On a Family of 4-Critical Graphs with Diameter Three

Francesco Barioli, Marc Loizeaux, and Lucas van der Merwe
University of Tennessee at Chattanooga

February 8, 2008

Abstract

A graph G is *k-total domination edge critical*, abbreviated to *k*-critical if confusion is unlikely, if the total domination number $\gamma_t(G)$ satisfies $\gamma_t(G) = k$ and $\gamma_t(G+e) < \gamma_t(G)$ for any edge $e \in E(\overline{G})$. Graphs that are 4-critical have diameter either 2, 3 or 4. In previous papers we characterized structurally the 4-critical graphs with diameter four, and found bounds on the order of 4-critical graphs with diameter two. In this paper we study a family $\mathcal{H}$ of 4-critical graphs with diameter three, in which every vertex is a diametrical vertex, and every diametrical pair dominates the graph. We also generalize the self-complementary graphs, and show that these graphs provide a special case of the family $\mathcal{H}$.

Keywords: edge critical, total domination, sub-self-complementary.

AMS subject classification: 05C99, 05C69

1 Introduction

The purpose of this paper is to explore the properties of a certain family of graphs with diameter three. We show that every graph H in this family contains two copies of some graph G, with a particular edge set between the two copies. We then define a property that we call *sub-self-complementary*, an extension of the property self-complementary, and show that both G and the complement of G give the same graph H if and only if G is sub-self-complementary.

We begin with some background definitions and information. A set $S \subseteq V(G)$ of a graph G is a *dominating set* if every vertex not in S is adjacent to a vertex in S. The *domination number* $\gamma(G)$ is the minimum cardinality of a domination set. A *total dominating set* in a graph G is a subset S of $V(G)$ such that every vertex in $V(G)$ is adjacent to a vertex of S. Every graph G without isolated vertices has a total dominating set, since $S = V(G)$ is such a set. The *total dominating number* $\gamma_t(G)$ is the minimum cardinality of a total dominating set. A total dominating set of G of cardinality $\gamma_t(G)$ is called a $\gamma_t(G)$*-set*. For sets S, $X \subseteq V$, if S dominates X, then we write $S \succ X$, while if S totally dominates X, we write written $S \succ_t X$. If $S = \{s\}$ or $X = \{x\}$, we also write $s \succ X$, $S \succ x$, etc. Domination-related concepts not defined here can be found in [2].

Let $G = (V, E)$ be a graph with order $|V| = n$. For $u, v \in V$, if u is adjacent to v, we write $u \perp v$. The *open neighborhood* of a vertex v is the set of vertices adjacent to v, that is, $N(v) = \{w \mid vw \in E(G)\}$, and the *closed neighborhood* of v is $N[v] = N(v) \cup \{v\}$.

Denote the distance from x to y as $d(x, y)$. If there is no path from x to y then $d(x, y) = \infty$ (and $diam(G) = \infty$). If G is a graph with $diam(G) = k \leq \infty$ and $d(u, v) = k$, then we say that u is a *diametrical vertex*, and $\{u, v\}$ is a *diametrical pair*. A shortest u-v path in G is a *diametrical path*, and $\{v : d(u, v) = k\}$ is the *diametrical set* for u.

A graph G is *total domination edge critical*, or just γ_t*-critical*, if $\gamma_t(G + e) < \gamma_t(G)$ for any edge $e \in E(\overline{G})$. If G is total domination edge critical and $\gamma_t(G) = k$, then we say G is k*-total domination edge critical*. (The phrase "k-total domination edge critical" is abbreviated to "k-critical" if confusion is unlikely.) Van der Merwe, Mynhardt, and Haynes [3] studied 3-critical graphs, that is, 3-total domination edge critical graphs. In [5], Van der Merwe and Loizeaux studied 4-critical graphs with diameter four, and showed that connected 4-critical graphs have diameter 2, 3, or 4. Figure 1 gives examples of such graphs. We also showed that disconnected 4-critical graphs have exactly two complete components, both with order at least two. In [6] we studied 4-critical graphs with diameter two.

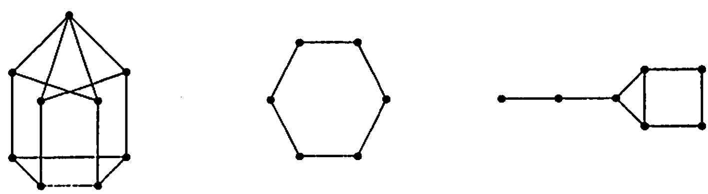

Figure 1: 4-critical graphs with diameters 2, 3, and 4 respectively.

It is shown in [3], and we restate it here for emphasis, that the addition of an edge to a graph can change the total domination number by at most two.

Proposition 1 [3] *For any edge $e \in E(\overline{G})$,*

$$\gamma_t(G) - 2 \leq \gamma_t(G+e) \leq \gamma_t(G).$$

Graphs G with the property $\gamma_t(G+e) = \gamma_t(G) - 2$ for any $e \in E(\overline{G})$ are called *supercritical* and are characterised in [4].

The following proposition, from [5], characterizes any pair of non-adjacent vertices in 4-critical graphs.

Proposition 2 *For any 4-critical graph G and non-adjacent vertices u and v, either*

1. *$\{u, v\} \succ G$, or*
2. *for either u or v, without loss of generality, say u, $\{w, u, v\} \succ G$, for some $w \in N(u)$ and $w \notin N(v)$, in which case we write $[uw, v] \succ G$, or*
3. *for either u or v, without loss of generality, say u, $\{x, y, u\} \succ G - v$, and x, y and u are connected. In this case we write $xyu \mapsto v$.*

In this paper we study 4-critical graphs with diamter three. The paper is organized as follows: In Section 2 we define a family $\mathcal{H}$ of 4-critical graphs with diameter three, and show that a graph in this family is an extension of the composition of two identical graphs. In Section 3 we define the concept of *sub-self-complementary*, and then give a necessary and sufficient condition for a graph H to be sub-self-complementary.

2 A Family of 4-Critical Graphs

Let $\mathcal{H}$ be the family of 4-critical graphs H with the properties that every $x \in V(H)$ is a diametrical vertex, and if y is a diametrical vertex for x, then the set $\{x, y\}$ dominates H. (We include here the possibility that the diameter of H is infinite.) It is clear that H has no cutvertex, and therefore no endvertex. As an example of a graph in $\mathcal{H}$, consider the cycle C_6.

Lemma 3 *If $H \in \mathcal{H}$, then $diam(H) = 3$ or H is composed of exactly two complete components, each with order at least two.*

Proof: Clearly, if $diam(H) = \infty$, then since H is 4-critical, H is composed of exactly two complete components, each with order at least two. Now suppose $diam(H) < \infty$. Let $u \in V(H)$, and let v be a diametrical vertex for u. If $diam(H) = 2$, then u and v have a common neighbor, say x, and then $\{u, x, v\} \succ_t H$, contradicting the fact that H is 4-critical. Now suppose that $diam(H) = 4$, again with u and v a diametrical pair. If $uxyzv$ is a shortest $u - v$ path, then y is not dominated by u or v, contradicting the fact that $\{u, v\}$ dominates H. Hence $diam(H) = 3$. □

Given a graph $G \in \mathcal{H}$, we can construct another graph in $\mathcal{H}$ in the following manner: for a given $x \in V(G)$, construct H by appending a vertex w to x, and adding edges between w and all the neighbors of x. $\mathcal{H}$ is closed under this construction, as shown in the following lemma.

Lemma 4 *Let $G \in \mathcal{H}$. Let H be such that $V(H) = V(G) \cup \{w\}$, and $E(H) = E(G) \cup \{wy : y \in N[x]$,* for some *$x \in V(G)\}$. Then $H \in \mathcal{H}$.*

Proof: By construction, $N[w] = N[x]$, so for any u, $v \in V(G)$, $d_G(u, v) = d_H(u, v)$, where $d_G(u, v)$ is the distance from u to v in G. Also, for any $u \in V(G)$, with $u \neq x$, $d_H(w, u) = d_H(x, u)$. Thus it follows that every vertex in H is a diametrical vertex, every diametrical pair in H dominates H, and $\gamma_t(H) = \gamma_t(G) = 4$.

Now let u and v be non-adjacent vertices in H. If $d_H(u, v) = \infty$, then $d_G(u, v) = \infty$. Since G is 4-critical, G is composed of two complete components, each of order at least two. Then by construction, H is also composed of two complete components, each of order at least two, and thus H is 4-critical. By Lemma 3, if G is connected, then $d_G(u, v) = d_H(u, v) \leq 3$. Now if $d_H(u, v) = 3$, then (u, v) is a diametrical pair, and thus $\{u, v\} \succ_t H + uv$. If $d_H(u, v) = 2$, then, if v' is a diametrical vertex to v, we must have $v' \perp u$ and so $\{v', u, v\} \succ_t H + uv$. Thus H is 4-critical. □

Let $Y \subset V(G)$, where $G \in \mathcal{H}$. If Y is a diametrical set for x, then for y_i and y_j in Y, both $\{y_i, x\} \succ G$ and $\{y_j, x\} \succ G$, hence $N[y_i] = N[y_j]$ (and in fact $\langle Y \rangle$ is complete). This implies that for every $v \in V(G)$, $d(v, y_i) = d(v, y_j)$, giving us the following lemma, stated without proof.

Lemma 5 *Let $G \in \mathcal{H}$, with $Y \subset V(G)$ and $X \subset V(G)$. Then Y is the diametrical set for every $x \in X$ if and only if X is the diametrical set for every $y \in Y$.*

Lemma 5 allows the partitioning of the vertices of G into pairs of diametrical sets (X^k, Y^k), $k = 1, \ldots, r$.

In [5] we show that 4-critical graphs have no forbidden subgraph characterization, i.e. any graph G can be used to construct a 4-critical graph $G^{\oplus}$. Let $\mathcal{F}$ be the family of graphs constructed as follows. Take two copies of $G \neq K_1$, label them G_1 and G_2, with corresponding vertices $u_1, u_2, \ldots, u_n \in G_1$ and $v_1, v_2, \ldots, v_n \in G_2$. For $i \neq j$, add edge $u_i v_j$ if and only if $u_i u_j \notin E(G_1)$. Call the resulting graph $G^{\oplus}$. See Figure 2.

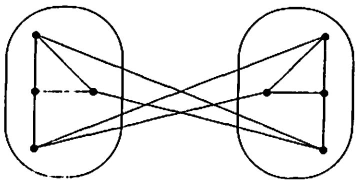

Figure 2: A 4-critical graph $G^{\oplus} \in \mathcal{F}$.

Theorem 6 (Van der Merwe and Loizeaux [5]) *$G^{\oplus}$ is 4-critical. In addition, if $G^{\oplus}$ is connected, then $diam(G^{\oplus}) = 3$.*

$G^{\oplus}$ is disconnected if and only if G is complete, or G is the union of two complete graphs. If $G = \overline{K_n}$, then $G^{\oplus}$ is $K_{n,n}$ minus a perfect matching. In particular, if $G = \overline{K_3}$, then $G^{\oplus}$ is C_6. In addition, it is easy to see that $(\overline{G})^{\oplus}$ is isomorphic to $\overline{(G^{\oplus})}$-p.m., the graph $\overline{(G^{\oplus})}$ minus a perfect matching, i.e. with edges between corresponding vertices (u_i, v_i) removed.

Theorem 7 *If $H \in \mathcal{F}$, then $H \in \mathcal{H}$.*

Proof: By Theorem 6, H is 4-critical. If H is connected, then again by Theorem 6, diam$(H) = 3$. In the construction of H, if x' is the copy of x, then $d(x, x') = 3$, so every vertex is a diametrical vertex. Now let x and y be a diametrical pair, and suppose $\{x, y\} \not\succ v$, for some $v \in H$. Then without loss of generality, x is adjacent to v' and y is adjacent to or coincident with v', and thus $d(x, y) \leq 2$, a contradiction. Thus every diametrical pair dominates H.

If H is not connected, then since H is 4-critical, H is composed of two complete components, each of order at least two. So $diam(H) = \infty$, every vertex is a diametrical vertex, and every diametrical pair dominates H. Thus $H \in \mathcal{H}$. □

Now consider a graph $H \in \mathcal{H}$. If H is connected, then for $x, y \in V(H)$ such that $N[x] = N[y]$, and $x \neq y$, the graph formed by removing y and all edges incident with y is also in $\mathcal{H}$. Then for each diametrical set $Y = \{y_1, y_2, \ldots, y_k\} \subset V(H)$, remove the vertices y_i, $i = 2, 3, \ldots, k$, and all incident edges to form the graph $H_r \in \mathcal{H}$. Since each diametrical set in H_r is a singleton, Lemma 5 implies that the order of H_r is even.

If $H \in \mathcal{H}$ is not connected, then the two complete components of H are two diametrical sets. In this case form the graph $H_r \in \mathcal{H}$ by removing all but two vertices, along with their incident edges, from each component.

In each case above, we call H_r the *reduction* of H in $\mathcal{H}$.

Theorem 8 *$H \in \mathcal{H}$ if and only if $H_r \in \mathcal{F}$.*

Proof: If $H_r \in \mathcal{F}$, Theorem 7 and Lemma 4 together imply that $H \in \mathcal{H}$.

Now suppose $H \in \mathcal{H}$, and form the graph H_r. If H is not connected, then H_r is the disjoint union of two K_2's, and thus $H_r \in \mathcal{F}$. Now suppose H is connected, and let $|V(H_r)| = 2n$. (Note that H connected implies that $n \geq 3$.) Without loss of generality, partition $V(H_r)$ into the sets $X = \{x_1, x_2, \ldots x_n\}$ and $Y = \{y_1, y_2, \ldots y_n\}$, such that x_i, y_i form a diametrical pair.

Now $x_i \perp x_j$ implies $x_i \not\perp y_j$, else $d(x_j, y_j) = 2$, a contradiction. This in turn implies $y_i \perp y_j$, since $\{x_i, y_i\} \succ H_r$. Furthermore, $x_i \not\perp x_j$ implies $x_i \perp y_j$ since $\{x_j, y_j\} \succ H_r$, and thus $y_i \not\perp y_j$, else $d(x_i, y_i) = 2$, again a contradiction.

Thus $\langle X \rangle$ is a copy of $\langle Y \rangle$, and for $i \neq j$, $x_i y_j \in H_r$ if and only if $x_i x_j \notin E(\langle X \rangle)$. But this is precisely the construction of a graph in $\mathcal{F}$, that is, $H_r = \langle X \rangle^{\oplus}$, and so we have $H_r \in \mathcal{F}$. □

If we let $\mathcal{H}_r = \{H_r : H \in \mathcal{H}\}$, then $\mathcal{H}_r \subset \mathcal{F} \subset \mathcal{H}$. Note that $\mathcal{H}_r$ is a proper subset of $\mathcal{F}$, since $P_4^{\oplus} \notin \mathcal{H}_r$. Also, by Lemma 4, $\mathcal{F}$ is a proper subset of $\mathcal{H}$.

3 Sub-self-complementary graphs

If the graphs G and H are isomorphic, we write $G \sim H$. The graph G is self-complementary if $\overline{G} \sim G$. In this section we introduce the concept of *sub-self-complementary* graphs, and show that there exists a sub-self-complementary graph of order n for every positive integer $n \not\equiv 3 (mod\ 4)$. We then give necessary and sufficient conditions for a graph G to be sub-self-complementary, showing that the sub-self-complementary graphs define a proper subset of the family $\mathcal{F}$.

We state the following lemma without proof.

Lemma 9 *If G is self-complementary then $G^{\oplus} \sim (\overline{G})^{\oplus}$.*

Let H and K be graphs, and let $H_E K$ be the graph formed by taking the disjoint union of H and K, and adding edges determined by the set E, where E is a subset of the set of edges $\{uv | u \in V(H), v \in V(K)\}$. For $G^{\oplus} \in \mathcal{F}$, let E be the set of edges between the two copies of G. Then $G^{\oplus} = G_E G$.

We say P' is a *weak partition* of S if $P' = P$ or $P' = P \cup \{\emptyset\}$, where P is a partition of S. For any graph G, let $\{S, T\}$ be a weak partition of the vertices of G, and let $E_{ST} = \{uv | u \in S, v \in T, uv \in E(G)\}$. Then $G = \langle S \rangle_{E_{ST}} \langle T \rangle$. We say that a graph G is *sub-self-complementary* if there is a weak partition $\{S, T\}$ of the vertices of G, such that $\langle S \rangle_{E_{ST}} \langle T \rangle \sim \overline{\langle S \rangle}_{E_{ST}} \overline{\langle T \rangle}$. Note that if G is self-complementary, then G is sub-self-complementary: take the weak partition $\{V(G), \emptyset\}$.

As an example of a sub-self-complementary graph which is not self-complementary, consider Figure 3. Here we see $G = (P_4)_{\emptyset}(K_1) \sim (\overline{P_4})_{\emptyset}(\overline{K_1})$, a graph on five vertices. As shown, we can also write $G = (K_1 \cup K_1)_E(P_3) \sim (\overline{K_1 \cup K_1})_E(\overline{P_3})$, where E consists of a single edge uv such that $u \in V(K_1 \cup K_1)$, and $v \in V(P_3)$ is an end vertex. Note that the isomorphisms on the left can be generalized as $G = H_{\emptyset} K$, where both H and K are self-complementary (but G may or may not be so).

Now consider a graph G which is sub-self-complementary, say $G = H_E K \sim \overline{H}_E \overline{K}$. If $h = |V(H)|$ and $k = |V(K)|$, then

$$|E(H)| + |E(\overline{H})| = \frac{h(h-1)}{2}, \text{ and } |E(K)| + |E(\overline{K})| = \frac{k(k-1)}{2}.$$

Now $H_E K = \overline{H}_E \overline{K}$ implies that

$$|E(H)| + |E(K)| = |E(\overline{H})| + |E(\overline{K})|,$$

and combining these two equations, we find that we must have

$$|E(H)| + |E(K)| = \frac{h(h-1) + k(k-1)}{4}.$$

The equation above constrains the possibilities for the number of vertices in H and K: we must have both h and k equivalent to 0 or 1(mod 4), or

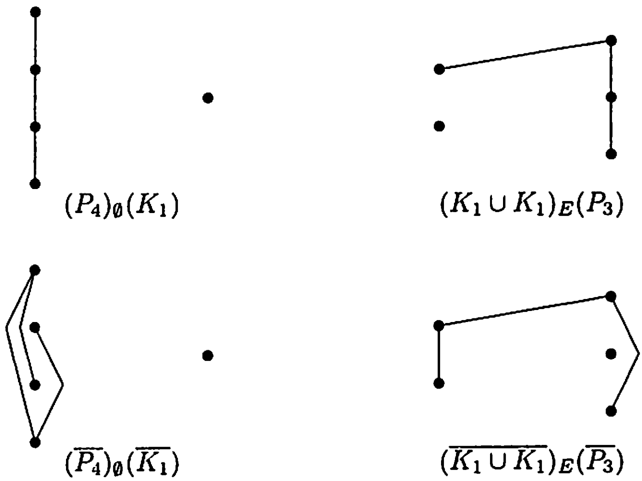

Figure 3: Four isomorphisms of a sub-self-complementary graph.

both h and k equivalent to 2 or $3(mod\ 4)$. This implies that $|V(H_E K)| = h + k \not\equiv 3(mod\ 4)$. On the left in Figure 3 we have $h = 4$, $k = 1$, and $|E(H)| + |E(K)| = 3$. On the right in Figure 3 we have $h = 2$, $k = 3$, and $|E(H)| + |E(K)| = 2$.

It is well known that there exist self-complementary (thus sub-self-complementary) graphs of order n for $n \equiv 0(mod\ 4)$ and $n \equiv 1(mod\ 4)$. (For a construction see Chartrand and Zhang [1].) For any even n, say $n = 2m$, $(G_m)_\emptyset(\overline{G_m})$ is a sub-self-complementary graph, where G_m is a graph on m vertices. Thus we state the following lemma:

Lemma 10 *There exists a graph of order n which is sub-self-complementary for every positive integer $n \not\equiv 3(mod\ 4)$.*

Let H and K be graphs, and let E be a set of edges between H and K. We take $\overline{E}$ to be the set of edges $\{uv : u \in V(H), v \in V(K), uv \notin E\}$.

Theorem 11 *If G is sub-self-complementary, then $G^\oplus \sim (\overline{G})^\oplus$.*

Proof: Suppose G is sub-self-complementary, say $G = (M_1)_P(N_1) \sim (\overline{M_1})_P(\overline{N_1})$. Then $\overline{G} = \overline{(\overline{M_1})_P(\overline{N_1})} = (M_1)_{\overline{P}}(N_1)$. If a copy of G is $G' = (M_2)_Q(N_2)$, with M_2, N_2, and Q copies of M_1, N_1 and P respectively, then

$$G^{\oplus} = G_E G' = \Big((M_1)_P(N_1)\Big)_E\Big((M_2)_Q(N_2)\Big),$$

and

$$(\overline{G})^{\oplus} = \overline{G}_{E'}\overline{G'} = \Big((M_1)_{\overline{P}}(N_1)\Big)_{E'}\Big((M_2)_{\overline{Q}}(N_2)\Big),$$

where E' is a copy of $\overline{E} - p.m.$ Let $\phi : G^{\oplus} \to (\overline{G})^{\oplus}$ be a mapping such that

$$\phi : M_1 \to M_1,$$

$$\phi : N_1 \to N_2,$$

$$\phi : M_2 \to M_2,$$

and

$$\phi : N_2 \to N_1$$

are isomorphisms. (See Figure 4.) Now let u and v be vertices in $G^{\oplus}$. If u and v are in M_1 and N_1 respectively, then $uv \in P$ if and only if $\phi(u)\phi(v) \in E'$. The case is similar for u and v in M_2 and N_2 respectively. If u and v are in M_1 and M_2 respectively, then $uv \in E$ if and only if $\phi(u)\phi(v) \in E'$. Again, the case is similar for u and v in N_1 and N_2 respectively. Finally, if $u \in M_1$ and $v \in N_2$, then $uv \in E$ if and only if $\phi(u)\phi(v) \in \overline{P}$, and if $u \in N_1$ and $v \in M_2$, then $uv \in E$ if and only if $\phi(u)\phi(v) \in \overline{Q}$. Thus $\phi : G^{\oplus} \to (\overline{G})^{\oplus}$ is an isomorphism. □

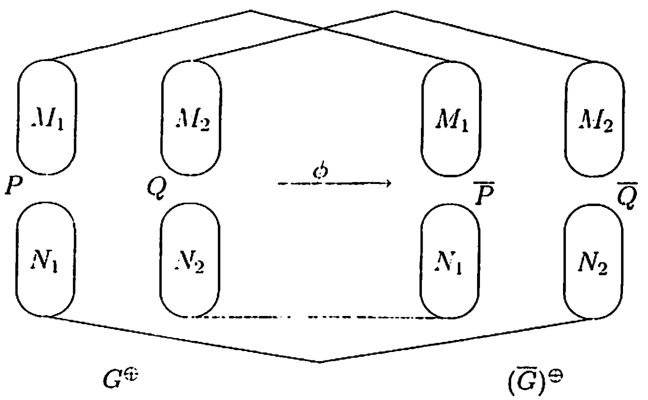

Figure 4: An isomorphism ϕ from $G^{\oplus}$ to $(\overline{G})^{\oplus}$, illustrating Theorem 11.

Theorem 11 shows that $G^{\oplus} \sim (\overline{G})^{\oplus}$ is a necessary condition for G to be sub-self-complementary. Theorem 13 below shows that this condition is also sufficient. Prior to this theorem, we show (Lemma 12) that an isomorphism from $G^{\oplus}$ to $(\overline{G})^{\oplus}$ can be assumed to map copies of vertices in $G^{\oplus}$ to copies of vertices in $(\overline{G})^{\oplus}$.

Lemma 12 *If $G^{\oplus} \sim (\overline{G})^{\oplus}$, then there is an isomorphism $\sigma : G^{\oplus} \to (\overline{G})^{\oplus}$ such that for each i, $\sigma(v_i)$ is the copy of $\sigma(u_i)$ in $(\overline{G})^{\oplus}$, where v_i is the copy of u_i in $G^{\oplus}$.*

Proof: Let $\phi : G^{\oplus} \to (\overline{G})^{\oplus}$ be an isomorphism. Let (X^k, Y^k), for $k = 1, \ldots, r$ be the pairs of diametrical sets in $G^{\oplus}$. Since $N[y_i] = N[y_j]$ for all $y_i, y_j \in Y^k$, it follows that $N[\phi(y_i)] = N[\phi(y_j)]$ for all $\phi(y_i), \phi(y_j) \in \phi(Y^k)$. Thus $\phi(G^{\oplus})$ is isomorphic to $\pi \circ \phi(G^{\oplus})$, where π allows a permutation of the vertices in $\phi(Y^k)$, for each k.

For each k, and for each $x_i \in X^k$, let $y_i \in Y^k$ be the copy of x_i. For $\phi(x_i) \in (\overline{G})^{\oplus}$, let $\pi(\phi(x_i)) = \phi(x_i)$. If $\phi(x_i)$ is the copy of $\phi(y_i)$, let $\pi(\phi(y_i)) = \phi(y_i)$. If $\phi(x_i)$ is the copy of $\phi(y_j)$, let $\pi(\phi(y_i)) = \phi(y_j)$. Then π is a permutation of the vertices in $\phi(Y^k)$, for each k, and so $\sigma = \pi \circ \phi$ is an isomorphism from $G^{\oplus}$ to $(\overline{G})^{\oplus}$, and $\sigma(v_i)$ is the copy of $\sigma(u_i)$ in $(\overline{G})^{\oplus}$, where v_i is the copy of u_i in $G^{\oplus}$. □

Theorem 13 *If $G^{\oplus} \sim (\overline{G})^{\oplus}$, then G is sub-self-complementary.*

Proof: Let $\phi : G^{\oplus} \to (\overline{G})^{\oplus}$ be an isomorphism. Let G_1 and G_2 be the two copies of G in $G^{\oplus}$, and let $\overline{G}_1$ and $\overline{G}_2$ be the two copies of $\overline{G}$ in $(\overline{G})^{\oplus}$. By Lemma 12, we can assume that $\phi(u_i)$ is the copy of $\phi(v_i)$ in $(\overline{G})^{\oplus}$ if and only if u_i is the copy of v_i in $G^{\oplus}$.

If $\phi(G_1) = \overline{G}_1$ then G is self-complementary, and we are done. So let $\phi(G_1) = (M_1)_Q(N_2)$, where M_1 is an induced subgraph of $\overline{G}_1$ and N_2 is an induced subgraph of $\overline{G}_2$.

Note that no vertex in N_2 is a copy of a vertex in M_1. It follows then that, with $N_1 = \overline{G}_1 - M_1$, $N_1 \sim N_2$. Let $y' \in \overline{G}_2$ be the copy of $y \in \overline{G}_1$, and let $P = \{xy : x \in M_1, y \in N_1, xy' \in Q\}$. Then $(M_1)_P(N_1) \sim (M_1)_Q(N_2)$. Now $M_1 = \overline{M}$ and $N_1 = \overline{N}$ for some induced subgraphs M and N of G_1, with $G_1 = M_R N$ for some edge set R. Also, $xy \in P$ if and only if $xy' \in Q$ if and only if $xy \notin \overline{G}_1$, which is the case if and only if $xy \in G_1$. Therefore P is an exact copy of R, and

$$M_R N = G = G_1 \sim (M_1)_P(N_1) \sim \overline{M}_P \overline{N}.$$

Thus G is sub-self-complementary. □

We have defined a sub-self-complementary graph G as a graph whose vertices can be partitioned into two sets, such that the graph G can be formed either from the graphs induced by these two vertex sets, or the complements of these induced graphs, together with a common edge set. We finish this paper with a generalization of this idea, which we hope will lead to additional research.

Let $\mathcal{P} = \{P_1, P_2, \ldots, P_m\}$ be a set of graphs. For $1 \leq i < j \leq m$, let E_{ij} be a subset of the set of edges $\{uv : u \in V(P_i), v \in V(P_j)\}$, and let $E = \cup_{i<j} E_{ij}$. Let $(\mathcal{P}, E)$ be the graph formed by taking the disjoint union of the graphs in $\mathcal{P}$, and adding edges determined by the edge set E.

We say that a graph G is *self-complementary of order* m, or $sc(m)$, if m is the smallest integer such that $G = (\mathcal{P}, E) \sim (\overline{\mathcal{P}}, E)$, for some set of graphs $\mathcal{P} = \{P_1, P_2, \ldots, P_m\}$ and corresponding edge set E, where $\overline{\mathcal{P}} = \{\overline{P_1}, \overline{P_2}, \ldots, \overline{P_m}\}$. Graphs which are self-complementary are $sc(1)$, and graphs which are sub-self-complementary, but not self-complementary, are $sc(2)$. Note that if G is $sc(m)$, then $m \leq |V(G)|$. For any integer r, the graphs of order r can be partitioned according to the value of their self-complementary order.

References

[1] G. Chartrand and P. Zhang, *Introduction to Graph Theory*, McGraw Hill, New York (2005).

[2] T. W. Haynes, S. T. Hedetniemi, and P. J. Slater, *Fundamentals of Domination in Graphs*, Marcel Dekker, Inc., New York (1998).

[3] T. W. Haynes, C. M. Mynhardt, and L. C. van der Merwe, Total domination edge critical graphs, *Utilitas Math.* 54 (1998) 229–240.

[4] T. W. Haynes, C. M. Mynhardt, and L. C. van der Merwe, Criticality index of total domination, *Congr. Numer.* 131 (1998) 67–73.

[5] L. C. van der Merwe, M. A. Loizeaux, 4-critical graphs with maximum diameter, *JCMCC*, 60 (2007) 65–80.

[6] L. C. van der Merwe, M. A. Loizeaux, Bounds on the order of 4-critical graphs with diameter two, *Utilitas Math.* (to appear).