On super $(a,1)$-edge-antimagic total labelings of grids and crowns

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Abstract

A graph $G(V,E)$ with order p and size q is called (a,d)-edge-antimagic total labeling graph if there exists a bijective function $f: V(G)\bigcup E(G) \to \{1,2,...,p+q\}$ such that the edge-weights $\lambda_f(uv) = f(u)+f(v)+f(uv), uv \in E(G)$, form an arithmetic sequence with first term a and common difference d. Such a labeling is called super if the p smallest possible labels appear at the vertices. In this paper, we study super $(a,1)$-edge-antimagic properties of $m(P_4 \square P_n)$ for $m,n \geq 1$ and $m(C_n \odot \overline{K_l})$ for n even and $m,l \geq 1$.

1 Introduction

All graph mentioned in this paper is finite, undirected and simple. For a graph G, $V(G)$ and $E(G)$ denote the vertex set and the edge set, respectively. A (p,q) graph G is a graph with order $|V(G)| = p$ and size $|E(G)| = q$. For a (p,q) graph G, a bijective mapping $f: V(G)\bigcup E(G) \to \{1,2,\cdots,p+q\}$ is a total labeling of G and the associated *edge-weights* are $\lambda_f(uv) = f(u)+f(v)+f(uv)$, for every $uv \in E(G)$. The total labeling is called an *(a,d)-edge-antimagic total labeling* (*(a,d)-EAT* labeling for short) of G if the set of all edge-weights equals $\{a, a+d, a+2d, \cdots, a+(q-1)d\}$, where $a > 0$ and $d \geq 0$ are two fixed integers. Furthermore, f is a *super (a,d)-EAT* labeling of G if the vertex labels are the integers $\{1,2,\cdots,p\}$. A graph that admits an (a,d)-EAT labeling or a super (a,d)-EAT labeling is called an *(a,d)-EAT graph* or a *super (a,d)-EAT graph*,

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respectively.

The definition of an (a,d)-EAT labeling was introduced by Simanjuntak et al. in [12]. In [14], the authors described how to construct super (a,d)-EAT labelings of all caterpillars for $d = 0,1,2$ and of certain caterpillars for $d = 3$. Some construction of super (a,d)-EAT labeling for disconnected graphs are presented using the notion of an α-labeling [1]. Bača et al. also studied super (a,d)-EAT labeling for path-like trees [6], mK_n [2], $mK_{n,n}$ [3], $mK_{n,n,n}$ [8] and mC_n, mP_n [9]. Even regular graph and odd regular graph with a 1-factor are super $(a,1)$-EAT, see [5]. The super (a,d)-EAT properties of $P_2\square P_n$, $m(C_n\odot\overline{K_l})$ and $mP_n\cup\mu C_n$ studied in [13] and [4].

In this paper, we deal with the existence of super $(a,1)$-EAT labeling of $m(P_4\square P_n)$ for $m,n\geq 1$ and $m(C_n\odot\overline{K_l})$ for n even and $m,l\geq 1$.

2 (a,1)-EAT labeling for grids

The Cartesian product of graphs G_1 and G_2, written $G_1\square G_2$, is the graph with vertex set $V(G_1)\times V(G_2)$ specified by putting (u,v) adjacent to (u',v') if and only if (1) $u=u'$ and $vv'\in E(G_2)$, or (2) $v=v'$ and $uu'\in E(G_1)$. Let P_n be a path on n vertices. We call $P_l\square P_n (l,n\geq 2)$ a grid.

Assuming that $l,n\geq 2$ are integers. Let P_l be a path with an ordered list of distinct vertices $u_1,u_2,\cdots,u_l$ such that $u_{i-1}u_i$ is an edge for all $2\leq i\leq l$. Similarly, we assume a path P_n as an ordered list $v_1,v_2,\cdots,v_n$ such that all $v_{i-1}v_i$ are edges. In the graph $P_l\square P_n$, the vertex $(u_i,v_j)\in V(P_l\square P_n)$ is represented by $w_{i,j}$, and the edge $[(u_i,v_j),(u_i,v_{j+1})]\in E(P_l\square P_n)$ is represented by $e_{i,j}$, and the edge $[(u_i,v_j),(u_{i+1},v_j)]\in E(P_l\square P_n)$ is represented by $\varepsilon_{i,j}$. Clearly, $|V((P_l\square P_n))| = ln$, $|E((P_l\square P_n))| = 2nl-l-n$.

Let A be a set, $A=\{a_1,a_2,\cdots,a_n,a_1,a_2,\cdots,a_n,\cdots,a_1,a_2,\cdots, a_n\}$, in which the amount of a_i is t for all $i=1,\cdots,n$, we simply denote A by $t\{a_1,a_2,\cdots,a_n\}$. As illustrations, if $A=\{a,b,c,a,b,c\}$, we denote A by $2\{a,b,c\}$.

Lemma 2.1 *Let A be a set, $A=\{c,c+2,c+4,\cdots,c+2k\}\cup t\{c+k-1,c+k,c+k+1\}\cup\{c+k\}$ for positive integers c,k and t. For a graph $G=(p,q)$, if there exists a bijective function h from $V(G)$ onto the set $\{1,2,\cdots,p\}$ such that the set of edge-sums $\{s_h(uv)=h(u)+h(v) : uv\in E(G)\}$ equals A, then G has a super $(c+p+k+1,1)$-EAT labeling.*

Proof. Let $E(G)=\{e_1,e_2,\cdots,e_q\}$, where $q=|A|=k+3t+2$. For our convenience, suppose that

$$\begin{cases} s_h(e_i) = c+2i-2, & \text{for } i = 1,2,\cdots,k+1 \\ s_h(e_i) = c+k+1, & \text{for } i = k+2,k+4,\cdots,k+2t \\ s_h(e_i) = c+k-1, & \text{for } i = k+3,k+5,\cdots,k+2t+1 \\ s_h(e_i) = c+k, & \text{for } i = k+2t+2,k+2t+3,\cdots,k+3t+2. \end{cases}$$

Define a bijective total labeling $f : V(G) \bigcup E(G) \to \{1,2,\cdots,p+q\}$ in the following way:

$$\begin{cases} f(u) = h(u), & \text{for } u \in V(G) \\ f(e_i) = p+k+2-i, & \text{for } i = 1,2,\cdots,k+1 \\ f(e_i) = p+i, & \text{for } i = k+2,k+3\cdots,k+3t+2. \end{cases}$$

The edge-weights λ_f of G under the labeling f, constitute the sets

$$\begin{aligned} W_1 =& \{\lambda_f(e_i) = s_h(e_i) + f(e_i) = c+p+k+i : i = 1,2,\cdots,k+1\} \\ =& \{c+p+k+1, c+p+k+2,\cdots,c+p+2k+1\}, \\ W_2 =& \{\lambda_f(e_i) = s_h(e_i) + f(e_i) = c+p+k+i+1 : i = k+2,k+4,\cdots,k+2t\} \\ =& \{c+p+2k+3, c+p+2k+5,\cdots,c+p+2k+2t+1\}, \\ W_3 =& \{\lambda_f(e_i) = s_h(e_i) + f(e_i) = c+p+k+i-1 : i = k+3,k+5,\cdots,k+2t+1\} \\ =& \{c+p+2k+2, c+p+2k+4,\cdots,c+p+2k+2t\}, \\ W_4 =& \{\lambda_f(e_i) = s_h(e_i) + f(e_i) = c+p+k+i : i = k+2t+2,\cdots,k+3t+2\} \\ =& \{c+p+2k+2t+2, c+p+2k+2t+3,\cdots,c+p+2k+3t+2\}. \end{aligned}$$

Hence, the set $\bigcup_{s=1}^{4} W_s = \{c+p+k+1, c+p+k+2,\cdots,c+p+2k+3t+2\}$ consists of consecutive integers. Thus f is a super $(c+p+k+1,1)$-EAT labeling of G. □

Theorem 2.2 *Let $n \geq 2$ be an integer. Then the graph $P_4 \Box P_n$ is a super $(8n+2,1)$-EAT.*

Proof. Note that $p = |V(P_4 \Box P_n)| = 4n$, $q = |E(P_4 \Box P_n)| = 7n-4$. We define the vertex labeling $h : V(P_4 \Box P_n) \to \{1,2,...,4n\}$ in the following way:

$$\begin{cases} h(w_{1,j}) = 2j-1, & \text{for } j = 1,2,\cdots,n \\ h(w_{2,j}) = 4n-2j+1, & \text{for } j = 1,2,\cdots,n \\ h(w_{3,j}) = 2j, & \text{for } j = 1,2,\cdots,n \\ h(w_{4,j}) = 4n-2j+2, & \text{for } j = 1,2,\cdots,n. \end{cases}$$

It is not difficult to see the vertex labeling f is a bijective function from $V(P_4 \Box P_n)$ onto the set $\{1,2,...,4n\}$. The set of edge-sums $\{s_h(uv) = h(u)+h(v) : uv \in E(P_4 \Box P_n)\}$ is given by:

$$\begin{cases} s_h(e_{1,j}) = h(w_{1,j}) + h(w_{1,j+1}) = 4j, & \text{for } j = 1,2,\cdots,n-1 \\ s_h(e_{2,j}) = h(w_{2,j}) + h(w_{2,j+1}) = 8n-4j, & \text{for } j = 1,2,\cdots,n-1 \\ s_h(e_{3,j}) = h(w_{3,j}) + h(w_{3,j+1}) = 4j+2, & \text{for } j = 1,2,\cdots,n-1 \\ s_h(e_{4,j}) = h(w_{4,j}) + h(w_{4,j+1}) = 8n-4j+2, & \text{for } j = 1,2,\cdots,n-1 \\ s_h(\varepsilon_{1,j}) = h(w_{1,j}) + h(w_{2,j}) = 4n, & \text{for } j = 1,2,\cdots,n \\ s_h(\varepsilon_{2,j}) = h(w_{2,j}) + h(w_{3,j}) = 4n+1, & \text{for } j = 1,2,\cdots,n \\ s_h(\varepsilon_{3,j}) = h(w_{3,j}) + h(w_{4,j}) = 4n+2, & \text{for } j = 1,2,\cdots,n. \end{cases}$$

That is,

$$\begin{aligned} A_1 =& \{s_h(e_{i,j}) : i = 1,2,3,4; j = 1,2,\cdots,n-1\} \cup \{s_h(\varepsilon_{1,n})\} \cup \{s_h(\varepsilon_{3,n})\} \\ =& \{4,6,8,\cdots,4n-2,4n+4,\cdots,8n-2\} \cup \{4n\} \cup \{4n+2\} \\ =& \{4,6,8,\cdots,8n-2\}, \text{ and} \\ A_2 =& \{s_h(\varepsilon_{i,j}) : i = 1,2,3; j = 1,2,\cdots,n-1\} \cup \{s_h(\varepsilon_{2,n})\} \\ =& (n-1)\{4n,4n+1,4n+2\} \cup \{4n+1\}. \end{aligned}$$

Hence, the set $A_1 \cup A_2$ satisfies the conditions in Lemma 2.1 with $c = 4$ and $k = 4n-3$. Thus by Lemma 2.1, $P_4 \Box P_n$ is a super $(8n+2,1)$-EAT graph. □

In the paper [7] is proved the following theorem:

Theorem 2.3[7] *Let G be a super $(a,1)$-EAT graph. Then the disjoint union of arbitrary number of copies of G, i.e. $mG, m \geq 1$, also admits a super $(b,1)$-EAT labeling.*

Directly from Theorem 2.2 and 2.3 follows:

Theorem 2.4 *Let $m \geq 1, n \geq 2$ be integers. Then the graph $m(P_4 \Box P_n)$ is a super $(b,1)$-EAT.*

3 (a,1)-EAT labeling for l-crowns

In [4], the authors studied the super (a,d)-EAT labeling of $m(C_n \odot \overline{K_l})$. They remained an open problem:

Open Problem [4] *For the graph $m(C_n \odot \overline{K_l})$, m odd and $n(l+1)$ even, it would determine if there is a super $(a,1)$-EAT labeling.*

In this section, we show the open problem is true for even n.

The l-crowns, denoted by $C_n \odot \overline{K_l}$, is the graph with vertex set $V(C_n \odot \overline{K_l}) = \{c_i : 1 \leq i \leq n\} \cup \{x_{i,k} : 1 \leq i \leq n, 1 \leq k \leq l\}$ and the edge set $E(C_n \odot \overline{K_l}) = \{c_i c_{i+1} : 1 \leq i \leq n-1\} \cup \{c_n c_1\} \cup \{c_i x_{i,k} : 1 \leq i \leq n, 1 \leq k \leq l\}$. Clearly, $|V(C_n \odot \overline{K_l})| = |E(C_n \odot \overline{K_l})| = n(l+1)$.

Lemma 3.1 *Let A be a set, $A = \{c, c+1, c+2, \cdots, c+k\} \cup \{c+\frac{k}{2}\}$, k even. For a graph $G = (p, q)$, if there exists a bijective function h from $V(G)$ onto the set $\{1, 2, \cdots, p\}$ such that the set of edge-sums $\{s_h(uv) = h(u) + h(v) : uv \in E(G)\}$ equals A, then G has a super $(c+p+\frac{k}{2}+1, 1)$-EAT labeling.*

Proof. Let $E(G) = \{e_0, e_1, \cdots, e_{q-1}\}$, where $q = |A| = k+2$. For our convenience, suppose that

$$\begin{cases} s_h(e_i) = c+i, & \text{for } i = 0, 1, 2, \cdots, k \\ s_h(e_{k+1}) = c+\frac{k}{2}. \end{cases}$$

Define a bijective total labeling $f : V(G) \bigcup E(G) \to \{1, 2, \cdots, p+q\}$ in the following way:

$$\begin{cases} f(u) = h(u), & \text{for } u \in V(G) \\ f(e_i) = p + \frac{k-i}{2} + 1, & \text{for } i = 0, 2, 4, \cdots, k \\ f(e_i) = p + k - \frac{i+1}{2} + 2, & \text{for } i = 1, 3, 5, \cdots, k-1 \\ f(e_{k+1}) = p + k + 2. \end{cases}$$

The edge-weights λ_f of G under the labeling f, constitute the sets:

$$W_1 = \{\lambda_f(e_i) = s_h(e_i) + f(e_i) = c + p + \frac{k+i}{2} + 1 : i = 0, 2, 4, \cdots, k\}$$
$$= \{c + p + \frac{k}{2} + 1, c + p + \frac{k}{2} + 2, \cdots, c + p + k + 1\},$$
$$W_2 = \{\lambda_f(e_i) = s_h(e_i) + f(e_i) = c + p + k + \frac{i+3}{2} : i = 1, 3, \cdots, k-1\}$$
$$= \{c + p + k + 2, c + p + k + 3, \cdots, c + p + \frac{3k}{2} + 1\},$$
$$W_3 = \{\lambda_f(e_{k+1}) = s_h(e_{k+1}) + f(e_{k+1}) = c + p + \frac{3k}{2} + 2\}.$$

Hence, the set $\bigcup_{s=1}^{3} W_s = \{c+p+\frac{k}{2}+1, c+p+\frac{k}{2}+2, \cdots, c+p+\frac{3k}{2}+2\}$ consists of consecutive integers. Thus f is a super $(c+p+\frac{k}{2}+1, 1)$-EAT labeling of G. □

Theorem 3.2 *Let l be a positive integer and $n \geq 2$ be even. Then the graph $C_n \odot \overline{K_l}$ is a super $(2n(l+1)+2, 1)$-EAT.*

Proof. Note that $|V(C_n \odot \overline{K_l})| = |E(C_n \odot \overline{K_l})| = n(l+1)$. We distinguish the proof into two cases for $n = 4t$ and $n = 4t+2$.

Case 1. $n = 4t$.

We define the vertex labeling $h : V(C_n \odot \overline{K_l}) \to \{1, 2, \cdots, n(l+1)\}$ in the following way:

$$\begin{cases} h(c_i) = 2lt + \frac{i+1}{2}, & \text{for } i = 1, 3, \cdots, 4t-1 \\ h(c_i) = 2t(l+1) + \frac{i}{2}, & \text{for } i = 2, 4, \cdots, 4t \\ h(x_{i,k}) = 2t(k+l+2) - s, & \text{for } i = 4s+1, k = 1, 2, \cdots, l \\ h(x_{i,k}) = 2t(k+l+1) + t - s, & \text{for } i = 4s+2, k = 1, 2, \cdots, l \\ h(x_{i,k}) = 2tk - s, & \text{for } i = 4s+3, k = 1, 2, \cdots, l \\ h(x_{i,k}) = 2tk - t - s, & \text{for } i = 4s+4, k = 1, 2, \cdots, l. \end{cases}$$

It is not difficult to see the vertex labeling h is a bijective function from $V(C_n \odot \overline{K_l})$ onto the set $\{1, 2, ..., n(l+1)\}$. The set of edge-sums $\{s_h(uv) = h(u) + h(v) : uv \in E(C_n \odot \overline{K_l})\}$ is given by:

$$\begin{cases} s_h(c_i c_{i+1}) = h(c_i) + h(c_{i+1}) = 2t(2l+1) + i + 1, & \text{for } i = 1, 2, \cdots, 4t-1 \\ s_h(c_{4t} c_1) = h(c_{4t}) + h(c_1) = 2t(2l+2) + 1, & \end{cases}$$

and for $k = 1, 2, \cdots, l$,

$$\begin{cases} s_h(c_i x_{i,k}) = h(c_i) + h(x_{i,k}) = 2t(2l+k+2) + s + 1, & \text{for } i = 4s+1 \\ s_h(c_i x_{i,k}) = h(c_i) + h(x_{i,k}) = 2t(2l+k+2) + t + s + 1, & \text{for } i = 4s+2 \\ s_h(c_i x_{i,k}) = h(c_i) + h(x_{i,k}) = 2t(l+k) + s + 2, & \text{for } i = 4s+3 \\ s_h(c_i x_{i,k}) = h(c_i) + h(x_{i,k}) = 2t(l+k) + t + s + 2, & \text{for } i = 4s+4. \end{cases}$$

That is,

$$\begin{aligned} A_1 =& \{s_h(c_i c_{i+1}) : i = 1, 2, \cdots, 4t-1\} \\ =& \{2t(2l+1) + 2, 2t(2l+1) + 3, \cdots, 2t(2l+1) + 4t\}, \\ A_2 =& \{s_h(c_i x_{i,k}) : i = 4s+1, k = 1, 2, \cdots, l\} \\ =& \{2t(2l+k+2) + s + 1 : k = 1, 2, \cdots, l; s = 0, 1, \cdots, t-1\} \\ =& \{2t(2l+3) + 1, 2t(2l+3) + 2, \cdots, 2t(2l+3) + t, \\ & 2t(2l+4) + 1, 2t(2l+4) + 2, \cdots, 2t(2l+4) + t, \\ & \cdots\cdots\cdots\cdots\cdots\cdots, \\ & 2t(3l+2) + 1, 2t(3l+2) + 2, \cdots, 2t(3l+2) + t\}, \\ A_3 =& \{s_h(c_i x_{i,k}) : i = 4s+2, k = 1, 2, \cdots, l\} \\ =& \{2t(2l+k+2) + t + s + 1 : k = 1, 2, \cdots, l; s = 0, 1, \cdots, t-1\} \\ =& \{2t(2l+3) + t + 1, 2t(2l+3) + t + 2, \cdots, 2t(2l+3) + 2t, \\ & 2t(2l+4) + t + 1, 2t(2l+4) + t + 2, \cdots, 2t(2l+4) + 2t, \\ & \cdots\cdots\cdots\cdots\cdots\cdots, \\ & 2t(3l+2) + t + 1, 2t(3l+2) + t + 2, \cdots, 2t(3l+2) + 2t\}, \\ A_4 =& \{s_h(c_i x_{i,k}) : i = 4s+3, k = 1, 2, \cdots, l\} \\ =& \{2t(l+k) + s + 2 : k = 1, 2, \cdots, l; s = 0, 1, \cdots, t-1\} \\ =& \{2t(l+1) + 2, 2t(l+1) + 3, \cdots, 2t(l+1) + t + 1, \\ & 2t(l+2) + 2, 2t(l+2) + 3, \cdots, 2t(l+2) + t + 1, \end{aligned}$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots,$$

$$2t(2l)+2, 2t(2l)+3, \cdots, 2t(2l)+t+1\},$$

$$\begin{aligned}A_5=&\{s_h(c_ix_{i,k}) : i=4s+4, k=1,2,\cdots,l\}\\ =&\{2t(l+k)+t+s+2 : k=1,2,\cdots,l; s=0,1,\cdots,t-1\}\\ =&\{2t(l+1)+t+2, 2t(l+1)+t+3, \cdots, 2t(l+1)+2t+1,\\ &2t(l+2)+t+2, 2t(l+2)+t+3, \cdots, 2t(l+2)+2t+1,\\ &\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots,\\ &2t(2l)+t+2, 2t(2l)+t+3, \cdots, 2t(2l)+2t+1\},\end{aligned}$$

$$A_6=\{s_h(c_{4t}c_1)=2t(2l+2)+1\}.$$

Hence, the set $A=(\bigcup_{i=1}^{5} A_i)\cup A_6=\{2t(l+1)+2, 2t(l+1)+3, \cdots, t(3l+2)+2t\}\cup\{2t(2l+2)+1\}$ satisfies the conditions in Lemma 3.1 with $c=2t(l+1)+2$ and $k=4t(l+1)-2$. Thus by Lemma 3.1, $C_{4t}\odot\overline{K_l}$ is a super $(c+p+\frac{k}{2}+1, 1)$-EAT graph, i.e. $C_n\odot\overline{K_l}$ is a super $(2n(l+1)+2, 1)$-EAT.

Case 2. $n=4t+2$.

We define the vertex labeling $h : V(C_n\odot\overline{K_l})\rightarrow\{1,2,...,n(l+1)\}$ in the following way:

$$\begin{cases} h(c_i)=(2t+1)l+\frac{i+1}{2}, & \text{for } i=1,3,\cdots,4t+1\\ h(c_i)=(2t+1)(l+1)+\frac{i}{2}, & \text{for } i=2,4,\cdots,4t+2\\ h(x_{i,k})=(2t+1)(k+l+2)-s, & \text{for } i=4s+1, k=1,2,\cdots,l\\ h(x_{i,k})=(2t+1)k-t-s, & \text{for } i=4s+2, k=1,2,\cdots,l\\ h(x_{i,k})=(2t+1)k-s, & \text{for } i=4s+3, k=1,2,\cdots,l\\ h(x_{i,k})=(2t+1)(k+l+1)+t-s, & \text{for } i=4s+4, k=1,2,\cdots,l. \end{cases}$$

It is not difficult to see the vertex labeling h is a bijective function from $V(C_n\odot\overline{K_l})$ onto the set $\{1,2,\cdots,n(l+1)\}$. The set of edge-sums $\{s_h(uv)=h(u)+h(v) : uv\in E(C_n\odot\overline{K_l})\}$ is given by:

$$\begin{cases} s_h(c_ic_{i+1})=h(c_i)+h(c_{i+1})=(2t+1)(2l+1)+i+1, & \text{for } i=1,2,\cdots,4t+1\\ s_h(c_{4t+2}c_1)=h(c_{4t+2})+h(c_1)=(2t+1)(2l+2)+1, \end{cases}$$

and for $k=1,2,\cdots,l$,

$$\begin{cases} s_h(c_ix_{i,k})=h(c_i)+h(x_{i,k})=(2t+1)(2l+k+2)+s+1, & \text{for } i=4s+1\\ s_h(c_ix_{i,k})=h(c_i)+h(x_{i,k})=(2t+1)(l+k)+t+s+2, & \text{for } i=4s+2\\ s_h(c_ix_{i,k})=h(c_i)+h(x_{i,k})=(2t+1)(l+k)+s+2, & \text{for } i=4s+3\\ s_h(c_ix_{i,k})=h(c_i)+h(x_{i,k})=(2t+1)(2l+k+2)+t+s+2, & \text{for } i=4s+4. \end{cases}$$

That is,

$$\begin{aligned}A_1=&\{s_h(c_ic_{i+1}) : i=1,2,\cdots,4t+1\}\\ =&\{(2t+1)(2l+1)+2, (2t+1)(2l+1)+3, \cdots, (2t+1)(2l+1)+4t+2\},\end{aligned}$$

$$A_2=\{s_h(c_ix_{i,k}) : i=4s+1, k=1,2,\cdots,l\}$$

$=\{(2t+1)(2l+k+2)+s+1 : k=1,2,\cdots,l; s=0,1,\cdots,t\}$
$=\{(2t+1)(2l+3)+1,(2t+1)(2l+3)+2,\cdots,(2t+1)(2l+3)+t+1,$
$(2t+1)(2l+4)+1,(2t+1)(2l+4)+2,\cdots,(2t+1)(2l+4)+t+1,$
$\cdots\cdots\cdots,$
$(2t+1)(3l+2)+1,(2t+1)(3l+2)+2,\cdots,(2t+1)(3l+2)+t+1\},$

$A_3=\{s_h(c_i x_{i,k}) : i=4s+2, k=1,2,\cdots,l\}$
$=\{(2t+1)(l+k)+t+s+2 : k=1,2,\cdots,l; s=0,1,\cdots,t\}$
$=\{(2t+1)(l+1)+t+2,(2t+1)(l+1)+t+3,\cdots,(2t+1)(l+1)+2t+2,$
$(2t+1)(l+2)+t+2,(2t+1)(l+2)+t+3,\cdots,(2t+1)(l+2)+2t+2,$
$\cdots\cdots\cdots,$
$(2t+1)(2l)+t+2,(2t+1)(2l)+t+3,\cdots,(2t+1)(2l)+2t+2\},$

$A_4=\{s_h(c_i x_{i,k}) : i=4s+3, k=1,2,\cdots,l\}$
$=\{(2t+1)(l+k)+s+2 : k=1,2,\cdots,l; s=0,1,\cdots,t-1\}$
$=\{(2t+1)(l+1)+2,(2t+1)(l+1)+3,\cdots,(2t+1)(l+1)+t+1,$
$(2t+1)(l+2)+2,(2t+1)(l+2)+3,\cdots,(2t+1)(l+2)+t+1,$
$\cdots\cdots\cdots,$
$(2t+1)(2l)+2,(2t+1)(2l)+3,\cdots,(2t+1)(2l)+t+1\},$

$A_5=\{s_h(c_i x_{i,k}) : i=4s+4, k=1,2,\cdots,l\}$
$=\{(2t+1)(2l+k+2)+t+s+2 : k=1,2,\cdots,l; s=0,1,\cdots,t-1\}$
$=\{(2t+1)(2l+3)+t+2,(2t+1)(2l+3)+t+3,\cdots,(2t+1)(2l+3)+2t+1,$
$(2t+1)(2l+4)+t+2,(2t+1)(2l+4)+t+3,\cdots,(2t+1)(2l+4)+2t+1,$
$\cdots\cdots\cdots,$
$(2t+1)(3l+2)+t+2,(2t+1)(3l+2)+t+3,\cdots,(2t+1)(3l+2)+2t+1\},$

$A_6=\{s_h(c_{4t+2}c_1)=(2t+1)(2l+2)+1\}.$

Hence, the set $A=(\bigcup_{i=1}^{5} A_i)\cup A_6=\{(2t+1)(l+1)+2,(2t+1)(l+1)+3,\cdots,(2t+1)(3l+2)+2t+1\}\cup\{(2t+1)(2l+2)+1\}$, which satisfies the conditions in Lemma 3.1 with $c=(2t+1)(l+1)+2$ and $k=(2t+1)(2l+2)-2$. Thus by Lemma 3.1, $C_{4t+2}\odot\overline{K_l}$ is a super $(c+p+\frac{k}{2}+1,1)$-EAT graph, i.e. $C_n\odot\overline{K_l}$ is a super $(2n(l+1)+2,1)$-EAT. □

Directly from Theorem 3.2 and 2.3 follows:

Theorem 3.3 *Let l,m be positive integers and $n\geq 2$ be even. Then the graph $m(C_n\odot\overline{K_l})$ is a super $(b,1)$-EAT.*

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