The number of rooted simple bipartite maps on the sphere *

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Abstract

This paper investigates the number of rooted simple bipartite maps on the sphere and presents some formulae for such maps with the number of edges and the valency of the root-face as two parameters.

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1. Introduction

The concept of rooted map was first introduced by Tutte. His series of census papers [23–26] laid the foundation for the theory. Since then, the theory has been developed by many scholars such as Arquès [1], Brown [7,8], Mullin et al. [21], Tutte [27], Bender et al. [2–6], Liskovets et al. [13,14], Gao [9,10] and Liu [15–20]. A good survey of results in this area can be found in [12]. In particular, bipartite maps, both rooted and unrooted, were enumerated in [14]. The maps investigated there and in most of the above-cited articles were allowed to have loops and/or multiple edges. In this article we treat rooted bipartite maps that are *simple* – that is, they have neither loops nor multiple edges.

Although much work has been done on enumerating rooted planar maps, many classes of simple maps are still untreated. In 1983, Liu [18] investigated for the first time the enumeration of general simple planar maps with the number of edges of the maps as one parameter and an explicit formula was obtained, and then Liu also discussed the enumeration of rooted general simple planar maps with the valency of root-vertex and the number of edges as parameters [19] and with the partition of face-degrees as parame-

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ters [20]. Some functional equations were provided. In 2002, Ren and Liu [22] investigated the enumeration of simple bipartite maps on the sphere and the projective plane according to the root-face valencies and the number of edges of the maps. An explicit formula with the number of edges as one parameter was given. But it was very complicated and a formula for the number of simple bipartite maps on the sphere with the root-face valency and the number of edges as parameters could not be obtained at that time. In addition, the result on rooted planar unicycled simple bipartite maps is in error.

In this paper, on the basis of what was obtained in [22] we obtain the parametric expressions of the functional equations as shown by Theorem A and Corollary 2.2 in [22]. By employing Lagrangian inversion the solutions are found. Further, formulae for the numbers of rooted simple bipartite maps on the sphere and rooted planar unicycled simple bipartite maps with the root-face valency and the number of edges as two parameters are obtained.

Now, we define some basic concepts and terms. A *map* is a connected graph cellularly embedded on a surface. A map is *rooted* if an edge and a direction along that edge are distinguished. If the root is the oriented edge from u to v, then u is the *root-vertex* while the face on the right side of the edge as seen by an observer on the edge facing away from u is defined as the *root-face*. In this paper, maps are always rooted and planar.

For convenience, we introduce the following generating function for the set $\mathcal{M}$ of maps:

$$f_{\mathcal{M}}(x, y, z) = \sum_{M \in \mathcal{M}} x^{m(M)} y^{s(M)} z^{n(M)},$$

where $m(M)$, $s(M)$ and $n(M)$ denote the root-face valency, the number of edges and the number of inner faces of M, respectively. In addition, we write that

$$g_{\mathcal{M}}(x, y) = f_{\mathcal{M}}(x, y, 1),$$
$$h_{\mathcal{M}}(y) = g_{\mathcal{M}}(1, y) = f_{\mathcal{M}}(1, y, 1).$$

For the power series $f(x)$, $f(x, y)$ and $f(x, y, z)$, we employ the following notations:

$$\partial_x^m f(x), \quad \partial_{(x,y)}^{(m,s)} f(x, y) \quad \textit{and} \quad \partial_{(x,y,z)}^{(m,s,n)} f(x, y, z)$$

to represent the coefficients of x^m in $f(x)$, $x^m y^s$ in $f(x, y)$ and $x^m y^s z^n$ in $f(x, y, z)$, respectively. Terminologies and notations not explained here can be found in [15].

2. Functional equations

Let $\mathscr{R}$ denote the set of rooted simple bipartite maps on the sphere and

$$f_{\mathscr{R}}(x, y, z) = \sum_{M \in \mathscr{R}} x^{m(M)} y^{s(M)} z^{n(M)} \tag{1}$$

be its enumerating function. The enumerating function $f = f_{\mathscr{A}}(x, y, z)$ satisfies the following functional equation as shown by Theorem A in [22]:

$$x^2yf^2 - \left(\frac{x^2yz}{1-x^2} + z(f^* - 1) + 1\right)f + \left(\frac{x^2yz}{1-x^2} + z\right)f^* + 1 - z = 0, \quad (2)$$

where $f^* = f(1, y, z)$.

If $f(x, y, z)$ is rewritten as

$$f(x, y, z) = \sum_{k\geq 0} \alpha_k z^k, \quad (3)$$

then α_k is the enumerating function of rooted planar simple bipartite maps with $k+1$ faces.

The enumerating function of rooted trees satisfies the following functional equation as shown by Corollary 2.1 in [22]:

$$x^2y\alpha_0^2 - \alpha_0 + 1 = 0. \quad (4)$$

If the coefficients of z is considered, then the enumerating function of rooted planar unicycled simple bipartite maps satisfies the following functional equation as shown by Corollary 2.2 in [22]:

$$(2x^2y\alpha_0 - 1)\alpha_1 + \frac{x^2y(\alpha_0(1, y) - \alpha_0)}{1-x^2} - (\alpha_0 - 1)(\alpha_0(1, y) - 1) = 0. \quad (5)$$

Let $z = 1$ in (2); then we have

Lemma 1. The enumerating function $g = g_{\mathscr{A}}(x, y)$ satisfies the following functional equation:

$$x^2(1-x^2)yg^2 - [x^2y + (1-x^2)h]g + (1 - x^2 + x^2y)h = 0, \quad (6)$$

where $h = g(1, y)$.

3. Enumeration

In this section we will find the explicit formulae for enumerating functions $g_{\mathscr{A}}(x, y)$, $h_{\mathscr{A}}(y)$ and $\alpha_1(x, y)$ by using Lagrangian inversion.

The discriminant of equation (6) is

$$\delta(x, y) = [x^2y + (1-x^2)h]^2 - 4x^2y(1-x^2)(1-x^2+x^2y)h. \quad (7)$$

Now, if we rewrite the discriminant in the form

$$\delta(x, y) = (h - ax^2)^2(1 - 2bx^2), \quad (8)$$

then by (7) and (8), we have

$$a + bh = y + h,$$
$$a^2 + 4abh = (y+h)^2 + 4y(1-y)h,$$
$$a^2 b = 2y(1-y)h. \tag{9}$$

Let

$$a = \frac{1-\lambda-\lambda^2}{(1+\lambda)(1-\lambda)^2}, \quad b = \frac{2\lambda}{(1+\lambda)^2}. \tag{10}$$

By (9) and (10), one may find that

$$y = \frac{\lambda(1-\lambda-\lambda^2)}{(1+\lambda)^2(1-\lambda)}, \quad h = \frac{1-\lambda-\lambda^2}{(1-\lambda)^2}. \tag{11}$$

Further, let $\lambda = \frac{\eta}{1+\eta}$. By (11) we have the parametric expression of the function $h = h_{\mathscr{R}}(y)$ as follows:

$$y = \frac{\eta(1+\eta-\eta^2)}{(1+2\eta)^2}, \quad h = 1+\eta-\eta^2. \tag{12}$$

Theorem 1. The enumerating function $h = h_{\mathscr{R}}(y)$ has the following explicit expression:

$$\begin{aligned} h_{\mathscr{R}}(y) =& 1 + \sum_{s\geq 1}\sum_{i=0}^{s-1}\sum_{j=0}^{\lfloor\frac{s-i-1}{2}\rfloor} \frac{2i+5j-2s+4}{s(j+2)}\binom{2s}{i} \\ & \times \binom{s+j-1}{j}\binom{s-i-j}{s-i-2j-1} y^s. \end{aligned} \tag{13}$$

Proof. Applying Lagrangian inversion with one parameter [28] to (12), we obtain

$$\begin{aligned} h_{\mathscr{R}}(y) =& 1 + \sum_{s\geq 1} \frac{y^s}{s!}\frac{d^{s-1}}{d\eta^{s-1}} \frac{(1+2\eta)^{2s}(1-2\eta)}{(1+\eta-\eta^2)^s}\Bigg|_{\eta=0} \\ =& 1 + \sum_{s\geq 1} \frac{y^s}{s!}\frac{d^{s-1}}{d\eta^{s-1}} \frac{(1+\eta)^s(1+\frac{\eta}{1+\eta})^{2s}(1-2\eta)}{(1-\frac{\eta^2}{1+\eta})^s}\Bigg|_{\eta=0} \\ =& 1 + \sum_{s\geq 1}\sum_{i=0}^{s-1}\sum_{j=0}^{\lfloor\frac{s-i-1}{2}\rfloor} \frac{1}{s}\binom{2s}{i}\binom{s+j-1}{j} \\ & \times \left[\binom{s-i-j}{s-i-2j-1} - 2\binom{s-i-j}{s-i-2j-2}\right] y^s \end{aligned}$$

$$=1+\sum_{s\geq 1}\sum_{i=0}^{s-1}\sum_{j=0}^{\lfloor\frac{s-i-1}{2}\rfloor}\frac{2i+5j-2s+4}{s(j+2)}\binom{2s}{i}$$
$$\times\binom{s+j-1}{j}\binom{s-i-j}{s-i-2j-1}y^s.$$

This completes the proof of Theorem 1. □

By Eq. (6) we have

$$g=\frac{x^2y+(1-x^2)h-\sqrt{\delta(x,y)}}{2x^2(1-x^2)y}. \tag{14}$$

Now, let

$$x^2=\theta(1-\theta\lambda)(1+\lambda)^2. \tag{15}$$

By (7–11), (14) and (15), one may find that

$$(1-x^2)g=1-\frac{\theta}{1-\lambda}+\frac{\theta^2\lambda^2}{(1-\lambda)(1-\theta\lambda)}. \tag{16}$$

By the first part of (11), (15) and (16), we have the following parametric expression of $g=g_{\mathscr{R}}(x,y)$:

$$\begin{aligned} x^2 &=\theta(1-\theta\lambda)(1+\lambda)^2, \quad y=\frac{\lambda(1-\lambda-\lambda^2)}{(1+\lambda)^2(1-\lambda)},\\ (1-x^2)g &=1-\frac{\theta}{1-\lambda}+\frac{\theta^2\lambda^2}{(1-\lambda)(1-\theta\lambda)} \end{aligned} \tag{17}$$

and from which we get

$$\Delta_{(\theta,\lambda)}=\begin{vmatrix}\frac{1-2\theta\lambda}{1-\theta\lambda} & * \\ 0 & \frac{1-3\lambda}{(1-\lambda^2)(1-\lambda-\lambda^2)}\end{vmatrix}=\frac{(1-2\theta\lambda)(1-3\lambda)}{(1-\theta\lambda)(1-\lambda^2)(1-\lambda-\lambda^2)}. \tag{18}$$

Theorem 2. The enumerating function $g=g_{\mathscr{R}}(x,y)$ has the following explicit expression:

$$g_{\mathscr{R}}(x,y)=1+x^2y+\sum_{m,s\geq 2}g_{m,n}x^{2m}y^s, \tag{19}$$

where

$$g_{m,n}=\sum_{i=0}^{\lfloor\frac{s}{2}\rfloor}\sum_{j=0}^{s-2i}\left[\frac{4i+2j-s}{s}\binom{2s}{s-2i-j}\binom{i+j+1}{j}\right.$$

$$+\sum_{k=0}^{m-1}\frac{(2m-2k-2)!(2s-2m+2k-1)!}{(m-k-1)!(m-k+1)!(l+1)!}$$

$$\times\frac{R_{i,j}}{(2s-2m-l+2k)!}\binom{i+j+2}{j}\Bigg]\binom{s+i}{i}, \tag{20}$$

in which

$$\begin{aligned}R_{i,j}=&3(2s-2m-4l+2k)(m-k-1)(l+1)\\&-(2s-2m-4l+2k-4)(m-k+1)(2s-2m-l+2k),\\ l=&s-2i-j-m+k.\end{aligned} \tag{21}$$

Proof. By employing Lagrangian inversion with two parameters [11], from (17) and (18) one may find that

$$\begin{aligned}g_{\mathscr{R}}(x,y)=&\sum_{m,s\geq 0}\partial^{(m,s)}_{(\theta,\lambda)}\frac{(1+\lambda)^{2s-2m-1}(1-\lambda)^{s-1}(1-2\theta\lambda)(1-3\lambda)}{(1-\theta\lambda)^{m+1}(1-\lambda-\lambda^2)^{s+1}}\\&\times\left[1-\frac{\theta}{1-\lambda}+\frac{\theta^2\lambda^2}{(1-\lambda)(1-\theta\lambda)}\right]\frac{x^{2m}y^s}{1-x^2}\\=&\sum_{m,s\geq 0}\sum_{k=0}^{m}\partial^{(m-k,s)}_{(\theta,\lambda)}\frac{(1+\lambda)^{2s-2m+2k-1}(1-\lambda)^{s-1}}{(1-\theta\lambda)^{m-k+1}}\\&\times\frac{(1-2\theta\lambda)(1-3\lambda)}{(1-\lambda-\lambda^2)^{s+1}}\left[1-\frac{\theta}{1-\lambda}+\frac{\theta^2\lambda^2}{(1-\lambda)(1-\theta\lambda)}\right]x^{2m}y^s\\=&1+x^2y+\sum_{m,s\geq 2}\sum_{k=0}^{m}\partial^{(m-k,s)}_{(\theta,\lambda)}\frac{(1+\lambda)^{2s-2m+2k-1}}{(1-\theta\lambda)^{m-k+1}(1-\lambda)^2}\\&\times\frac{(1-2\theta\lambda)(1-3\lambda)}{(1-\frac{\lambda^2}{1-\lambda})^{s+1}}\left[1-\frac{\theta}{1-\lambda}+\frac{\theta^2\lambda^2}{(1-\lambda)(1-\theta\lambda)}\right]x^{2m}y^s\\=&1+x^2y+\sum_{m,s\geq 2}\sum_{k=0}^{m}\sum_{i=0}^{\lfloor\frac{s}{2}\rfloor}\binom{s+i}{i}\partial^{(m-k,s-2i)}_{(\theta,\lambda)}\frac{(1+\lambda)^{2s-2m+2k-1}}{(1-\theta\lambda)^{m-k+1}}\\&\times\frac{(1-2\theta\lambda)(1-3\lambda)}{(1-\lambda)^{i+2}}\left[1-\frac{\theta}{1-\lambda}+\frac{\theta^2\lambda^2}{(1-\lambda)(1-\theta\lambda)}\right]x^{2m}y^s\\=&1+x^2y+\sum_{m,s\geq 2}\sum_{k=0}^{m}\sum_{i=0}^{\lfloor\frac{s}{2}\rfloor}\binom{s+i}{i}\end{aligned}$$

$$\times \left[\partial_{(\theta,\lambda)}^{(m-k,s-2i)} \frac{(1+\lambda)^{2s-2m+2k-1}(1-2\theta\lambda)(1-3\lambda)}{(1-\theta\lambda)^{m-k+1}(1-\lambda)^{i+2}}\right.$$

$$- \partial_{(\theta,\lambda)}^{(m-k-1,s-2i)} \frac{(1+\lambda)^{2s-2m+2k-1}(1-2\theta\lambda)(1-3\lambda)}{(1-\theta\lambda)^{m-k+1}(1-\lambda)^{i+3}}$$

$$\left.+ \partial_{(\theta,\lambda)}^{(m-k-2,s-2i-2)} \frac{(1+\lambda)^{2s-2m+2k-1}(1-2\theta\lambda)(1-3\lambda)}{(1-\theta\lambda)^{m-k+2}(1-\lambda)^{i+3}}\right] x^{2m}y^s$$

$$=1 + x^2y + \sum_{m,s\geq 2} \sum_{i=0}^{\lfloor \frac{s}{2} \rfloor} \binom{s+i}{i} \left[\partial_\lambda^{s-2i} \frac{(1+\lambda)^{2s-1}(1-3\lambda)}{(1-\lambda)^{i+2}}\right.$$

$$- \sum_{k=0}^{m-1} \frac{(2m-2k-2)!}{(m-k-1)!(m-k)!} \partial_\lambda^{s-2i-m+k+1} \frac{(1+\lambda)^{2s-2m+2k-1}(1-3\lambda)}{(1-\lambda)^{i+3}}$$

$$+ \sum_{k=0}^{m-2} \frac{3(2m-2k-2)!}{(m-k-2)!(m-k+1)!}$$

$$\left.\times\, \partial_\lambda^{s-2i-m+k} \frac{(1+\lambda)^{2s-2m+2k-1}(1-3\lambda)}{(1-\lambda)^{i+3}}\right] x^{2m}y^s$$

$$=1 + x^2y + \sum_{m,s\geq 2} \sum_{i=0}^{\lfloor \frac{s}{2} \rfloor} \sum_{j=0}^{s-2i} \binom{s+i}{i} \left\{ \binom{i+j+1}{j} \partial_\lambda^{s-2i-j} (1+\lambda)^{2s-1}(1-3\lambda)\right.$$

$$- \sum_{k=0}^{m-1} \frac{(2m-2k-2)!}{(m-k-1)!(m-k)!} \binom{i+j+2}{j} \partial_\lambda^{l+1} (1+\lambda)^{2s-2m+2k-1}(1-3\lambda)$$

$$+ \sum_{k=0}^{m-2} \frac{3(2m-2k-2)!}{(m-k-2)!(m-k+1)!} \binom{i+j+2}{j}$$

$$\left.\times\, \partial_\lambda^{l} (1+\lambda)^{2s-2m+2k-1}(1-3\lambda)\right\} x^{2m}y^s,$$

where $l = s - 2i - j - m + k$. So

$$g_{\mathscr{A}}(x,y) = 1 + x^2y + \sum_{m,s\geq 2} \sum_{i=0}^{\lfloor \frac{s}{2} \rfloor} \sum_{j=0}^{s-2i} \left[\frac{4i+2j-s}{s} \binom{2s}{s-2i-j} \binom{i+j+1}{j}\right.$$

$$+ \sum_{k=0}^{m-1} \frac{(2m-2k-2)!(2s-2m+2k-1)! R_{i,j}}{(m-k-1)!(m-k+1)!(l+1)!(2s-2m-l+2k)!}$$

$$\left.\times \binom{i+j+2}{j}\right] \binom{s+i}{i} x^{2m}y^s,$$

in which

$$\begin{aligned} R_{i,j} =& 3(2s-2m-4l+2k)(m-k-1)(l+1) \\ & -(2s-2m-4l+2k-4)(m-k+1)(2s-2m-l+2k). \end{aligned} \tag{22}$$

This completes the proof of Theorem 2. □

By (4) we have the parametric expression of the function $\alpha_0 = \alpha_0(x, y)$ as follows:

$$x^2y = \eta(1-\eta), \quad \alpha_0(x,y) = \frac{1}{1-\eta}. \tag{23}$$

By using Lagrangian inversion with one variable [28], from (23) one may find that the enumerating function $\alpha_0 = \alpha_0(x, y)$ has the following explicit expression:

$$\alpha_0(x,y) = \sum_{s\geq 0} \frac{(2s)!}{s!(s+1)!} x^{2s} y^s, \tag{24}$$

which is just as same as that in [27].

By (5) we have

$$\alpha_1 = \frac{1}{1-2x^2y\alpha_0}\left[\frac{x^2y(\alpha_0(1,y)-\alpha_0)}{1-x^2} - (\alpha_0-1)(\alpha_0(1,y)-1)\right]. \tag{25}$$

Let

$$\begin{aligned} P(x,y) =& \frac{1}{1-2x^2y\alpha_0}, \\ Q(x,y) =& \frac{x^2y(\alpha_0(1,y)-\alpha_0)}{1-x^2} - (\alpha_0-1)(\alpha_0(1,y)-1). \end{aligned} \tag{26}$$

By (23) and the first part of (26), we have the following parametric expression of $P = P(x, y)$:

$$x^2y = \eta(1-\eta), \quad P = \frac{1}{1-2\eta}. \tag{27}$$

Applying Lagrangian inversion with one parameter [28] to (27), we obtain

$$\begin{aligned} P(x,y) &= 1 + \sum_{m\geq 1} \frac{2(x^2y)^m}{m!} \frac{d^{m-1}}{d\eta^{m-1}}\left[(1-\eta)^{-m}(1-2\eta)^{-2}\right]\Big|_{\eta=0} \\ &= 1 + \sum_{m\geq 1}\sum_{j=0}^{m-1} \frac{2^{j+1}(j+1)(2m-j-2)!}{m!(m-j-1)!} x^{2m} y^m. \end{aligned} \tag{28}$$

Now, combining (24) with the second part of (26), one may find that

$$Q(x,y) = \sum_{s\geq 2}\sum_{k=2}^{s} \frac{(2s)!}{s!(s+1)!} x^{2k} y^{s+1} - \sum_{s\geq 3}\sum_{k=2}^{s-1} \frac{(2k)!(2s-2k)!}{k!(k+1)!(s-k)!(s-k+1)!} x^{2k} y^{s}. \qquad (29)$$

Theorem 3. The enumerating function $\alpha_1 = \alpha_1(x,y)$ has the following explicit expression:

$$\begin{aligned}\alpha_1(x,y) =& \sum_{s\geq 2}\sum_{k=2}^{s} \frac{(2s)!}{s!(s+1)!} x^{2k} y^{s+1} \\ &+ \sum_{\substack{m\geq 1\\ s\geq 2}}\sum_{j=0}^{m-1}\sum_{k=2}^{s} \frac{2^{j+1}(j+1)(2m-j-2)!(2s)!}{m!(m-j-1)!s!(s+1)!} x^{2(m+k)} y^{m+s+1} \\ &- \sum_{s\geq 3}\sum_{k=2}^{s-1} \frac{(2k)!(2s-2k)!}{k!(k+1)!(s-k)!(s-k+1)!} x^{2k} y^{s} \\ &- \sum_{\substack{m\geq 1\\ s\geq 3}}\sum_{j=0}^{m-1}\sum_{k=2}^{s-1} \frac{2^{j+1}(j+1)(2m-j-2)!}{m!(m-j-1)!k!(k+1)!} \\ &\times \frac{(2k)!(2s-2k)!}{(s-k)!(s-k+1)!} x^{2(m+k)} y^{m+s}. \end{aligned} \qquad (30)$$

Proof. By (25), (26), (28) and (29), we have

$$\begin{aligned}\alpha_1(x,y) =& \left[1 + \sum_{m\geq 1}\sum_{j=0}^{m-1} \frac{2^{j+1}(j+1)(2m-j-2)!}{m!(m-j-1)!} x^{2m} y^{m}\right] \\ &\times \left[\sum_{s\geq 2}\sum_{k=2}^{s} \frac{(2s)!}{s!(s+1)!} x^{2k} y^{s+1}\right. \\ &\left. - \sum_{s\geq 3}\sum_{k=2}^{s-1} \frac{(2k)!(2s-2k)!}{k!(k+1)!(s-k)!(s-k+1)!} x^{2k} y^{s}\right] \\ =& \sum_{s\geq 2}\sum_{k=2}^{s} \frac{(2s)!}{s!(s+1)!} x^{2k} y^{s+1} \\ &+ \sum_{\substack{m\geq 1\\ s\geq 2}}\sum_{j=0}^{m-1}\sum_{k=2}^{s} \frac{2^{j+1}(j+1)(2m-j-2)!(2s)!}{m!(m-j-1)!s!(s+1)!} x^{2(m+k)} y^{m+s+1} \end{aligned}$$

$$-\sum_{s\geq 3}\sum_{k=2}^{s-1}\frac{(2k)!(2s-2k)!}{k!(k+1)!(s-k)!(s-k+1)!}x^{2k}y^{s}$$

$$-\sum_{\substack{m\geq 1\\ s\geq 3}}\sum_{j=0}^{m-1}\sum_{k=2}^{s-1}\frac{2^{j+1}(j+1)(2m-j-2)!(2k)!(2s-2k)!}{m!(m-j-1)!k!(k+1)!(s-k)!(s-k+1)!}x^{2(m+k)}y^{m+s}.$$

This completes the proof of Theorem 3. □

Remark. This result corrects Corollary 4.2 in [22].

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