λ-fold 4-cycle system called the ith metamorphosis of (X, B). The full metamorphosis of (X, B) is the set of three metamorphoses $\{M_1, M_2, M_3\}$.

In what follows, a λ-fold block design will always mean a λ-fold block design with block size 4 and we will denote K_4 by its vertex set and the 4-cycle with the edge set $\{\{a,b\},\{b,c\},\{c,d\},\{d,a\}\}$ by (a,b,c,d) or (a,d,c,b), or any cyclic shift of these. The following example illustrates the full metamorphosis of a block design of order 25.

Example 1.1 (*Full metamorphosis of a block design of order* 25 *into a* 4-*cycle system.*)

Let (X, B) be the block design with vertex set $X = \mathbb{Z}_{25}$ and block set B, where

$B = \{[1,0,12,5], [2,1,13,6], [3,2,14,7], [4,3,10,8], [0,4,11,9], [6,5,17,10], [7,6,18,11], [8,7,19,12], [9,8,15,13], [5,9,16,14], [11,10,22,15], [12,11,23,16], [13,12,24,17], [14,13,20,18], [10,14,21,19], [16,15,2,20], [17,16,3,21], [18,17,4,22], [19,18,0,23], [15,19,1,24], [21,20,7,0], [22,21,8,1], [23,22,9,2], [24,23,5,3], [20,24,6,4], [2,0,24,10], [3,1,20,11], [4,2,21,12], [0,3,22,13], [1,4,23,14], [7,5,4,15], [8,6,0,16], [9,7,1,17], [5,8,2,18], [6,9,3,19], [12,10,9,20], [13,11,5,21], [14,12,6,22], [10,13,7,23], [11,14,8,24], [17,15,14,0], [18,16,10,1], [19,17,11,2], [15,18,12,3], [16,19,13,4], [22,20,19,5], [23,21,15,6], [24,22,16,7], [20,23,17,8], [21,24,18,9]\}$. Then

$D_1^* = \{(2,3,1,0), (7,5,3,0), (14,13,4,0), (23,7,16,0), (10,11,2,1), (8,6,4,1), (24,10,17,1), (18,15,4,2), (20,11,9,2), (21,22,4,3), (19,5,12,3), (9,7,6,5), (21,24,8,5), (22,20,9,6), (15,22,13,6), (14,10,8,7), (17,18,9,8), (13,11,12,10), (18,16,14,11), (19,15,13,12), (21,23,14,12), (17,19,16,15), (23,24,17,16), (20,21,19,18), (24,22,23,20)\}$.

$D_2^* = \{(8,13,5,0),(11,21,10,0),(19,9,15,0),(22,17,20,0),(9,14,6,1),$
$(16,6,11,1),\ (21,7,12,1),\ (23,12,15,1),\ (7,10,5,2),\ (24,9,12,2),$
$(17,3,13,2),\ (22,11,16,2),\ (18,21,6,3),\ (14,19,8,3),\ (23,15,20,3),$
$(10,18,7,4),\ (18,23,9,4),\ (19,24,14,4),\ (24,16,21,4),\ (20,10,15,5),$
$(24,13,16,5),\ (20,14,17,6),\ (22,12,17,7),\ (22,19,11,8),\ (23,13,18,8)\}.$

$D_3^* = \{(9,10,6,0),(13,20,12,0),(18,5,17,0),(24,15,21,0),(7,11,5,1),$
$(19,7,13,1),\ (20,8,14,1),\ (22,14,18,1),\ (8,12,6,2),\ (15,3,10,2),$
$(23,5,14,2),\ (21,13,19,2),\ (16,20,7,3),\ (11,17,9,3),\ (24,18,22,3),$
$(16,22,5,4),\ (12,16,8,4),\ (17,23,11,4),\ (23,19,20,4),\ (24,12,18,6),$
$(23,10,19,6),\ (24,11,15,7),\ (21,14,15,8),\ (21,17,13,9),\ (22,10,16,9)\}.$

The purpose of this paper is to give a complete solution to the following problem: For which n and λ does there exist a λ-fold block design having a full metamorphosis into a λ-fold 4-cycle system? If we can give a complete solution of this problem for $\lambda = 1,2,3,4,6$, and 12, we can paste these solutions together to get a solution for all other values of λ. We will organize our results into four sections, followed by a summary.

Finally, the interested reader is referred to [3, 4, 5, 6, 8, 9, 12] for related work on metamorphosis problems and [7] on the full metamorphosis problem for triple systems.

2 $\lambda = 1$

It is well-known that the spectrum for block designs is precisely the set of all $n \equiv 1$ or 4 (mod 12) and the spectrum for 4-cycle systems is the set of all $n \equiv 1$ (mod 8). Hence a necessary condition for the existence of a block design having a full metamorphosis into a 4-cycle system is $n \equiv 1$ (mod 24).

Example 1.1 gives a solution for $n = 25$ and the following examples give solutions for the cases $n = 49, 73$, and 97.

Example 2.1 (*Full metamorphosis of a block design of order* 49 *into a* 4-*cycle system.*)

Let (X, B) be the block design with vertex set $X = \mathbb{Z}_{49}$, and block set B with starter blocks $[0,1,3,8]$, $[0,18,4,29]$, $[0,21,6,33]$, $[0,9,19,32]$.

Then the starter 4-cycles for D_1^* are $(0,5,4,22)$, $(0,9,34,13)$; for D_2^* are $(0,3,7,26)$, $(0,6,18,7)$; and for D_3^* are $(0,15,31,2)$, $(0,8,18,32)$.

Example 2.2 (*Full metamorphosis of a block design of order* 73 *into a* 4-*cycle system.*)

Let (X, B) be the block design with vertex set $X = \mathbb{Z}_{73}$, block set B with starter blocks $[0,1,4,6]$, $[20,7,28,0]$, $[33,9,44,0]$, $[15,25,47,0]$, $[12,46,0,30]$, $[14,50,0,31]$.

Then the starter 4-cycles for D_1^* are $(0,1,3,13)$, $(0,26,54,24)$, $(0,29,65,31)$; for D_2^* are $(0,4,12,5)$, $(0,9,25,11)$, $(0,12,44,19)$; and for D_3^* are $(0,3,21,6)$, $(0,17,44,21)$, $(0,20,55,22)$.

Example 2.3 (*A* $4-GDD$ *of type* $(2t)^4$ $(t \geq 2,\ t \neq 3)$ *having a full metamorphosis into a* 4-*cycle decomposisiton of* $K_{2t,2t,2t,2t}$.)

Let $\mathcal{G} = (X, G, B)$ be a $4-GDD$ of type $(2t)^4$, $t \geq 2$, $t \neq 3$ with $X = \{1,2,...,2t\} \times \{1,2,3,4\}$, groups $G_j = \{1,2,...,2t\} \times \{j\}$, $j = 1,2,3,4$, and $B = \{(a,1),(b,2),(a \circ_1 b,3),(a \circ_2 b,4)\}$, where $(\{1,2,...,2t\},\circ_1)$ and $(\{1,2,...,2t\},\circ_2)$ are two orthogonal quasigroups of order $2t$ [1]. Then for each block $b \in B$, consider $F_1 = \{\{(a,1),(b,2)\},\{(a \circ_1 b,3),(a \circ_2 b,4)\}\}$, $F_2 = \{\{(a,1),(a \circ_1 b,3)\},\{(b,2),(a \circ_2 b,4)\}\}$, and $F_3 = \{\{(a,1),(a \circ_2 b,4)\},\{(b,2),(a \circ_1 b,3)\}\}$. These deleted edges can be easily rearranged into 4-cycles since there exists a 4-cycle decompositon of $K_{2t,2t}$ [11].

Example 2.4 (*Full metamorphosis of a block design of order* 97 *into a* 4-*cycle system.*)

Set $X = \{\infty\} \cup \{(i,j) \mid 1 \le i \le 24, 1 \le j \le 4\}$.

(i) For each $j = 1, 2, 3$, and 4, place a block design of order 25 having a full metamorphosis into a 4-cycle system on the set $\{\infty\} \cup \{(i,j) \mid 1 \le i \le 24\}$.

(ii) Place a $4 - GDD$ of type $(24)^4$ having a full metamorphosis into a 4-cycle decomposition of $K_{24,24,24,24}$ on $\{(i,j) \mid 1 \le i \le 24, 1 \le j \le 4\}$ with groups $G_j = \{(i,j) \mid 1 \le i \le 24\}$, $j = 1, 2, 3, 4$.

Combining (i) and (ii) gives a block design of order 97 having a full metamorphosis into a 4-cycle system.

With these examples in hand we can give a general construction for all of the remaining cases.

The 4mu + 1 Construction. Let $n = 4mu + 1$ and suppose there exist a 4-GDD of type m^u and a λ-fold block design of order $4m + 1$ having a full metamorphosis into a 4-cycle system. Set $X = \{\infty\} \cup \{(i,j) \mid 1 \le i \le mu, 1 \le j \le 4\}$. Then

(i) On each set $\{\infty\} \cup \{(im + 1, j), (im + 2, j), ..., (im + m, j) \mid 1 \le j \le 4\}$, $0 \le i \le u - 1$, place a block design of order $4m + 1$ having a full metamorphosis into a 4-cycle system.

(ii) Take a 4-GDD of type m^u on $\{1, 2, ..., um\}$ with groups $\{im+1, im+2, ..., im+m\}$, $0 \le i \le u-1$. For each block $\{x, y, z, w\}$ in the 4-GDD, place a copy of $\mathcal{G}$ in Example 2.3 on the set $\{x, y, z, w\} \times \{1, 2, 3, 4\}$ with groups $\{x\} \times \{1, 2, 3, 4\}$, $\{y\} \times \{1, 2, 3, 4\}$, $\{z\} \times \{1, 2, 3, 4\}$, and $\{w\} \times \{1, 2, 3, 4\}$.

Combining (i) and (ii) gives a λ-fold block design of order n having a full metamorphosis into a 4-cycle system. □

Lemma 2.5 *There exists a block design of order n having a full metamorphosis into a 4-cycle system if and only if $n \equiv 1 \pmod{24}$.*

Proof There exist a 4-GDD of type 6^u for every $u \geq 5$ [1] and a block design of order 25 having a full metamorphosis into a 4-cycle system, therefore $m = 6$ in the $4mu + 1$ Construction gives a λ-fold block design of order $24u + 1$ having a full metamorphosis into a 4-cycle system for all $u \geq 5$. The cases when $u = 1, 2, 3$, and 4 are obtained in Examples 1.1, 2.1, 2.2, and 2.4. □

3 $\lambda = 2$

It is well-known that the spectrum for 2-fold block designs is the set of all $n \equiv 1 \pmod 3$ and the spectrum for 2-fold 4-cycle systems is the set of all $n \equiv 0$ or $1 \pmod 4$. Hence $n \equiv 1$ or $4 \pmod{12}$ is necessary for the existence of a 2-fold block design having a full metamorphosis into a 2-fold 4-cycle system. The following lemma shows that this necessary condition is sufficient.

Lemma 3.1 *There exists a 2-fold block design having a full metamorphosis into a 2-fold 4-cycle system if and only if $n \equiv 1$ or $4 \pmod{12}$.*

Proof Let (X, B) be the 2-fold block design of order 4 with blocks $[1,2,3,4]$ and $[1,3,4,2]$. Then $M_i = (X, C_i \cup D_i^*)$, where $D_1^* = \{(1,2,4,3)\}$, $D_2^* = \{(1,4,2,3)\}$, and $D_3^* = \{(1,2,3,4)\}$ forms a full metamorphosis of a 2-fold block design of order 4 into a 2-fold 4-cycle system of order 4. Since the spectrum for block designs is $n \equiv 1$ or $4 \pmod{12}$, doubling each block of a block design of the required order, labeling each pair of blocks similar to the block design of order 4, and placing a full metamorphosis of a 2-fold block design of order 4 into a 2-fold 4-cycle system of order 4 gives a full metamorphosis of a 2-fold block design into a 2-fold 4-cycle system. □

4 $\lambda = 3$

The spectrum for 3-fold block designs is the set of all $n \equiv 0$ or $1 \pmod 4$ and the spectrum for 3-fold 4-cycle systems is the set of all $n \equiv 1 \pmod 8$. Hence $n \equiv 1 \pmod 8$ is necessary for the existence of a 3-fold block design having a full metamorphosis into a 3-fold 4-cycle system. We will consider n in modulo 24. So the cases to be considered are $n \equiv 1, 9$ and $17 \pmod{24}$. When $n \equiv 1 \pmod{24}$ three copies of the solution in the case $\lambda = 1$ will give the required full metamorphosis. For the remaining two cases we will need the following designs.

Example 4.1 (*Full metamorphosis of a 3-fold block design of order* 9 *into a 4-cycle system.*)

Let (X, B) be the block design with vertex set $X = \mathbb{Z}_9$ and block set B with starter blocks $[0,2,1,4]$ and $[0,4,1,6]$.

Then the starter 4-cycle for D_1^* is $(0,4,8,2)$; for D_2^* is $(0,1,3,2)$; and for D_3^* is $(0,3,6,1)$.

Example 4.2 (*Full metamorphosis of a 3-fold block design of order* 17 *into a 4-cycle system.*)

Let (X, B) be the block design with vertex set $X = \mathbb{Z}_{17}$ and block set B with starter blocks $[0,1,4,6]$, $[0,8,12,2]$, $[0,4,7,16]$, and $[0,14,8,15]$.

Then the starter 4-cycles for D_1^* are $(0,8,16,3)$, $(0,1,3,10)$; for D_2^* are $(1,0,5,10)$, $(0,5,1,7)$; and for D_3^* are $(0,1,7,11)$, $(0,3,5,2)$.

Example 4.3 (*Full metamorphosis of a 3-fold block design of order* 33 *into a 4-cycle system.*)

Set $X = \{\infty\} \cup \{(i,j) \mid 1 \le i \le 8, 1 \le j \le 4\}$.

(i) For each $j = 1, 2, 3$, and 4, place a 3-fold block design of order 9 having a full metamorphosis into a 4-cycle system on each set $\{\infty\} \cup \{(i,j) \mid 1 \leq i \leq 8\}$.

(ii) Place a $4 - GDD$ of type 8^4 having a full metamorphosis into a 4-cycle decomposition of $3K_{8,8,8,8}$ on $\{(i,j) \mid 1 \leq i \leq 8, 1 \leq j \leq 4\}$ with groups $\{(i,j) \mid 1 \leq i \leq 8\}$, $j = 1, 2, 3$, and 4 (3 copies of a solution in Example 2.3).

Combining (i) and (ii) gives a 3-fold block design of order 33 having a full metamorphosis into a 4-cycle system.

Lemma 4.4 *If $n \equiv 9$ (mod 24) then there exists a 3-fold block design of order n having a full metamorphosis into a 3-fold 4-cycle system.*

Proof There exists a 4-GDD of type 2^u for every $u \equiv 1 \pmod 3$ except $u =$ 1 and 4 [1]. Also a 3-fold block design of order 9 having a full metamorphosis into a 4-cycle system has been constructed in Example 4.1. Therefore taking $m = 2$ and $u \equiv 1 \pmod 3$ in the $4mu + 1$ Construction gives a 3-fold block design of order $8u + 1$ having a full metamorphosis into a 4-cycle system for all u except $u = 1$ and 4. The cases when $u = 1$, and 4 are obtained in Examples 4.1 and 4.3. □

The remaining case $n \equiv 17 \pmod{24}$ will be split into the three cases $n \equiv 17 \pmod{48}$, $n \equiv 41 \pmod{96}$, and $n \equiv 89 \pmod{96}$.

Lemma 4.5 *If $n \equiv 17$ (mod 48) then there exists a 3-fold block design of order n having a full metamorphosis into a 3-fold 4-cycle system.*

Proof There exists a 4-GDD of type 4^u for every $u \equiv 1 \pmod 3$ except $u = 1$ [1]. Also a 3-fold block design of order 17 having a full metamorphosis into a 4-cycle system has been constructed in Example 4.2. Therefore taking $m = 4$ and $u \equiv 1 \pmod 3$ in the $4mu + 1$ Construction gives a 3-fold block design of order $16u + 1$ having a full metamorphosis into a 4-cycle system for all u, except $u = 1$. □

Lemma 4.6 *If $n \equiv 41 \pmod{96}$ then there exists a 3-fold block design of order n having a full metamorphosis into a 3-fold 4-cycle system.*

Proof Set $X = \{\infty\} \cup \{(i,j) \mid 1 \leq i \leq 12s+5, 1 \leq j \leq 8\}$.

(i) For each $i = 1, 2, ..., 12s + 5$, place a 3-fold block design of order 9 having a full metamorphosis into a 4-cycle system on each set $\{\infty\} \cup \{(i,j) \mid 1 \leq j \leq 8\}$.

(ii) Take a 3-fold block design (S, B) of order $12s+5$ [2]. For each block $\{x, y, z, w\}$ in B, place a 4−GDD of type 8^4 having a full metamorphosis into a 4-cycle decomposition of $3K_{8,8,8,8}$ (see Example 2.3) on $\{\{x\} \times \{1,2,...,8\}\} \cup \{\{y\} \times \{1,2,...,8\}\} \cup \{\{z\} \times \{1,2,...,8\}\} \cup \{\{w\} \times \{1,2,...,8\}\}$ with groups $\{x\} \times \{1,2,...,8\}$, $\{y\} \times \{1,2,...,8\}$, $\{z\} \times \{1,2,...,8\}$, and $\{w\} \times \{1,2,...,8\}$.

Combining (i) and (ii) gives a 3-fold block design of order $96s + 41$ having a full metamorphosis into a 4-cycle system. □

Finally, we will construct a 3-fold block design of order n having a full metamorphosis into a 3-fold 4-cycle system for $n = 89$, $n = 185$, and for all $n \equiv 89 \pmod{96}$, $n \geq 281$.

Example 4.7 *There exists a 3-fold block design of order 89 having a full metamorphosis into a 3-fold 4-cycle system.*

Let (P, G, B) be a transversal design $TD(5,5)$ (5-GDD of type 5^5), where the groups are $G_i = \{1,2,3,4,5\} \times \{i\}$ for $1 \leq i \leq 5$. Consider the set of blocks $\widetilde{B} = \{\widetilde{b_1}, \widetilde{b_2}, ..., \widetilde{b_5}\}$ containing the point $(1,1)$. Now by deleting the points $(1,1)$, $(2,1)$, $(3,1)$ in G_1 and considering the new groups as $\{(4,1), (5,1)\}$ and $\{\widetilde{b_i} \setminus \{(1,1)\}\}$ for $1 \leq i \leq 5$, we obtain a $\{4,5\}$−GDD of type $2^1 4^5$. (Note that the blocks of the new GDD are G_i for $2 \leq i \leq 5$

with size 5 and the remaining blocks of (P, G, B); so that if they contain the point $(4,1)$ or $(5,1)$ they have size 5, otherwise 4.) Next we will apply Wilson's Construction and blow up every point by 4 and add an extra ∞ point. That is;

(1) For each group of the $\{4,5\}-$GDD of type $2^1 4^5$, blow every point in the group up by 4 and place a 3-fold block design of order 9 or 17 having a full metamorphosis into a 3-fold 4-cycle system on the points in the group together with the point ∞.

(2) For each block of the $\{4,5\}-$GDD of type $2^1 4^5$, place a $4-$GDD of type 4^4 or 4^5 having a full metamorphosis into a 4-cycle decomposition of $3K_{4,4,4,4}$ or $3K_{4,4,4,4,4}$ on the blown up points of the block.

Combining (1) and (2) gives a 3-fold block design of order 89 having a full metamorphosis into a 4-cycle system.

Example 4.8 *There exists a 3-fold block design of order* 185 *having a full metamorphosis into a 3-fold 4-cycle system.*

Let (P, G, B) be a transversal design $TD(5,5)$ (5-GDD of type 5^5), where the groups are $G_i = \{1, 2, ..., 5\} \times \{i\}$ for $1 \leq i \leq 5$. Now by deleting the points $(1,1)$ and $(2,1)$ in G_1 we obtain a $\{4,5\}-$GDD of type $3^1 5^4$. (Note that the blocks are the remaining blocks of (P, G, B) and if these blocks contain a point in $\{(3,1), (4,1), (5,1)\}$ they have size 5, otherwise 4.) Next we will apply Wilson's Construction and blow up every point by 8 and add an extra ∞ point. That is;

(1) For each group of the $\{4,5\}-$GDD of type $3^1 5^4$, blow every point in the group up by 8 and place a 3-fold block design of order 25 or 41 having a full metamorphosis into a 3-fold 4-cycle system on the points in the group together with the point ∞.

(2) For each block of the $\{4,5\}-$GDD of type $3^1 5^4$, place a $4-$GDD of type 8^4 or 8^5 having a full metamorphosis into a 4-cycle decomposition of

$3K_{8,8,8,8}$ or $3K_{8,8,8,8,8}$ on the blown up points of the block.

Combining (1) and (2) gives a 3-fold block design of order 185 having a full metamorphosis into a 4-cycle system.

Lemma 4.9 *If $n \equiv 89$ (mod 96) then there exists a 3-fold block design of order n having a full metamorphosis into a 3-fold 4-cycle system.*

Proof Let (P, G, B) be a transversal design $TD(5, 6s+4)$ (5-GDD of type $(6s+4)^5$) with $s \geq 2$, where the groups are $G_i = \{1, 2, ..., 6s+4\} \times \{i\}$ for $1 \leq i \leq 5$. Now by deleting $(x, 1)$, $1 \leq x \leq 6s-2$ in G_1, we obtain a $\{4,5\}$−GDD of type $6^1(6s+4)^4$. (Note that the blocks are the remaining blocks of (P, G, B) and if these blocks contain a point in $\{(6s-1, 1), (6s, 1), (6s+1, 1), (6s+2, 1), (6s+3, 1), (6s+4, 1)\}$ they have size 5, otherwise 4.) Next we will apply Wilson's Construction and blow up every point by 4 and add an extra ∞ point. That is;

(1) For each group of the $\{4,5\}$−GDD of type $6^1(6s+4)^4$, blow every point in the group up by 4 and place a 3-fold block design of order 25 or $24s+17$ on the points in the group together with the point ∞ (we obtain such a 3-fold block design of order $24s+17$ either recursively throughout this construction or in Lemmas 4.5 and 4.6).

(2) For each block of the $\{4,5\}$−GDD of type $6^1(6s+4)^4$, place a 4−GDD of type 4^4 or 4^5 having a full metamorphosis into a 4-cycle decomposition of $3K_{4,4,4,4}$ or $3K_{4,4,4,4,4}$ on the blown up points of the block. Note also that a full metamorphosis into a 4-cycle decomposition of $3K_{4,4,4,4,4}$ and $3K_{8,8,8,8,8}$ can be easily obtained as in (ii) of the proof of Lemma 4.6.

Combining (1) and (2) and Examples 4.7 and 4.8 gives a 3-fold block design of order $96s+89$ having a full metamorphosis into a 4-cycle system.
□

5 $\lambda = 4, 5$, and 6

The spectrum for 4-fold block designs is the set of all $n \equiv 1 \pmod 3$ and the spectrum for 4-fold 4-cycle systems is the set of all $n \geq 4$. Hence $n \equiv 1 \pmod 3$ is necessary for the existence of a 4-fold block design having a full metamorphosis into a 4-fold 4-cycle system. The following lemma shows that this necessary condition is sufficient.

Lemma 5.1 *There exists a 4-fold block design having a full metamorphosis into a 4-fold 4-cycle system if and only if $n \equiv 1 \pmod 3$.*

Proof The spectrum for 2-fold block designs is $n \equiv 1 \pmod 3$. So let (X, B) be a 2-fold block design and let $(X, 2B)$ be a 4-fold block design obtained by doubling each block in B. The remainder of the proof is exactly the same as the proof of Lemma 3.1. □

The spectrum for 6-fold block designs is the set of all $n \geq 4$ and the spectrum for 6-fold 4-cycle systems is the set of all $n \equiv 0$ or $1 \pmod 4$. Hence $n \equiv 0$ or $1 \pmod 4$ is necessary for the existence of a 6-fold block design having a full metamorphosis into a 6-fold 4-cycle system. The following lemma shows that this necessary condition is sufficient.

Lemma 5.2 *There exists a 6-fold block design having a full metamorphosis into a 6-fold 4-cycle system if and only if $n \equiv 0$ or $1 \pmod 4$.*

Proof The spectrum for 3-fold block designs is $n \equiv 0$ or $1 \pmod 4$. So let (X, B) be a 3-fold block design and let $(X, 2B)$ be a 6-fold block design obtained by doubling each block in B. The remainder of the proof is exactly the same as the proof of Lemma 3.1. □

The spectrum for both 12-fold block designs and 12-fold 4-cycle systems is the set of all $n \geq 4$. Hence $n \geq 4$ is necessary for the existence of a 12-fold block design having a full metamorphosis into a 12-fold 4-cycle system. The following lemma shows that this necessary condition is sufficient.

Lemma 5.3 *There exists a* 12*-fold block design having a full metamorphosis into a* 12*-fold* 4*-cycle system if and only if* $n \geq 4$.

Proof The spectrum for 6-fold block designs is $n \geq 4$. So let (X, B) be a 6-fold block design and let $(X, 2B)$ be a 12-fold block design obtained by doubling each block in B. The remainder of the proof is exactly the same as the proof of Lemma 3.1. □

6 Summary

We can paste together solutions for $\lambda = 1, 2, 3, 4, 6$, and 12 to obtain solutions for all other values of λ. The following table gives a summary of the results in this paper.

λ (mod 12)	Spectrum of λ-fold block designs having a full metamorphosis into a λ-fold 4-cycle system
$1, 5, 7$ or 11	$n \equiv 1 \pmod{24}$
2 or 10	$n \equiv 1$ or $4 \pmod{12}$
3 or 9	$n \equiv 1 \pmod 8$
4 or 8	$n \equiv 1 \pmod 3$
6	$n \equiv 0$ or $1 \pmod 4$
0	$n \geq 4$

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