k-Generalized Order-k Perrin Number Presentation by Matrix Method

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Abstract

In this paper, we give matrix representations of the k-generalized order-k Perrin Numbers and we obtain relationships between these sequences and matrix. In addition, we calculate the determinant of this matrix.

Key words: Perrin sequence, order-k Perrin number, Fibonacci sequence, order-k Fibonacci number.

1. Introduction

1.1. Perrin Numbers

Perrin sequences were analyzed by Edouard LUCAS in 1978 (American Journal of Mathematics, vol.1, page 230ff). In 1899, the same sequence was mentioned by R. Perrin (L'Intermediaire Des Mathematicians), whose name was given to the sequence[5].

Studies on this sequence have been done in two main fields. The first one is the conjecture of Lucas. If R_n is the n^{th} Perrin number then "$n \mid R(n) \Leftrightarrow n$ prime", *e.g.* $R(19)=209$ and $19|209$. Lucas conjectured that this is true for all values so that the Perrin sequence can be used as a test for non-primality; any number n that doesn't divide $R(n)$ is composite. But later it was found that there are composite integers n that divide $R(n)$. Such composite numbers are called Perrin pseudoprimes, the lowest being $n=521^2=271441$.

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The second is on Binet-like formula. The Perrin numbers can be written in terms of powers of the roots of the equation

$$x^3 - x - 1 = 0.$$

This equation has 3 roots; one real root p (known as the plastic number) and two complex conjugate roots q and r. Given these three roots, the Perrin sequence analogue of the Fibonacci sequence Binet formula is

$$R(n) = p^n + q^n + r^n.$$

Since the magnitudes of the complex roots q and r are both less than 1, the powers of these roots approach 0 for large n. For large n, the formula reduces to

$$R(n) \simeq p^n.$$

This formula can be used to quickly calculate values of the Perrin sequence for large n.

The ratio of successive terms in the Perrin sequence approaches p, which has a value of approximately 1.324718. This constant bears the same relationship to the Perrin sequence and the Padovan sequence as the golden ratio does to the Fibonacci sequence.[6].

1.2. Fibonacci Numbers

For $n \geq 2$, Fibonacci sequence is defined by

$$F_n = F_{n-1} + F_{n-2}$$

with initial conditions $F_0 = 0$, $F_1 = 1$. Some terms of Fibonacci sequence are

$$0, 1, 1, 2, 3, 5, 8, 13, 21, 34, \ldots.$$

For $A = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$, one can obtain Fibonacci sequence by using equality

$$A^n \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} F_n \\ F_{n+1} \end{bmatrix}.$$

Kalman[2] generalized the Fibonacci sequence as

$$F_{n+k} = c_0 F_n + c_1 F_{n+1} + \cdots + c_{k-1} F_{n+k-1}$$

where c_i $(i = 0, 1, ..., (k-1))$ are constants. For $k = 2$ and $c_0 = c_1 = 1$, the generalized sequence reduces to the ordinary Fibonacci sequence.

In addition, Kalman[2] obtained a closed form formula for the generalized sequence by matrix method.

Er[1] defined k sequences of the generalized order-k Fibonacci numbers as

$$g_n^i = \sum_{j=1}^{k} c_j g_{n-j}^i \quad \text{for } n > 0 \text{ and } 1 \leq i \leq k$$

with boundary conditions

$$g_n^i = \begin{cases} 1, & i = 1 - n \\ 0, & \text{otherwise} \end{cases} \quad \text{for } 1 - k \leq n \leq 0$$

where c_j, $1 \leq j \leq k$ are constant coefficients and g_n^i is the n^{th} term of the i^{th} sequence. When $k = 2$, the generalized order-k Fibonacci sequence reduces to the conventional Fibonacci sequence.

Furthermore, Er[1] defined companion matrix A like Kalman:

$$A = \begin{bmatrix} c_1 & c_2 & c_3 & \cdots & c_{k-1} & c_k \\ 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 0 \end{bmatrix}_{k \times k}$$

and got the equality

$$\begin{bmatrix} g_{n+1}^i \\ g_n^i \\ g_{n-1}^i \\ \vdots \\ g_{n-k+2}^i \end{bmatrix} = A \begin{bmatrix} g_n^i \\ g_{n-1}^i \\ g_{n-2}^i \\ \vdots \\ g_{n-k+1}^i \end{bmatrix}.$$

Then, the author defined the matrix

$$G_n = \begin{bmatrix} g_n^1 & g_n^2 & \cdots & g_n^k \\ g_{n-1}^1 & g_{n-1}^2 & \cdots & g_{n-1}^k \\ \vdots & \vdots & \ddots & \vdots \\ g_{n-k+1}^1 & g_{n-k+1}^2 & \cdots & g_{n-k+1}^k \end{bmatrix}$$

and showed

$$G_{n+1} = AG_n.$$

Also, Karaduman[3] demonstrated that $G_1 = A$ and $G_n = A^n$. Thus, he proved that

$$\det G_n = \begin{cases} (-1)^n, & k \text{ even} \\ 1, & k \text{ odd} \end{cases}.$$

2. Main Results

Similar to the definitions of Er[1] and Kılıç and Taşçı[4], we also can define k-sequences of the generalized order-k Perrin numbers.

Definition 2.1. k-sequences of the generalized order-k Perrin numbers for $n > 0$ and $1 \le i \le k$ are

$$R_n^i = \sum_{j=1}^{k-1} R_{n-j-1}^i = R_{n-2}^i + R_{n-3}^i + \cdots + R_{n-k-2}^i = \sum_{j=1}^{k} R_{n-j}^i - R_{n-1}^i.$$

For example, for $k = 5$ we have,

$$R_n^i = \sum_{j=1}^{4} R_{n-j-1}^i = R_{n-2}^i + R_{n-3}^i + R_{n-4}^i + R_{n-5}^i.$$

Initial conditions of these sequences for $n \le 0$ are

$$R_n^i = \begin{cases} 1, & k+n+i=2 \\ (-1), & k+n+i=3 \\ 3, & n+i=1 \\ 2, & n+i=3 \\ 0, & \text{otherwise} \end{cases}$$

where R_n^i is the n^{th} term of the i^{th} sequence. For $i = 1$ and $k = 3$ the generalized sequence reduces to the ordinary Perrin sequence.

Let us obtain terms of sequence by matrix multiplications. Define $k \times k$ square matrix A such that

$$A = \begin{bmatrix} 0 & 1 & \cdots & 1 & 1 \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \end{bmatrix}. \tag{1}$$

So, we have

$$\begin{bmatrix} R_{n+1}^i \\ R_n^i \\ \vdots \\ R_{n-k+2}^i \end{bmatrix} = \begin{bmatrix} 0 & 1 & \cdots & 1 & 1 \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \end{bmatrix} \begin{bmatrix} R_n^i \\ R_{n-1}^i \\ \vdots \\ R_{n-k+1}^i \end{bmatrix} \tag{2}$$

It is possible to find any term of the sequence, if any successive k terms are given.

To deal with the $k-$sequences of the generalized order-k Perrin sequences simultaneously, we define $k \times k$ square matrix R_n as

$$R_n = \begin{bmatrix} R_n^1 & R_n^2 & \cdots & R_n^k \\ R_{n-1}^1 & R_{n-1}^2 & \cdots & R_{n-1}^k \\ \vdots & \vdots & \ddots & \vdots \\ R_{n-k+1}^1 & R_{n-k+1}^2 & \cdots & R_{n-k+1}^k \end{bmatrix}. \tag{3}$$

Generalizing equation (2) we get the following theorem:

Theorem 1. Let R_n be the matrix of the form (3). Then, we have

$$R_{n+1} = AR_n. \tag{4}$$

Proof.

$$AR_n = \begin{bmatrix} 0 & 1 & \cdots & 1 \\ 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix} \begin{bmatrix} R_n^1 & R_n^2 & \cdots & R_n^k \\ R_{n-1}^1 & R_{n-1}^2 & \cdots & R_{n-1}^k \\ \vdots & \vdots & \ddots & \vdots \\ R_{n-k+1}^1 & R_{n-k+1}^2 & \cdots & R_{n-k+1}^k \end{bmatrix}$$

$$= \begin{bmatrix} R_{n-1}^1 + \cdots + R_{n-k+1}^1 & R_{n-1}^2 + \cdots + R_{n-k+1}^2 & \cdots & R_{n-1}^k + \cdots + R_{n-k+1}^k \\ R_n^1 & R_n^2 & \cdots & R_n^k \\ \vdots & \vdots & \ddots & \vdots \\ R_{n-k+2}^1 & R_{n-k+2}^2 & \cdots & R_{n-k+2}^k \end{bmatrix}$$

$$= \begin{bmatrix} R_{n+1}^1 & R_{n+1}^2 & \cdots & R_{n+1}^k \\ R_n^1 & R_n^2 & \cdots & R_n^k \\ \vdots & \vdots & \ddots & \vdots \\ R_{n-k+2}^1 & R_{n-k+2}^2 & \cdots & R_{n-k+2}^k \end{bmatrix} = R_{n+1}.$$

Lemma 2. Let K be a $k \times k$ square matrix as

$$K = \begin{bmatrix} x & 0 & y & \cdots & 0 & 0 \\ 0 & x & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & y \\ -1 & 0 & 0 & \cdots & x & 0 \\ 1 & -1 & 0 & \cdots & 0 & x \end{bmatrix}. \quad (5)$$

Then,

$$detK = \begin{cases} y^{k-2} + x^k + (-1)^p 2x^p y^{p-1}, & k, \text{ even} \\ y^{k-2} + x^k + (-1)^p x^p y^p, & k, \text{ odd} \end{cases}$$

where $p = \lfloor \frac{k}{2} \rfloor$ ($\lfloor a \rfloor$ the largest integer not larger than a).

∎

Proof. Let us expand the determinant with respect to the first column,

$$\begin{aligned} \det K &= x.(-1)^2.\,|B| + (-1).(-1)^k.\,|C| + 1.(-1)^{k+1}\,|D| \\ &= x.\,|B| + (-1)^{k+1}\,|C| + (-1)^{k+1}\,|D|. \end{aligned}$$

Then, by writing determinants of $(k-1) \times (k-1)$ matrices B, C and D for even and odd integers, we obtain,

$$|B| = \begin{vmatrix} x & 0 & y & \cdots & 0 & 0 \\ 0 & x & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & y \\ 0 & 0 & 0 & \cdots & x & 0 \\ -1 & 0 & 0 & \cdots & 0 & x \end{vmatrix} = \begin{cases} x^{k-1} + (-1)^p x^{p-1} y^{p-1}, & k, \text{ even} \\ x^{k-1}, & k, \text{ odd} \end{cases}$$

$$|C| = \begin{vmatrix} 0 & y & \cdots & 0 & 0 \\ x & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & y \\ -1 & 0 & \cdots & 0 & x \end{vmatrix} = \begin{cases} -y^{k-2} + (-1)^{p-1} x^p y^{p-1}, & k, \text{ even} \\ y^{k-2}, & k, \text{ odd} \end{cases}$$

$$|D| = \begin{vmatrix} 0 & y & \cdots & 0 & 0 \\ x & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & y \\ 0 & 0 & \cdots & x & 0 \end{vmatrix} = \begin{cases} 0, & k, \text{ even} \\ (-1)^p x^p y^p, & k, \text{ odd} \end{cases}$$

and by substituting $|B|$, $|C|$ and $|D|$ we get the result. ∎

Theorem 3. For $p = \lfloor \frac{k}{2} \rfloor$,

$$\det R_n = \begin{cases} (-1)^n \left[2^{k-2} + 3^k + (-1)^p 6^p\right], & k, \text{ even} \\ \left[2^{k-2} + 3^k + (-1)^p 6^p\right] & k, \text{ odd} \end{cases}.$$

Proof. First, we calculate the determinant of the matrix A (1), by expanding the determinant with respect to the last column,

$$\det A = (-1)^{k+1} \begin{vmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{vmatrix}_{(k-1)\times(k-1)} = (-1)^{k+1}.$$

Now,

$$\begin{aligned} \det R_n &= \det(A^n R_0) \\ &= \det(A^n)\det(R_0) \\ &= (\det A)^n \det(R_0) \\ &= ((-1)^{k+1})^n \det(R_0). \end{aligned}$$

To calculate R_0, substitute $x = 3$ and $y = 2$ in (5) and get the result

$$\begin{aligned} \det R_n &= \begin{cases} ((-1)^{k+1})^n \left[y^{k-2} + x^k + (-1)^p 2x^p y^{p-1}\right], & k, \text{ even} \\ ((-1)^{k+1})^n \left[y^{k-2} + x^k + (-1)^p x^p y^p\right], & k, \text{ odd} \end{cases} \\ &= \begin{cases} (-1)^n \left[2^{k-2} + 3^k + (-1)^p 6^p\right], & k, \text{ even} \\ \left[2^{k-2} + 3^k + (-1)^p 6^p\right] & k, \text{ odd} \end{cases}. \end{aligned}$$

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