The Rainbow Connectivities of Small Cubic Graphs

Futaba Fujie-Okamoto
Mathematics Department
University of Wisconsin-La Crosse
La Crosse, WI 54601, USA

Garry L. Johns
Department of Mathematical Sciences
Saginaw Valley State University
University Center, MI 48710-0001, USA

Ping Zhang
Department of Mathematics
Western Michigan University
Kalamazoo, MI 49008, USA

ABSTRACT

A path P in an edge-colored graph (not necessarily a proper edge-coloring) is a rainbow path if no two edges of P are assigned the same color. For a connected graph G with connectivity $\kappa(G)$ and an integer k with $1 \leq k \leq \kappa(G)$, the rainbow k-connectivity $\mathrm{rc}_k(G)$ of G is the minimum number of colors needed in an edge-coloring of G such that every two distinct vertices u and v of G are connected by at least k internally disjoint $u - v$ rainbow paths. In this paper, the rainbow 2-connectivity of the Petersen graph as well as the rainbow connectivities of all cubic graphs of order 8 or less are determined.

Key Words: edge coloring, rainbow path, rainbow connectivity.
AMS Subject Classification: 05C15, 05C38, 05C40.

1 Introduction

Let G be a nontrivial connected graph on which is defined an edge-coloring $c : E(G) \to \{1, 2, \ldots, k\}$, $k \in \mathbb{N}$, where adjacent edges may be colored the same. A path P in G is a *rainbow path* (*with respect to* c) if no two edges of P are colored the same. The graph G is *rainbow-connected* (*with respect to* c) if every two distinct vertices of G are connected by a rainbow path (see

[2]). In this case, the coloring c is called a *rainbow edge-coloring* (or simply a *rainbow coloring*) of G. The minimum integer k for which there exists a k-edge-coloring of G that results in a rainbow-connected graph is called the *rainbow connection number* $\mathrm{rc}(G)$ of G. In [1] the *rainbow connectivity* $\kappa_r(G)$ of a graph G with connectivity $\kappa(G) = \ell \geq 1$ is defined as the minimum number of colors needed in an edge-coloring of G such that every two distinct vertices u and v of G are connected by at least ℓ internally disjoint $u - v$ rainbow paths.

Suppose that G is an ℓ-connected graph ($\ell \geq 1$). It then follows from a well-known theorem of Whitney [7] that for every integer k with $1 \leq k \leq \ell$ and every two distinct vertices u and v of G, the graph G contains k internally disjoint $u - v$ paths. The *rainbow k-connectivity* $\mathrm{rc}_k(G)$ of G is the minimum integer j for which there exists a j-edge-coloring of G such that for every two distinct vertices u and v of G, there exist at least k internally disjoint $u-v$ rainbow paths. Thus $\mathrm{rc}_1(G) = \mathrm{rc}(G)$ is the rainbow connection number of G and if $\kappa(G) = \ell$, then $\mathrm{rc}_\ell(G) = \kappa_r(G)$. By coloring the edges of G with distinct colors, we see that every two vertices of G are connected by at least ℓ internally disjoint rainbow paths and so $\mathrm{rc}_k(G)$ is defined for every integer k with $1 \leq k \leq \ell$. Furthermore, $\mathrm{rc}_{k_1}(G) \leq \mathrm{rc}_{k_2}(G)$ for $1 \leq k_1 \leq k_2 \leq \ell$.

The rainbow connection number and rainbow connectivity of complete graphs and complete bipartite graphs were studied in [2, 3]. The chromatic index of a graph G is denoted by $\chi'(G)$.

Theorem 1 *For $n \geq 2$, $\mathrm{rc}(K_n) = 1$ and $\kappa_r(K_n) = \chi'(K_n) = 2\lceil n/2 \rceil - 1$.*

It was shown in [3] that $\mathrm{rc}_k(K_n) = 2$ if $k = 2$ and $n \geq 4$, or $k = 3$ and $n \geq 5$, or $k = 4$ and $n \geq 8$. More generally, the following was verified in [3].

Theorem 2 *For every integer $k \geq 2$, there exists an integer $f(k)$ such that if $n \geq f(k)$, then $\mathrm{rc}_k(K_n) = 2$.*

For integers s and t with $2 \leq s \leq t$, it was shown in [2] that

$$\mathrm{rc}(K_{s,t}) = \min\left\{\left\lceil \sqrt[s]{t} \right\rceil, 4\right\}. \tag{1}$$

For $2 \leq k \leq r$, it was shown in [3] that $\mathrm{rc}_k(K_{r,r}) = 3$ if $k = 2$ and $r \geq 3$ or if $k = 3$. The following more general result was established in [3].

Theorem 3 *For every integer $k \geq 2$, there exists an integer r such that $\mathrm{rc}_k(K_{r,r}) = 3$.*

For a connected graph G and an integer k with $1 \leq k \leq \kappa(G)$, the *k-diameter* $\mathrm{diam}_k(G)$ of G is the maximum of the minimum length of a longest path in any set of k internally disjoint $u-v$ paths, where the maximum is taken over all pairs u, v of distinct vertices of G. Thus $\mathrm{diam}_1(G) = \mathrm{diam}(G)$ for every connected graph G. The following observation is often useful.

Observation 4 *For every connected graph G and every integer k with $1 \le k \le \kappa(G)$,*

$$\mathrm{rc}_k(G) \ge \mathrm{diam}_k(G).$$

As a consequence of Observation 4, we have the following.

Observation 5 *If G is a 2-connected graph with girth g, then*

$$\mathrm{rc}_2(G) \ge g - 1.$$

The cubic graphs of minimum order having girth g are commonly referred to as *g-cages*. It is well known that the unique 3-cage is the complete graph K_4, the unique 4-cage is the complete bipartite graph $K_{3,3}$, and the unique 5-cage is the famous Petersen graph P. By Theorem 1 and (1), $\mathrm{rc}_1(K_4) = 1$ and $\mathrm{rc}_1(K_{3,3}) = 2$. Furthermore, it was shown in [2] that $\mathrm{rc}_1(P) = 3$. Since the connectivity of every g-cage, $g \ge 3$, is 3, the rainbow connectivity of every g-cage is the minimum number of colors needed in an edge-coloring of G such that every two distinct vertices u and v of G are connected by three internally disjoint $u-v$ rainbow paths. It was shown in [3] that $\mathrm{rc}_3(G) = \kappa_r(G) = 3$ if $G \in \{K_4, K_{3,3}\}$ and in [1] that $\mathrm{rc}_3(P) = \kappa_r(P) = 5$.

In this paper, we show that the previously missing rainbow 2-connectivity of the Petersen graph is 5 and determine $\mathrm{rc}_k(G)$ for all connected cubic graphs G of order 8 or less and for every integer k with $1 \le k \le \kappa(G)$. We refer to the book [4, 5] for graph theory notation and terminology not described in this paper.

2 The Rainbow Connectivities of Cubic Graphs of Order 8

If G is a connected cubic graph of order $n \le 8$, then $n = 4, 6, 8$. If $n = 4$, then $G = K_4$ and $\kappa(K_4) = 3$. It is known that $\mathrm{rc}_1(K_4) = 1$, $\mathrm{rc}_2(K_4) = 2$, and $\mathrm{rc}_3(K_4) = \kappa_r(K_4) = 3$ (see [3]). If $n = 6$, then $G = K_{3,3}$ or $G = K_3 \times K_2$ and so $\kappa(G) = 3$ in each case. Also, it is known that $\mathrm{rc}_1(K_{3,3}) = 2$, $\mathrm{rc}_2(K_{3,3}) = 3$, and $\mathrm{rc}_3(K_{3,3}) = \kappa_r(K_{3,3}) = 3$ (see [3]). Furthermore, it was shown in [3] that $\mathrm{rc}_1(K_3 \times K_2) = 2$, $\mathrm{rc}_2(K_3 \times K_2) = 3$, and $\mathrm{rc}_3(K_3 \times K_2) = \kappa_r(K_3 \times K_2) = 6$. Thus, we now consider the rainbow connectivities of the five connected cubic graphs of order 8. Figure 1 shows these graphs (see [6]).

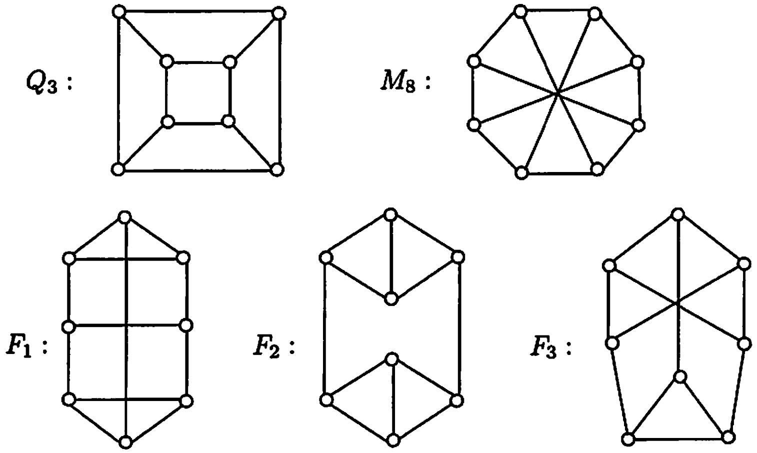


Figure 1: Connected cubic graphs of order 8

2.1 The Rainbow Connectivities of Q_3

The connectivity of the 3-dimensional cube Q_3 is 3 and $\mathrm{rc}_k(Q_3)$ exists for $1 \leq k \leq 3$. Since $\mathrm{diam}_1(Q_3) = 3$ and there is a 3-edge-coloring of Q_3 such that every two distinct vertices of Q_3 are connected by a rainbow path (see Figure 2), it then follows by Observation 4 that $\mathrm{rc}_1(Q_3) = 3$. Thus we now determine $\mathrm{rc}_2(Q_3)$ and $\mathrm{rc}_3(Q_3) = \kappa_r(Q_3)$, beginning with $\mathrm{rc}_2(Q_3)$.

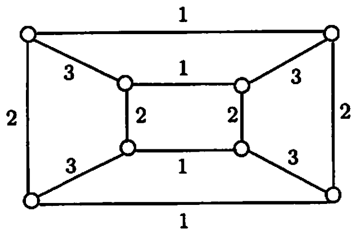


Figure 2: A rainbow 3-edge-coloring of Q_3

Theorem 6 $\mathrm{rc}_2(Q_3) = 4$.

Proof. Since $\mathrm{diam}_2(Q_3) = 3$, it follows that $\mathrm{rc}_2(Q_3) \geq 3$. We claim that $\mathrm{rc}_2(Q_3) \geq 4$. Assume, to the contrary, that Q_3 has a 3-edge-coloring such that every two distinct vertices of Q_3 are connected by two internally disjoint rainbow paths. Let such a 3-edge-coloring c of Q_3 be given. Then every 4-cycle in Q_3 contains at least two edges that are assigned the same color. We consider two cases.

Case 1. *Two adjacent edges in a* 4*-cycle of* Q_3 *are assigned the same color, say* u_1u_2 *and* u_2u_3 *in* $C : u_1, u_2, u_3, u_4, u_1$ *are colored* a. (See Figure 3(a).) Then one of the two $u_1 - u_3$ rainbow paths has length 4, which is impossible.

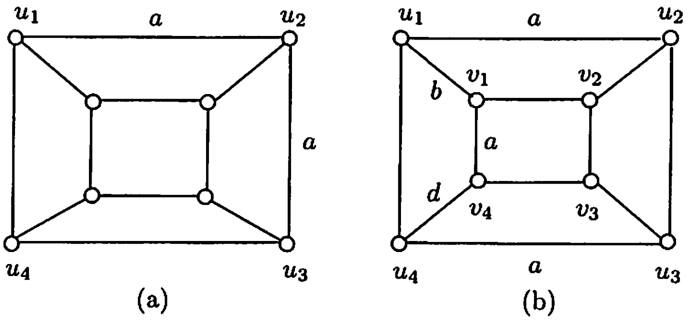


Figure 3: Steps in the proof of Theorem 6

Case 2. *No two adjacent edges in a* 4*-cycle of* Q_3 *are assigned the same color.* Thus c is a proper edge-coloring. Hence we may assume that u_1u_2 and u_3u_4 in C are colored a. Consider u_1 and u_4. Since u_1, u_2, u_3, u_4 is not a $u_1 - u_4$ rainbow path, u_1, v_1, v_4, u_4 must be a $u_1 - u_4$ rainbow path. This implies that $c(u_1v_1) = b$, $c(v_1v_4) = a$, and $c(u_4v_4) = d$, where a, b, and d are the three colors used by c. (See Figure 3(b).) Then $c(u_1u_4) \in \{a, b, d\}$, which is a contradiction since c is a proper edge-coloring. Thus, as claimed, $\text{rc}_2(Q_3) \geq 4$.

Since the 4-edge-coloring of Q_3 given in Figure 4 has the property that every two distinct vertices of Q_3 are connected by two internally disjoint rainbow paths, it follows that $\text{rc}_2(Q_3) = 4$. ■

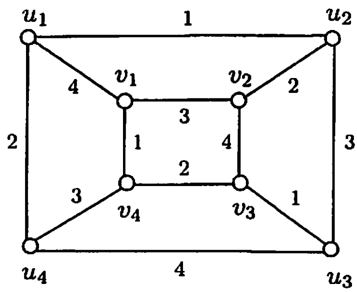


Figure 4: The rainbow 2-connectivity of Q_3

Theorem 7 $rc_3(Q_3) = \kappa_r(Q_3) = 7$.

Proof. The 7-edge-coloring of Q_3 shown in Figure 5 has the property that every two distinct vertices of Q_3 are connected by three internally disjoint rainbow paths. Therefore, $\kappa_r(Q_3) \leq 7$.

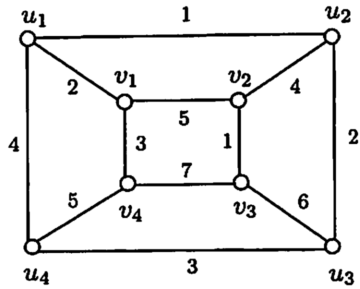


Figure 5: $\kappa_r(Q_3) \leq 7$

Consider an edge-coloring c of Q_3 such that for every two distinct vertices x and y of Q_3, there exist three internally disjoint $x - y$ rainbow paths. If x and y are adjacent, then every set of three internally disjoint $x - y$ rainbow paths consists of the path x, y and either (i) the two paths of length 3 or (ii) one path of length 3 and one path of length 5. This implies that every two nonadjacent edges in a 4-cycle in Q_3 must be assigned distinct colors. On the other hand, if x and y are vertices of Q_3 such that $d(x, y) = 2$, then every set of three internally disjoint $x - y$ rainbow paths consists of the two paths of length 2 and one path of length 4. Therefore, the edges of each 4-cycle in Q_3 must be assigned distinct colors and so c is a proper edge-coloring of Q_3.

Suppose that $c : E(Q_3) \to \{1, 2, \ldots, \kappa_r(Q_3)\}$ is a $\kappa_r(Q_3)$-edge-coloring of Q_3 possessing the desired property. Considering u_1 and u_3, we see that at least one of the paths $P : u_1, v_1, v_2, v_3, u_3$ and $P' : u_1, v_1, v_4, v_3, u_3$ is a rainbow path, say the former. Hence the four edges u_1v_1, v_1v_2, v_2v_3, and v_3u_3 are assigned distinct colors. Furthermore,

$$c(u_2v_2) \notin \{c(u_1v_1), c(v_1v_2), c(v_2v_3), c(v_3u_3)\}.$$

Therefore, $\kappa_r(Q_3) \geq 5$. We now consider two cases, according to whether three edges of Q_3 are colored the same by c or not.

Case 1. *At least three edges of* Q_3 *are colored the same by* c. We may assume, without loss of generality, that $c(u_1u_2) = c(u_3v_3) = c(v_1v_4) = 1$. Also, by the argument given above, we may further assume that $c(u_1v_1) = 2$, $c(v_1v_2) = 3$, $c(v_2v_3) = 4$, and $c(u_2v_2) = 5$ (see Figure 6). On the other hand, consider the pair u_3 and v_1 and observe that the path u_3, u_2, u_1, v_1 must be a rainbow path. Thus, $c(u_2u_3) \neq 2$. If $\kappa_r(Q_3) = 5$, then $c(u_2u_3) = 3$. However then, there is no $u_3 - v_4$ rainbow path of length 4, a contradiction. Therefore, $\kappa_r(Q_3) \geq 6$ and $c(u_2u_3) \neq 3$.

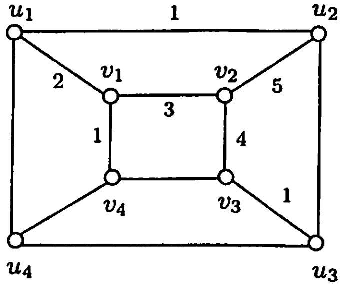


Figure 6: A step in the proof of Theorem 7

Suppose, in fact, that $\kappa_r(Q_3) = 6$. Then $c(u_2u_3) = 6$. Observe that each of the paths

$$P : u_1, u_2, v_2, v_3, v_4, \quad P' : u_1, v_1, v_4, v_3, \quad \text{and} \quad P'' : u_2, u_3, v_3, v_4$$

must be a rainbow path and so $c(v_3v_4) \notin \{1, 2, \ldots, 6\}$. Thus $c(v_3v_4) \geq 7$, contradicting the assumption that $\kappa_r(Q_3) = 6$. Therefore, if three edges of Q_3 are colored the same by c, then $\kappa_r(Q_3) = 7$.

Case 2. *No three edges of Q_3 are colored the same by c.* Hence no color is used more than twice, which implies that $\kappa_r(Q_3) \geq 6$. Suppose that $\kappa_r(Q_3) = 6$. Then each color is assigned to exactly two edges of Q_3. Assume, without loss of generality, that $c(u_1v_1) = 1$, $c(v_1v_2) = 2$, $c(v_2v_3) = 3$, $c(v_3u_3) = 4$, and $c(u_2v_2) = 5$ (see Figure 7). We consider the edge u_4v_4. Since $c(u_4v_4) \in \{2, 3, 6\}$, there are two subcases.

Subcase 2.1. $c(u_4v_4) = 6$. Then either $c(u_1u_2) = 6$ or $c(u_2u_3) = 6$, say the former. Also, $c(v_3v_4) = \{1, 5\}$. The path u_2, v_2, v_3, v_4 is a rainbow path, implying that $c(v_3v_4) \neq 5$. However then, $c(v_3v_4) = 1$ and there is no $u_2 - v_3$ rainbow path of length 4, which is a contradiction.

Subcase 2.2. $c(u_4v_4) \in \{2, 3\}$. By symmetry, we may assume that $c(u_4v_4) = 2$. Observe that the path u_2, v_2, v_3, v_4, u_4 is a rainbow path, implying that $c(v_3v_4) \in \{1, 6\}$. Also, u_4, u_3, u_2, v_2 is a rainbow path and so $c(u_3u_4) \neq 5$. Furthermore, $c(u_2u_3) \in \{1, 6\}$. If $c(u_2u_3) = 1$, then $c(v_3v_4) = 6$, which in turn implies that $c(u_1u_4) = 6$. Therefore, $c(v_1v_4) = 5$. However then, there is no $u_3 - v_4$ rainbow path of length 4, which is impossible. Thus, $c(u_2u_3) = 6$. Then observe that $c(v_3v_4) = 6$, $c(u_3u_4) = 1$,

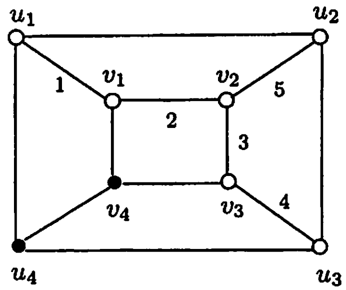


Figure 7: A step in the proof of Theorem 7

and $c(u_1u_2) = 3$. However, this implies that there is no $u_1 - v_4$ rainbow path of length 4.

Therefore, we conclude that $\kappa_r(Q_3) \neq 6$ and so $\kappa_r(Q_3) = 7$. ■

2.2 The Rainbow Connectivities of M_8

For each integer $n \geq 3$, the graph M_n is called the *Möbius ladder* of order n. For $n = 8$, the graph M_8 is a cubic graph of order 8, diameter 2, and connectivity 3. In this section we determine the rainbow connectivities of M_8.

Theorem 8 $\text{rc}_1(M_8) = 2$, $\text{rc}_2(M_8) = 4$, *and* $\text{rc}_3(M_8) = \kappa_r(M_8) = 5$.

Proof. We first show that $\text{rc}_1(M_8) = 2$. Since $\text{diam}_1(M_8) = 2$, it follows that $\text{rc}_1(M_8) \geq 2$. On the other hand, there exists a rainbow 2-coloring of M_8 (see Figure 8(a)) and so $\text{rc}_1(M_8) \leq 2$. Hence $\text{rc}_1(M_8) = 2$.

Next, we show that $\text{rc}_2(M_8) = 4$. Since there exists a 4-edge-coloring of M_8 which has the property that every two distinct vertices of M_8 are connected by two internally disjoint rainbow paths (see Figure 8(b)), we have $\text{rc}_2(M_8) \leq 4$.

For the vertices v_1 and v_2, every set of two internally disjoint $v_1 - v_2$ paths contains a path of length at least 3, implying that $\text{rc}_2(M_8) \geq 3$ (= $\text{diam}_2(M_8)$). We now show that $\text{rc}_2(M_8) > 3$. Assume, to the contrary, that there exists a 3-edge-coloring $c : E(M_8) \to \{1, 2, 3\}$ of M_8 with the desired property. Consider the vertices v_1 and v_2. Then the path v_1, v_5, v_6, v_2 must be a rainbow path. Without loss of generality, suppose that $c(v_1v_5) = 1$, $c(v_5v_6) = 2$, and $c(v_2v_6) = 3$. Similarly, for the vertices v_5 and v_6, the path v_5, v_1, v_2, v_6 must be a rainbow path, implying that $c(v_1v_2) = 2 = c(v_5v_6)$. Applying the same argument, we obtain $c(v_1v_8) = c(v_4v_5)$. Now consider v_1 and v_5. Since neither v_1, v_2, v_6, v_5 nor v_1, v_8, v_4, v_5 is a rainbow path, it

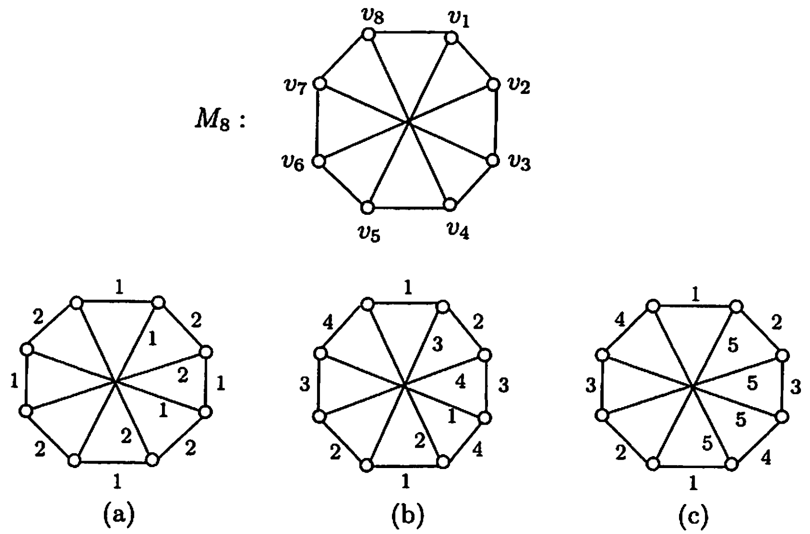


Figure 8: Rainbow colorings of M_8

follows that each set of two internally disjoint $v_1 - v_5$ rainbow paths contains a path of length at least 4, which is impossible. Therefore, $\text{rc}_2(M_8) = 4$.

Finally, we show that $\kappa_r(M_8) = 5$. Observe that there exists a 5-edge-coloring of M_8 such that every two distinct vertices are connected by three internally disjoint rainbow paths (see Figure 8(c)). Therefore, $4 \leq \kappa_r(M_8) \leq 5$.

We show that $\kappa_r(M_8) \geq 5$. Assume, to the contrary, that there exists a 4-edge-coloring $c : E(M_8) \to \{1, 2, 3, 4\}$ of M_8 having the desired property. Let $C : v_1, v_2, \ldots, v_8, v_1$ be a Hamiltonian cycle in M_8 and consider a pair of vertices x and y such that $d_C(x, y) = 3$, say v_1 and v_4. Then every set of three internally disjoint $v_1 - v_4$ paths must contain the paths v_1, v_5, v_4 and v_1, v_8, v_4. So the third path must be either

$$P : v_1, v_2, v_3, v_4 \quad \text{or} \quad P' : v_1, v_2, v_6, v_7, v_3, v_4.$$

Since P' has length 5, it cannot be a rainbow path. Therefore, the path P must be a rainbow path. By symmetry, this implies that for every pair of vertices x and y with $d_C(x, y) = 3$, the $x - y$ path of length 3 consisting of three consecutive edges in C must be a rainbow path. Also, this implies that c is a proper coloring.

Since C contains 8 edges and every three consecutive edges in C must be colored differently, it follows that exactly two edges in C are colored the same. Without loss of generality, suppose that $c(v_i v_{i+1}) = i$ for $i = 1, 2, 3$. Then $c(v_4 v_5) \in \{1, 4\}$. We now consider these two cases.

Case 1. $c(v_4v_5) = 1$. Then $c(v_5v_6) = c(v_1v_8) = 4$ and $\{c(v_6v_7), c(v_7v_8)\} = \{2,3\}$. Since c is a proper coloring, $c(v_2v_6) = 3$ and so $c(v_6v_7) = 2$ and $c(v_7v_8) = 3$. This implies that $c(v_4v_8) = 2$ and so $c(v_1v_5) = 3 = c(v_2v_6)$ or $c(v_1v_5) = 2 = c(v_4v_8)$. If $c(v_1v_5) = c(v_2v_6)$, then there is no set of three internally disjoint $v_1 - v_2$ rainbow paths; while if $c(v_1v_5) = c(v_4v_8)$, then there is no set of three internally disjoint $v_1 - v_8$ rainbow paths. Therefore, this case cannot occur.

Case 2. $c(v_4v_5) = 4$. Then exactly one of the edges v_5v_6 and v_6v_7 is colored 1. If $c(v_6v_7) = 1$, then it follows that $c(v_5v_6) = 2$ and $c(v_1v_5) = 3$, implying that $c(v_1v_8) = 4$, $c(v_7v_8) = 3$, and $c(v_3v_7) = 4$. Performing the permutation (1432) on the colors, we obtain the coloring described in Case 1. Therefore, $c(v_6v_7) \neq 1$ and so $c(v_5v_6) = 1$. Hence exactly one of the edges v_6v_7 and v_7v_8 is colored 2. If $c(v_7v_8) = 2$, then $c(v_6v_7) = 3$ and $c(c_1v_8) = 4$. Performing the permutation (13)(24) on the colors, we obtain the coloring described in Case 1. Hence suppose that $c(v_6v_7) = 2$. If $c(v_7v_8) = 4$ and $c(v_1v_8) = 3$, then by applying the permutation (123) on the colors, we obtain the coloring described in Case 1. Therefore, we may assume that $c(v_7v_8) = 3$ and $c(v_1v_8) = 4$.

Observe that $c(v_1v_5) \in \{2,3\}$ and without loss of generality, we may assume that $c(v_1v_5) = 3$. In order to guarantee the existence of three internally disjoint $v_1 - v_2$ rainbow paths, the path v_1, v_5, v_6, v_2 must be a rainbow path, implying that $c(v_2v_6) = 4$. Similarly, considering v_2 and v_3, it follows that $c(v_3v_7) = 1$. However then, there is no set of three internally disjoint $v_1 - v_3$ rainbow paths, which is a contradiction. Hence, this case cannot occur either.

Therefore, $\kappa_r(M_8) > 4$, implying that $\kappa_r(M_8) = 5$. ■

2.3 The Rainbow Connectivities of F_i $(1 \leq i \leq 3)$

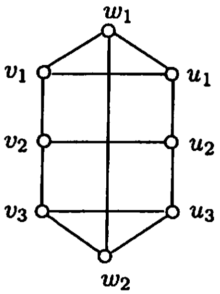
We now determine the rainbow connectivities of the graphs F_i $(1 \leq i \leq 3)$ shown in Figure 1, beginning with F_1, as shown in Figure 9. Observe that $\kappa(F_1) = 3$.

Theorem 9 $\mathrm{rc}_1(F_1) = 3$, $\mathrm{rc}_2(F_1) = 4$, *and* $\mathrm{rc}_3(F_1) = \kappa_r(F_1) = 7$.

Proof. We first show that $\mathrm{rc}_1(F_1) = 3$. Since $\mathrm{diam}_1(F_1) = 3$, it follows that $\mathrm{rc}_1(F_1) \geq 3$. On the other hand, there exists a rainbow 3-coloring of F_1 (see Figure 9(a)) and so $\mathrm{rc}_1(F_1) \leq 3$, that is, $\mathrm{rc}_1(F_1) = 3$.

Next, we show that $\mathrm{rc}_2(F_1) = 4$. Since there exists a 4-edge-coloring of F_1 which has the property that every two distinct vertices of F_1 are connected by two internally disjoint rainbow paths (see Figure 9(b)), we have $\mathrm{rc}_2(F_1) \leq 4$. On the other hand, for the vertices w_1 and w_2, every set of two internally disjoint $w_1 - w_2$ paths contains a path of length at least 4, implying that $\mathrm{rc}_2(F_1) \geq 4$ $(= \mathrm{diam}_2(F_1))$. Therefore, $\mathrm{rc}_2(F_1) = 4$.

F_1 :



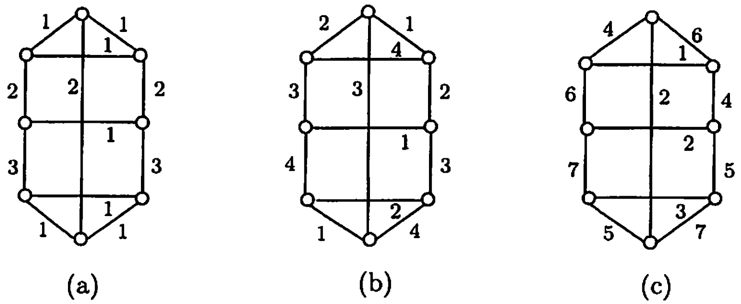


Figure 9: Rainbow colorings of F_1

Finally, we show that $\kappa_r(F_1) = 7$. Observe that there exists a 7-edge-coloring of F_1 which has the property that every two distinct vertices of F_1 are connected by three internally disjoint rainbow paths (see Figure 9(c)). Therefore, $\kappa_r(F_1) \leq 7$.

Now we show that $\kappa_r(F_1) \geq 7$. Let there be given a k-edge-coloring $c : E(F_1) \to \{1, 2, \ldots, k\}$ of F_1 with the desired property. Observe the following:

(a) Each of the three edges u_iv_i $(1 \leq i \leq 3)$ must be colored differently. (Consider the pair u_1 and u_2 of vertices, for example.)

(b) $c(u_1u_2), c(u_2, u_3) \notin \{c(u_iv_i) : 1 \leq i \leq 3\}$.

(c) The four edges belonging to each of the two $w_1 - w_2$ paths of length 4 must be colored differently. (Consider the pair w_1 and w_2 of vertices, for example.)

(d) None of the four edges w_1u_1, w_1v_1, w_2u_3, and w_2v_3 shares a color with the edges u_iv_i $(1 \leq i \leq 3)$. (Consider the pair w_1 and u_2 of vertices, for example.)

By (a) and (b), suppose, without loss of generality, that $c(u_iv_i) = i$ $(1 \leq i \leq 3)$, $c(u_1u_2) = 4$ and $c(u_2u_3) = 5$. By (c) and (d) then, $c(w_1u_1) \neq c(w_2u_3)$

and $c(w_1u_1), c(w_2u_3) \notin \{1, 2, \ldots, 5\}$. This implies that a new color must be available for each of the edges w_1u_1 and w_2u_3 in order to guarantee the existence of three internally disjoint $x - y$ rainbow paths for every two distinct vertices x and y of F_1. Therefore, $\mathrm{rc}_3(F_1) \geq 7$ and so $\mathrm{rc}_3(F_1) = 7$. ∎

Next, we determine the rainbow connectivities of the graph F_2 in Figure 10. Since $\kappa(F_2) = 2$, only $\mathrm{rc}_1(F_2)$ and $\mathrm{rc}_2(F_2)$ exist.

Theorem 10 $\mathrm{rc}_1(F_2) = 3$ *and* $\mathrm{rc}_2(F_2) = \kappa_r(F_2) = 5$.

Proof. We first show that $\mathrm{rc}_1(F_2) = 3$. Since $\mathrm{diam}_1(F_2) = 3$, it follows that $\mathrm{rc}_1(F_2) \geq 3$. On the other hand, there exists a rainbow 3-coloring of F_2 (see Figure 10(a)) and so $\mathrm{rc}_1(F_2) \leq 3$, that is, $\mathrm{rc}_1(F_2) = 3$.

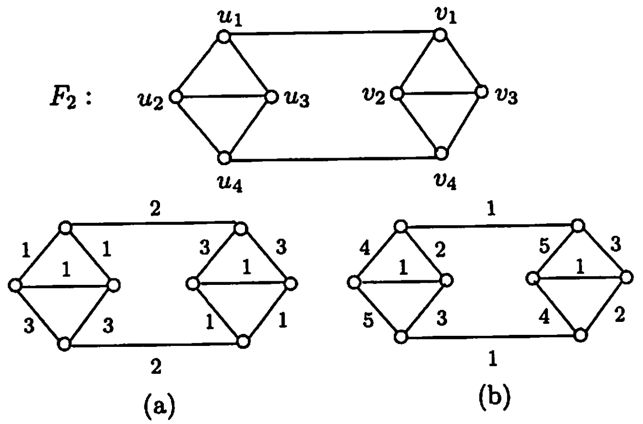


Figure 10: Rainbow colorings of F_2

Next, we show that $\kappa_r(F_2) = 5$. Since there exists a 5-edge-coloring of F_2 which has the property that every two distinct vertices are connected by two internally disjoint rainbow paths (see Figure 10(b)), $\kappa_r(F_2) \leq 5$. On the other hand, every set of two internally disjoint $u_1 - v_1$ paths consists of a path of length 1 and a path of length at least 5, implying that $\kappa_r(F_2) \geq 5$ $(= \mathrm{diam}_2(F_2))$. Therefore, we conclude that $\kappa_r(F_2) = 5$. ∎

We now determine the rainbow connectivities of F_3 as shown in Figure 11. Observe that $\kappa(F_3) = 3$.

Theorem 11 $\mathrm{rc}_1(F_3) = 3$, $\mathrm{rc}_2(F_3) = 4$, *and* $\mathrm{rc}_3(F_3) = \kappa_r(F_3) = 6$.

Proof. We first show that $\mathrm{rc}_1(F_3) = 3$. Since there exists a rainbow 3-coloring of F_3 (see Figure 11(a)), it follows that $\mathrm{rc}_1(F_3) \leq 3$. On the other hand, since $\mathrm{diam}_1(F_3) = 2$, it follows that $\mathrm{rc}_1(F_3) \geq 2$. We show that $rc_1(F_3) \neq 2$. Assume, to the contrary, that there exists a rainbow 2-coloring $c : E(F_3) \to \{1,2\}$ of F_3. Without loss of generality, suppose that $c(u_2v_2) = 1$. Since u_i, u_2, v_2 is a rainbow path for $i = 1,3$, it follows that $c(u_1u_2) = c(u_2u_3) = 2$. Since u_2, u_i, v_i is a rainbow path for $i = 1,3$, it follows that $c(u_1v_1) = c(u_3v_3) = 1$. On the other hand, since u_i, v_i, w_j is a rainbow path for $i \in \{1,3\}$ and $j \in \{1,2\}$, it follows that $c(v_1w_1) = c(v_1w_2) = c(v_3w_1) = c(v_3w_2) = 2$. However then, there is no $v_1 - v_3$ rainbow path, a contradiction. Therefore, $rc_1(F_3) \geq 3$ and so $rc_1(F_3) = 3$.

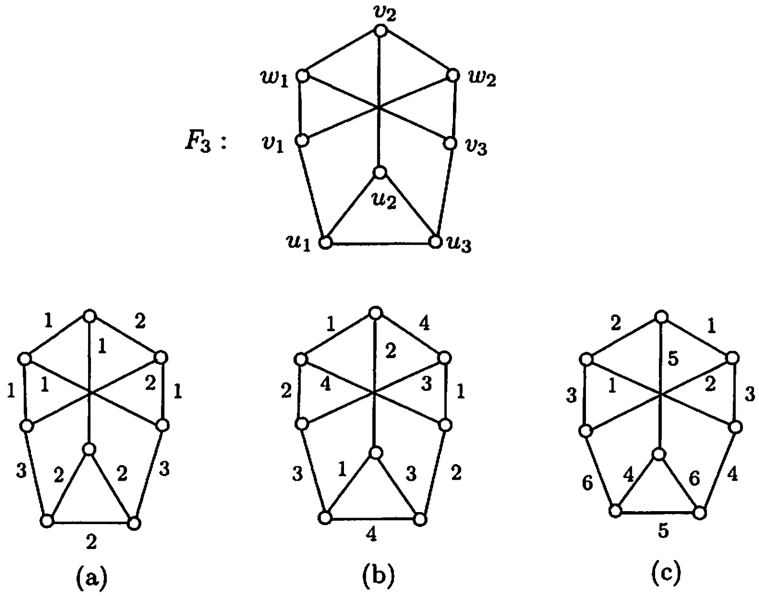


Figure 11: Rainbow colorings of F_3

Next, we show that $rc_2(F_3) = 4$. Since there exists a 4-edge-coloring of F_3 which has the property that every two distinct vertices of F_3 are connected by two internally disjoint rainbow paths (see Figure 11(b)), we have $rc_2(F_3) \leq 4$. Furthermore, for the vertices u_1 and v_1, every set of two internally disjoint $u_1 - v_1$ paths contains a path of length at least 4 and so $rc_2(F_3) \geq 4$ $(= \text{diam}_2(F_3))$. Therefore, $rc_2(F_3) = 4$.

Finally, we show that $\kappa_r(F_3) = 6$. There exists a 6-edge-coloring of F_3 such that every two distinct vertices of F_3 are connected by three internally disjoint rainbow paths (see Figure 11(c)), so $\kappa_r(F_3) \leq 6$. On the other hand, observe that $\kappa_r(F_3) \geq rc_2(F_3) \geq 4$.

We first show that $\kappa_r(F_3) \geq 5$. Let $c : E(F_3) \to \{1,2,\ldots,k\}$ be a k-edge-coloring of F_3 such that every two distinct vertices of F_3 are connected by three internally disjoint rainbow paths and consider the five edges u_1u_2, u_2u_3, u_2v_2, v_2w_1, and v_2w_2. Observe first that c must be a proper coloring. Therefore, $c(u_1u_2) \neq c(u_2u_3)$. Without loss of generality, assume that $c(u_1u_2) = 1$ and $c(u_2u_3) = 2$. Now consider the pairs $\{u_i, w_j\}$, where $i \in \{1,3\}$ and $j \in \{1,2\}$. Observe that the path u_i, u_2, v_2, w_j must be a rainbow path, implying that (i) $c(u_2v_2), c(v_2w_j) \notin \{1,2\}$ and (ii) $c(u_2v_2) \neq c(v_2w_j)$ for $j = 1,2$. Therefore, we may assume, without loss of generality, that $c(u_2v_2) = 3$ and $c(v_2w_1) = 4$. Since w_1, v_2, w_2 must be a rainbow path, it then follows that $c(v_2w_2) \notin \{1,2,3,4\}$, implying that we need another color for the edge v_2w_2. Therefore, $\kappa_r(F_3) \geq 5$ and so $\kappa_r(F_3) \in \{5,6\}$.

Next we show that $\kappa_r(F_3) > 5$. Assume, to the contrary, that there exists a 5-edge-coloring c of F_3 having the desired property. Since one path in every set of three internally disjoint $u_i - u_j$ rainbow paths ($1 \leq i < j \leq 3$) must contain the path containing the edges u_iv_i and u_jv_j, it follows that the three edges u_1v_1, u_2v_2, and u_3v_3 must be colored differently. We have seen that the five edges u_1u_2, u_2u_3, u_2v_2, v_2w_1, and v_2w_2 must be colored differently and without loss of generality, suppose that $c(u_1u_2) = 1$, $c(u_2u_3) = 2$, $c(u_2v_2) = 3$, $c(v_2w_1) = 4$, and $c(v_2w_2) = 5$. By symmetry, there are two other sets of five edges, namely $\{u_1u_2, u_1u_3, u_1v_1, v_1w_1, v_1w_2\}$ and $\{u_1u_3, u_2u_3, u_3v_3, v_3w_1, v_3w_2\}$, for which we need to use five different colors. This implies that the colors 1, 2, and $c(u_1u_2)$ cannot be used more than twice. (Note that $c(u_1u_2) \neq 1$ and $c(u_1u_2) \neq 2$.) Consequently, two colors are used three times each and three colors (the colors used for the three edges u_iu_j ($1 \leq i < j \leq 3$)) are used twice each. Then at least one of the colors 4 and 5 must be used three times, say the color 4 is used three times. Then $c(u_1u_3) \in \{3,5\}$. Furthermore, since c is a proper coloring, the other two edges colored 4 are either (i) v_1w_2 and u_3v_3 or (ii) v_3w_2 and u_1v_1. However, by symmetry, these colorings are essentially the same and so we only consider the coloring described in (i).

Case 1. $c(u_1u_3) = 3$. Then the color 5 must be used three times and so $c(u_1v_1) = c(v_3w_1) = 5$. However, there is no set of three internally disjoint $u_1 - u_3$ rainbow paths, which is a contradiction.

Case 2. $c(u_1u_3) = 5$. Then $c(v_1w_1) = c(v_3w_2) = 3$, implying that $c(u_1v_1) = 2$ and $c(v_3w_1) = 1$. However, there is no set of three internally disjoint $u_1 - v_3$ rainbow paths, another contradiction.

Therefore, there is no 5-edge-coloring of F_3 satisfying the requirement, implying that $\kappa_r(F_3) \geq 6$ and so $\kappa_r(F_3) = 6$. ∎

3 The Rainbow 2-Connectivity of the Petersen Graph

The connectivity of the Petersen graph is 3. It was shown in [2] that $\mathrm{rc}_1(P) = 3$ and in [3] that $\mathrm{rc}_3(P) = \kappa_r(P) = 5$. In this section, we show that the previously missing rainbow 2-connectivity of the Petersen graph is 5.

Theorem 12 $\mathrm{rc}_2(P) = 5$.

Proof. Figure 12 shows a 5-edge-coloring of P such that every two distinct vertices of P are connected by two internally disjoint rainbow paths. Thus $\mathrm{rc}_2(P) \leq 5$. On the other hand, by Observation 5, $\mathrm{rc}_2(P) \geq 4$. Therefore, $\mathrm{rc}_2(P) \in \{4, 5\}$. We show that $\mathrm{rc}_2(P) \neq 4$. Suppose that the vertices of P are labeled as shown in Figure 12.

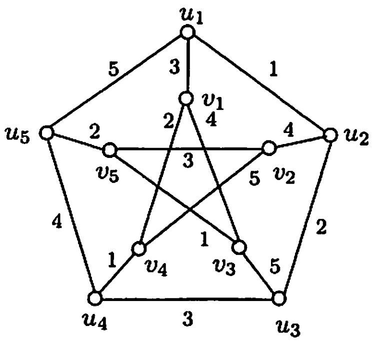


Figure 12: A rainbow 5-edge-coloring of the Petersen graph

Assume, to the contrary, that there exists a 4-edge-coloring $c : E(P) \to \{1, 2, 3, 4\}$ having the desired property. First note that since there are fifteen edges, at least one of the four colors is used for at most three edges. Without loss of generality, suppose that the color 4 is used at most three times. We first make some observations.

A. If x and y are adjacent vertices, then there are exactly four $x - y$ paths of length 4. Therefore, at least one of these four paths must be a rainbow path and so must contain one edge colored 4.

B. If x and y are nonadjacent vertices, then there are exactly five $x - y$ paths of length at most 4. More specifically, there is one path of length 2 (the $x - y$ geodesic), two paths of length 3, and two paths of length 4. If the $x - y$ geodesic is not a rainbow path, then because no $x - y$ path of length 3 and $x - y$ path of length 4 are internally disjoint, there must be two $x - y$ rainbow paths of length 3 or two of length 4. Furthermore, in the latter case, each path must contain one edge colored 4.

C. There must be at least two edges colored 4, for if the edge u_1u_2, say, is the only edge colored 4, then there is no $u_4 - v_4$ rainbow path of length 4.

Hence there are either exactly two or exactly three edges of P colored 4. We claim that P does not contain two adjacent edges colored 4. Suppose that there are two adjacent edges colored 4, say, without loss of generality, $c(u_1u_2) = c(u_1u_5) = 4$. We may further assume that u_1, u_2, u_3, u_4, u_5 is a $u_1 - u_5$ rainbow path of length 4 with $c(u_iu_{i+1}) = i - 1$ for $i = 2, 3, 4$. Observe that

(a) $c(u_1v_1) \neq 4$, for otherwise there is no $u_3 - u_4$ rainbow path of length 4.

(b) $c(v_2v_5) \neq 4$, for otherwise there is no $v_2 - v_5$ rainbow path of length 4.

(c) $c(v_1v_3) \neq 4$, for otherwise there is no $u_5 - v_5$ rainbow path of length 4. By symmetry, $c(v_1v_4) \neq 4$.

Suppose that $c(u_2v_2) = 4$. Considering v_2 and v_4, it follows that $c(u_4v_4) = 3$. Then there is no $v_1 - v_4$ rainbow path of length 4, so $c(u_2v_2) \neq 4$. By symmetry, $c(u_5v_5) \neq 4$.

Suppose that $c(u_3v_3) = 4$. For u_2 and u_3, we have

$$\{c(u_2v_2), c(v_2v_5), c(v_3v_5)\} = \{1, 2, 3\}.$$

On the other hand, for u_2 and u_5,

$$\{c(u_2v_2), c(u_5v_5), c(v_2v_5)\} = \{1, 2, 3\}.$$

Then $c(u_5v_5) = c(v_3v_5)$. However then, the $u_3 - u_5$ geodesic is the only $u_3 - u_5$ rainbow path, a contradiction. Hence $c(u_3v_3) \neq 4$ and by symmetry, $c(u_4v_4) \neq 4$.

Suppose that $c(v_2v_4) = 4$. Considering u_2 and v_2, it follows that $c(u_4v_4) = 3$. However, this means that the $u_5 - v_2$ geodesic is the only path that is possibly a $u_5 - v_2$ rainbow path, a contradiction. Hence $c(v_2v_4) \neq 4$ and by symmetry, $c(v_3v_5) \neq 4$.

Therefore, the two edges u_1u_2 and u_1u_5 are the only edges colored 4. However, this implies that there is no $u_3 - u_4$ rainbow path of length 4. Hence, the claim is verified and consequently, no two edges of P colored 4 are adjacent.

Since for every edge xy colored 4, there is an $x-y$ path of length 4 containing an edge colored 4, we may assume that $c(u_2u_3) = c(u_4u_5) = 4$ and u_2, u_1, u_5, u_4, u_3 is a $u_2 - u_3$ rainbow path of length 4. We may further assume that $c(u_1u_2) = 1$, $c(u_3u_4) = 2$, and $c(u_1u_5) = 3$.

We claim that there are three edges colored 4, for suppose that u_2u_3 and u_4u_5 are the only edges colored 4. Considering u_1 and u_2, we see that $\{c(u_1v_1), c(u_3v_3), c(v_1v_3)\} = \{1,2,3\}$. Similarly, considering u_1 and u_5, we have $\{c(u_1v_1), c(u_4v_4), c(v_1v_4)\} = \{1,2,3\}$. Considering v_1 and v_3, we see that $\{c(u_1v_1), c(u_3v_3)\} = \{2,3\}$ and so $c(v_1v_3) = 1$. Similarly with v_1 and v_4, we have $c(v_1v_4) = 3$. However, this is impossible since there is no $u_1 - v_1$ rainbow path of length 4. Therefore, as claimed, there is a third edge colored 4. We now show that there are certain edges that are not colored 4.

(d) $c(v_1v_3) \neq 4$ *and* $c(v_1v_4) \neq 4$. If $c(v_1v_3) = 4$, then there is no $u_1 - u_2$ rainbow path of length 4, producing a contradiction. By symmetry, $c(v_1v_4) \neq 4$.

(e) $c(u_1v_1) \neq 4$. Suppose that $c(u_1v_1) = 4$. For u_3 and u_4, either u_3, u_2, v_2, v_4, u_4 or u_3, v_3, v_5, u_5, u_4 must be a rainbow path of length 4, say the former. Hence $\{c(u_2v_2), c(u_4v_4), c(v_2v_4)\} = \{1,2,3\}$. On the other hand, considering u_1 and u_2, $\{c(u_2v_2), c(v_1v_4), c(v_2v_4)\} = \{1,2,3\}$, implying that $c(u_4v_4) = c(v_1v_4)$. However then, P does not contain two internally disjoint $u_4 - v_1$ rainbow paths, a contradiction.

(f) $c(v_2v_4) \neq 4$ *and* $c(v_3v_5) \neq 4$. Suppose that $c(v_2v_4) = 4$. For u_3 and u_4, it follows that $\{c(u_3v_3), c(u_5v_5), c(v_3v_5)\} = \{1,2,3\}$, while for u_5 and v_5, $\{c(u_3v_3), c(v_3v_5)\} = \{1,3\}$. Therefore, $c(u_5v_5) = 2$. On the other hand, for u_4 and v_4, $\{c(u_1v_1), c(v_1v_4)\} = \{1,2\}$. Then for u_2 and v_2, $\{c(u_3v_3), c(v_2v_5), c(v_3v_5)\} = \{1,2,3\}$. Therefore, $c(v_2v_5) = 2 = c(u_5v_5)$, implying that there do not exist two internally disjoint $u_5 - v_2$ rainbow paths, producing a contradiction. By symmetry, $c(v_3v_5) \neq 4$.

Since no two adjacent edges of P are colored 4, it follows that $c(v_2v_5) = 4$. Considering u_1 and v_1, we see that either $\{c(u_3v_3), c(v_1v_3)\} = \{2,3\}$ or $\{c(u_4v_4), c(v_1v_4)\} = \{1,2\}$. Without loss of generality, suppose that $\{c(u_3v_3), c(v_1v_3)\} = \{2,3\}$. We consider two cases.

Case 1. $c(u_3v_3) = 2$ *and* $c(v_1v_3) = 3$. Then considering u_5 and v_5, we see that $c(u_2v_2) = 2$. Also, considering the two pairs $\{v_3, v_5\}$ and $\{v_2, v_4\}$, we have

$$\{c(v_1v_4), c(v_2v_4)\} = \{c(v_1v_4), c(v_3v_5)\} = \{1,2\},$$

implying that $c(v_2v_4) = c(v_3v_5)$. Then for v_1 and v_3, it follows that $c(u_1v_1) = 3$. However, this implies that there is no $u_4 - v_4$ rainbow path of length 4, a contradiction.

Case 2. $c(u_3v_3) = 3$ *and* $c(v_1v_3) = 2$. We consider three subcases, according to the color assigned to the edge u_1v_1.

Subcase 2.1. $c(u_1v_1) = 1$. Considering u_3 and v_3, we have

$$\{c(u_5v_5), c(v_3v_5)\} = \{1, 3\}.$$

Since $c(u_5v_5) \neq 2$, it follows by considering the pair u_2, v_2 that

$$\{c(u_4v_4), c(v_2v_4)\} = \{1, 3\}.$$

Then with u_3 and u_4, it follows that $c(u_2v_2) = 2$, which in turn implies that $c(v_1v_4) = 2$ by considering u_4 and v_4. Then there is no $v_2 - v_4$ rainbow path of length 4, a contradiction.

Subcase 2.2. $c(u_1v_1) = 2$. Considering the two pairs $\{u_1, u_2\}$ and $\{u_1, v_3\}$, we see that

$$\{c(u_2v_2), c(u_5v_5)\} = \{c(u_5v_5), c(v_3v_5)\} = \{1, 2\}$$

and so $c(u_2v_2) = c(v_3v_5)$. Then the $u_2 - v_3$ geodesic is the only $u_2 - v_3$ rainbow path, a contradiction.

Subcase 2.3. $c(u_1v_1) = 3$. Considering the two pairs $\{u_4, v_4\}$ and $\{v_1, v_4\}$, we have

$$\{c(u_2v_2), c(v_2v_4)\} = \{c(v_2v_4), c(v_3v_5)\} = \{1, 3\}$$

and so $c(u_2v_2) = c(v_3v_5) \in \{1, 3\}$. Then by looking at u_5 and v_5, we see that $c(v_3v_5) = 1$. Therefore, $c(u_2v_2) = c(v_3v_5) = 1$ and $c(v_2v_4) = 3$. However then, there do not exist two internally disjoint $u_1 - v_2$ rainbow paths, a contradiction again.

Therefore, this case cannot occur either and so $\mathrm{rc}_2(P) > 4$. Hence, $\mathrm{rc}_2(P) = 5$ as desired. ■

The following table summarizes the rainbow connectivities of the cubic graphs obtained in this paper, where $d_i = \text{diam}_i(G)$ for $1 \le i \le 3$.

n	G	$\kappa(G)$	d_1	d_2	d_3	$\text{rc}_1(G)$	$\text{rc}_2(G)$	$\text{rc}_3(G)$
4	K_4	3	1	2	2	1	2	3
6	$K_3 \times K_2$	3	2	3	3	2	3	6
	$K_{3,3}$	3	2	3	3	2	3	3
8	Q_3	3	3	3	4	3	4	7
	M_8	3	2	3	4	2	4	5
	F_1	3	3	4	4	3	4	7
	F_2	2	3	5	-	3	5	-
	F_3	3	2	4	4	3	4	6
10	P	3	2	4	4	3	5	5

4 Acknowledgments

We are grateful to Professor Gary Chartrand for kindly providing useful information on this topic.

References

[1] G. Chartrand, G. L. Johns, K. A. McKeon, and P. Zhang. On the rainbow connectivity of cages. *Congr. Numer.* **184** (2007) 209-222.

[2] G. Chartrand, G. L. Johns, K. A. McKeon, and P. Zhang. Rainbow connection in graphs. *Math. Bohem.* **133** (2008) 85-98.

[3] G. Chartrand, G. L. Johns, K. A. McKeon, and P. Zhang. The rainbow connectivity of a graph. *Networks* **54** (2009) 75-81.

[4] G. Chartrand, L. Lesniak and P. Zhang, *Graphs & Digraphs: 5th Edition*, Chapman & Hall/CRC, Boca Raton, FL (2010).

[5] G. Chartrand and P. Zhang, *Chromatic Graph Theory.* Chapman & Hall/CRC, Boca Raton, FL (2008).

[6] R. Read and R. Wilson, *An Atlas of Graphs.* Oxford University Press, New York (1998).

[7] H. Whitney, Congruent graphs and the connectivity of graphs. *Amer. J. Math.* **54** (1932) 150-168.