On 4-cutwidth critical trees

Zhenkun Zhang[1] Yixun Lin[2]

1. Department of Mathematics, Huanghuai University, Zhumadian 463000, China; 2. Department of Mathematics, Zhengzhou University, Zhengzhou 450052, China

Abstract. Arising from the VLSI design and network communication, the cutwidth problem for a graph G is to embed G into a path such that the maximum number of overlap edges (i.e., the congestion) is minimized. The characterization of forbidden subgraphs or critical graphs is meaningful in the study of a graph-theoretic parameter. This paper characterizes the set of 4-cutwidth critical trees by twelve specified ones.

Keywords: Graph labeling; Cutwidth; Critical tree

1 Introduction

The cutwidth problem for a graph G is to embed G into a path such that the maximum number of overlap edges (i.e., the congestion) is minimized. It is known that the problem for general graphs is NP-complete [5] while it is polynomially solvable for trees [16]. The cutwidth problem has significant applications in the circuit layout design and the network communication [4]. And its theoretic interest comes up in connection with other graph-theoretic parameters such as bandwidth, pathwidth and treewidth (see [2, 4, 7]).

Let $G = (V, E)$ be a simple graph with vertex set V, $|V| = n$, and edge set E. A *labeling* of G is a bijection $f : V \rightarrow \{1, 2, ..., n\}$, which can be regarded as an embedding of G into a path P_n. For a given labeling f of G, the cutwidth of G with respect to f is

$$c(G, f) = \max_{1 \leq j < n} |\{uv \in E : f(u) \leq j < f(v)\}|,$$

which represents the congestion of the embedding. The *cutwidth* of G is

E-mail address: zhzhkun-2006@163.com or linyixun@zzu.edu.cn.

defined by

$$c(G) = \min_f c(G, f),$$

where the minimum is taken over all labelings f. A labeling f attaining the above minimum value is called an *optimal* labeling.

In the embedding version, we may denote $u_j = f^{-1}(j)$ $(1 \leq j \leq n)$. Then the labeling f can be regarded as an ordering of vertices $u_1, u_2, ..., u_n$ arranged on a line. Let $S_j = \{u_1, u_2, ..., u_j\}$ be the set of the first j vertices. The cut $\nabla(S_j) = \{u_i u_k \in E : i \leq j < k\}$ is called the cut at $[j, j+1]$. Then the cutwidth $c(G, f)$ is the maximum size of all these cuts $\nabla(S_j)$, $j = 1, 2, ..., n-1$, i.e.,

$$c(G, f) = \max_{1 \leq j < n} |\nabla(S_j)|.$$

Concerning the cutwidth for graphs, some exact results on special graphs, e.g., the complete graphs K_n, the complete bipartite graphs $K_{m,n}$, the n-cubes Q_n, the complete k-ary trees, the trees with diameter at most 4, and the meshes $P_m \times P_n, P_m \times C_n, C_m \times C_n, K_m \times P_n, K_m \times C_n, K_m \times K_n$, have been obtained in the literature [9, 10, 11, 13]. The relations between cutwidth and other graph-theoretic parameters were studied in various aspects [2, 7]. The critical graphs with cutwidth at most 3 have been investigated in literature [8].

In a theoretical point of view, the cutwidth has the following basic properties.

Proposition 1.1 [8]. (1) If G' is a subgraph of G, then $c(G') \leq c(G)$. (2) If G' is homeomorphic to G (i.e., they can both be obtained from the same graph by inserting new vertices of degree two into its edges, called a subdivision of the graph), then $c(G') = c(G)$.

Based on these properties, we may define the cutwidth critical graphs as follows. A graph G is said to be *k-cutwidth critical* if

(1) $c(G) = k$;

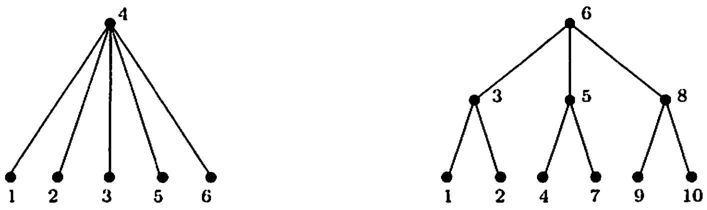
(2) for every proper subgraph G' of G, $c(G') < k$;

(3) G is homeomorphically minimal, that is, G is not a subdivision of any simple graph.

Lemma 1.2 [8]. The unique 1-cutwidth critical graph is K_2. The only 2-cutwidth critical graphs are K_3 and $K_{1,3}$. All 3-cutwidth critical graphs are H_1, H_2, H_3, H_4 and H_5, where H_1 is star $K_{1,5}$; H_2 is a tree with diameter 4 obtained by identifying a pendant vertex in three copies of star

$K_{1,3}$; H_3 is obtained from H_2 by replacing a claw $K_{1,3}$ by a triangle K_3; H_4 is a 'crown' made of a cycle C_3 with a pendant edge in each vertex of it; H_5 is a cycle C_4 with a chord.

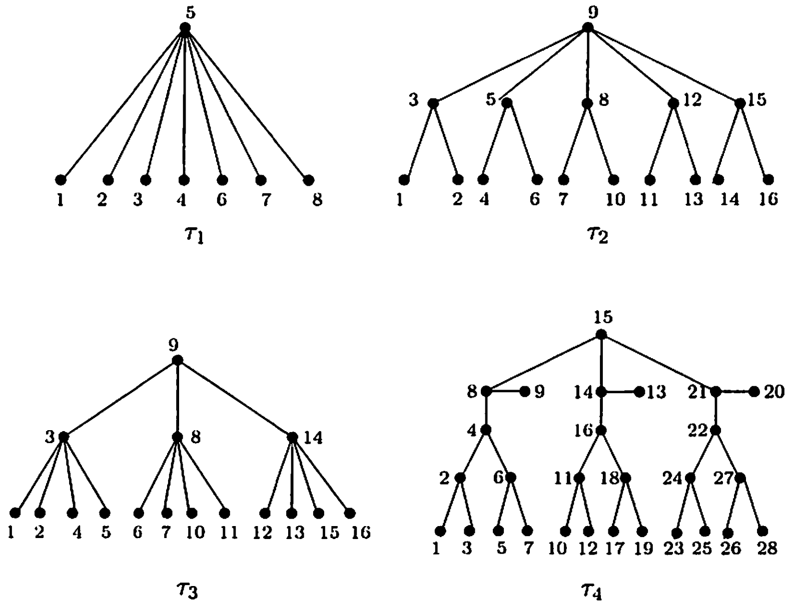
In particular, the only 3-cutwidth critical trees are H_1 and H_2 shown in Figure 1.



H_1 H_2

Figure 1. The 3-cutwidth critical trees

The main result of this paper is a characterization of the 4-cutwidth critical trees which is a further study of literature [8]. All of them are the 12 trees illustrated in Figure 2 (the numbers in each tree represent an *optimal* labeling).



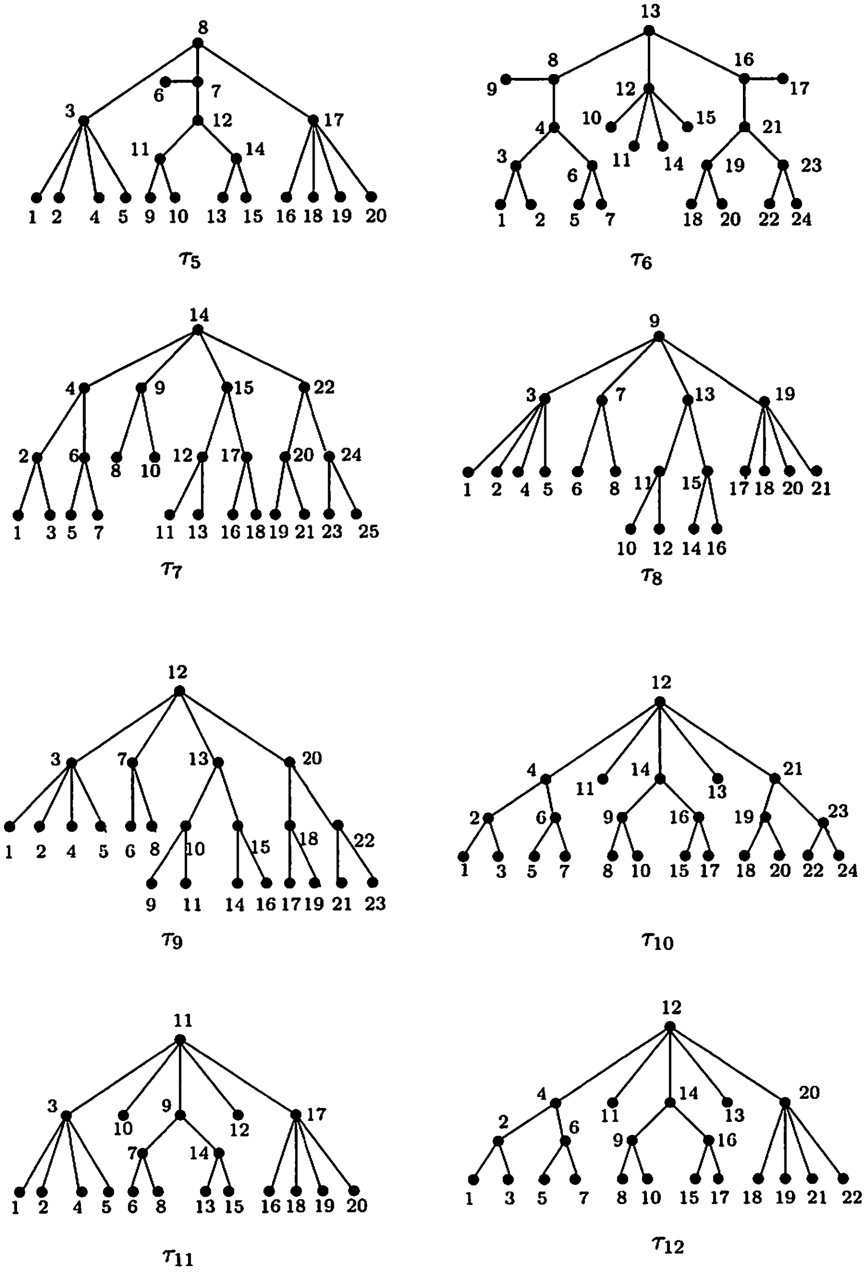


Figure 2. The 4-cutwidth critical trees

From this, we obtain the forbidden subtrees characterization of trees with cutwidth three as follows: A tree T has cutwidth 3 if and only if it is not homeomorphic to caterpillar (a tree which yields a path when all its pendent vertices are removed) and it does not contain any subtree homeomorphic to one of $\{\tau_i : i = 1, ..., 12\}$ in Figure 2.

A similar work has been done for the treewidth and pathwidth. A graph G is said to be k-treewidth (pathwidth) critical if G has treewidth (pathwidth) k and there is no proper minor G' of G having treewidth (pathwidth) k. The set of critical graphs with treewidth 3 consists of 4 elements [1, 14], and the critical graph set for pathwidth 3 is comprised of 110 graphs including 10 trees [6, 15]. The critical graphs for other parameters are worthy of further study; And these critical graphs are bound to be obtained, since they are finite [12].

The rest of this paper is organized as follows. Section 2 presents some preliminary results. Section 3 is devoted to the proof of the main results. Section 4 gives a summary.

2 Preliminaries

The following is an obvious lower bound.

Lemma 2.1. Let $\Delta(G)$ denote the maximum degree of G. Then $c(G) \geq \lceil \Delta(G)/2 \rceil$.

The bound is attainable by a star $K_{1,n}$ whose cutwidth is $c(K_{1,n}) = \lceil n/2 \rceil$, or a caterpillar T, a tree which yields a path (the spine) when all its pendant vertices are removed, whose cutwidth is $c(T) = \lceil \Delta(T)/2 \rceil$ [8].

Let $T = (V, E)$ be a tree and $\{v; v_1, v_2, ..., v_r\} \subseteq V$. Define $T(v; v_1, v_2, ..., v_r)$ as the largest subtree of T that contains v but does not contain any of $v_1, v_2, ..., v_r$.

Lemma 2.2 [3]. Let T be a tree. Then $c(T) \leq k$ if and only if every vertex v of degree at least 2 has neighbors v_1,v_2 such that $c(T(v; v_1, v_2)) \leq k - 1$.

This Lemma has an equivalent version as follows.

Corollary 2.3. Let T be a tree. Then $c(T) \geq k$ if and only if there exists a nonpendant vertex v such that $c(T(v; v_1, v_2)) \geq k-1$ holds for any two neighbors v_1, v_2 of v. □

Theorem 2.4. Let T_1, T_2 and T_3 be $(k-1)$-cutwidth trees. If tree T is formed by identifying a pendant vertex in T_1, T_2 and T_3, then T is a

k-cutwidth tree.

Proof Let v_0 be the identified vertex and pendant vertices x, y, z its neighbors in T_1, T_2, T_3 respectively. We first show that $c(T) \leq k$. In fact, we can embed $T - \{v_0x, v_0z\}$ in the order

$$V(T_1 - v_0), V(T_2), V(T_3 - v_0)$$

and using the $(k-1)-$cutwidth embedding in each part. And then put the edges v_0x and v_0z back to the embedding. This increases the congestion at most one. Thus we get an embedding with cutwidth at most k. So $c(T) \leq k$. On the other hand, $c(T) \geq k$ is obvious by Corollary 2.3, since the vertex v_0 satisfies $c(T(v_0; v_1, v_2)) = k - 1$ for any two neighbors of x, y or z. This completes the proof. □

A tree T is of diameter 4 if it yields a star when all its pendant vertices are removed. Denote by v_0 the unique center of T, and $v_1, v_2, ..., v_m$ the neighbors of v_0. For $1 \leq i \leq m$, let $v_{i1}, v_{i2}, ..., v_{in_i-1}$ be the pendant vertices adjacent to v_i (where $n_i \geq 2$), then the star induced by $v_0, v_i, v_{i1}, v_{i2}, ..., v_{in_i-1}$ is denoted by T_i. It is obvious that $|V(T_i)| = n_i + 1$ $(1 \leq i \leq m)$ and $|V(T)| = \sum_{i=1}^{m} n_i + 1 = n$. We denote this tree by $T(m; n_1, n_2, ..., n_m)$.

Lemma 2.5 [10]. Let T=$T(m; n_1, n_2, ..., n_m)$ be a tree of diameter at most 4, where $n_1 \geq n_2 \geq ... \geq n_m \geq 2$. Then

$$c(T) = \max_{1 \leq i \leq m} (\lfloor \frac{i-1}{2} \rfloor + \lceil \frac{n_i}{2} \rceil).$$

3 4-Cutwidth Critical Trees

We consider the trees τ_1 — τ_{12} in turn. They can be classified into four groups. The first group consists of τ_1 and τ_2, where τ_1 is a star $K_{1,7}$ and τ_2 is obtained by identifying a pendant vertex of 5 copies of $K_{1,3}$.

Lemma 3.1. Trees τ_1 and τ_2 are 4-cutwidth critical.

Proof. Note that τ_1 is a star with $\Delta(\tau_1) = 7$. By lemma 2.1, we have $c(\tau_1) = 4$. In addition, since the maximal proper subtree is star $K_{1,6}$ whose cutwidth $c(K_{1,6}) = 3$, the cutwidth of any proper subtree is at most 3. So, τ_1 is 4-cutwidth critical.

Noting that τ_2 is of diameter 4, we can easily obtain that $c(\tau_2) = 4$ by the formula in Lemma 2.5. On the other hand, since any maximal proper subtree of τ_2 can be obtained by deleting a pendant vertex v_{ij} of v_i (see the definition of tree of diameter 4), which is homeomorphic to the proper subtree τ_2' in Figure 3 (a). The labeling of τ_2' shows that $c(\tau_2') \leq 3$. So we can assert that the cutwidth of any proper subtree of τ_2 is at most three. Therefore, tree τ_2 is also 4-cutwidth critical. Thus the proof is completed. $\Box$

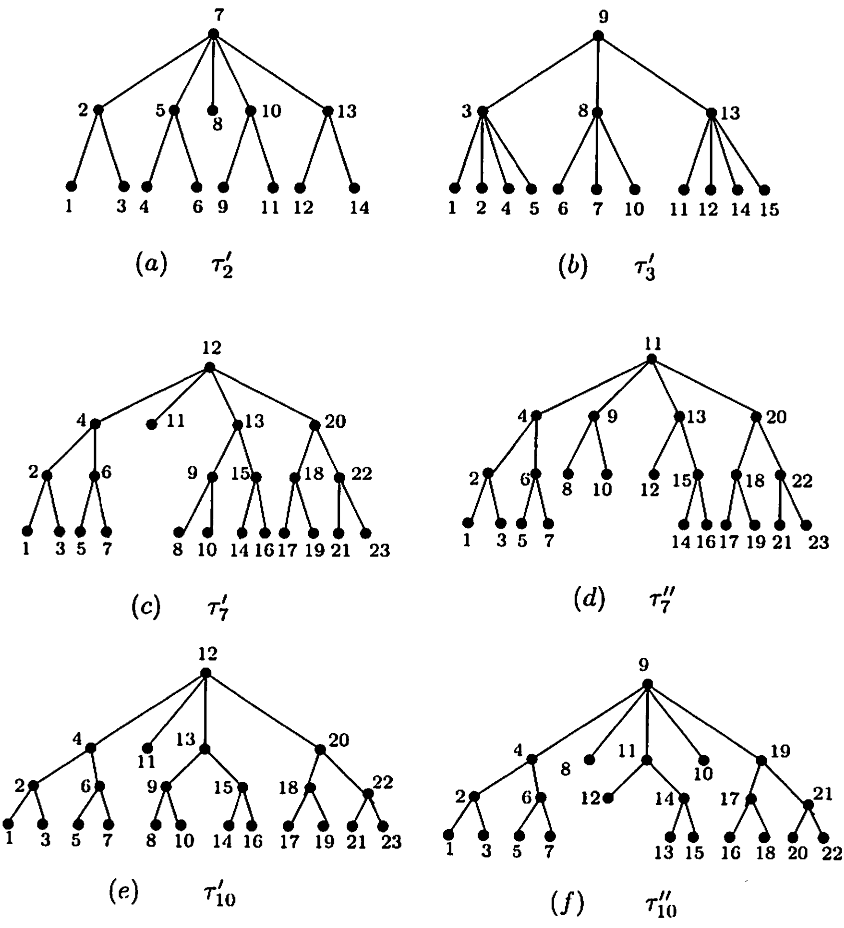


Figure 3. Proper subtrees of τ_2, τ_3, τ_7 and τ_{10}

The second group consists of τ_3, τ_4, τ_5 and τ_6 which are obtained by identifying a pendant vertex of three copies of H_1 or H_2.

Lemma 3.2. Trees τ_3, τ_4, τ_5 and τ_6 are 4-cutwidth critical.

Proof. Since τ_3, τ_4, τ_5 and τ_6 are obtained by identifying a pendant vertex of three copies of H_1 or H_2 respectively, we have $c(\tau_i) = 4$ for $i = 3, 4, 5, 6$ by Theorem 2.4.

Any maximal proper subtree of τ_3 is obtained by deleting a pendant vertex So that it is homeomorphic to τ_3' in Figure 3(b). Since the labeling of τ_3' in Figure 3(b) shows that the cutwidth $c(\tau_3')$ is at most three, we assert that the cutwidth of any proper subtree of τ_3 is at most 3. Therefore, tree τ_3 is 4-cutwidth critical tree. Similarly , we can also show that τ_4, τ_5, τ_6 are all 4-cutwidth critical. This completes the proof. □

The third group consists of τ_7, τ_8, τ_9 in which four branches are leading from a vertex as follows: one branch is a $K_{1,3}$; at least one other branch and this $K_{1,3}$ from a subtree homeomorphic to H_2 (with a subdivision on a nonpendant edge); the remaining branches (at most two) are H_1.

Lemma 3.3. Trees τ_7, τ_8 and τ_9 are 4-cutwidth critical.

Proof. The labeling f of τ_7 in Figure 2 asserts that cutwidth $c(\tau_7) \leq 4$. We now prove that $c(\tau_7) \geq 4$. Denote the vertex of degree 4 in τ_7 by v_0 and its neighbors by v_1, v_2, v_3, v_4. Since for any v_i and v_j $(1 \leq i < j \leq 4)$, the tree $\tau_7(v_0; v_i, v_j)$ contains an H_2, it follows that $c(\tau_7(v_0; v_i, v_j)) \geq 3$. Thus, by Corollary 2.3, we have $c(\tau_7) \geq 4$. Consequently, $c(\tau_7) = 4$.

On the other hand, τ_7 has two classes of maximal proper subtrees, one of which is obtained by deleting a pendant vertex with distance 2 from v_0, another by deleting a pendant vertex with distant 3. Since each maximal proper subtree has a vertex of 2 degree, the first class is homeomorphic to τ_7' in Figure 3(c), and the second class homeomorphic to τ_7'' in Figure 3(d). The labbelings of τ_7' and τ_7'' in Figure 3(c) and (d) show that $c(\tau_7') \leq 3$ and $c(\tau_7'') \leq 3$. By Proposition 1.1, the cutwidth of any subtree of τ_7 is at most three. Therefore, tree τ_7 is 4-cutwidth critical.

In the same way, we can show that τ_8 and τ_9 are also 4-cutwidth critical tree. The lemma follows. □

The fourth group consists of $\tau_{10}, \tau_{11}, \tau_{12}$ which are obtained from the previous group by contracting an edge.

Lemma 3.4. Trees τ_{10}, τ_{11} and τ_{12} are 4-cutwidth critical.

Proof. The labeling f of τ_{10} in Figure 2 implies that cutwidth $c(\tau_{10}) \leq 4$. We now prove that $c(\tau_{10}) \geq 4$. Denote the vertex of degree 5 in τ_{10} by v_0 and its neighbors by v_1, v_2, v_3, v_4, v_5. Since for any v_i and v_j $(1 \leq i < j \leq 5)$, it is apparent that the tree $\tau_{10}(v_0; v_i, v_j)$ contains an H_2, $c(\tau_{10}(v_0; v_i, v_j)) \geq 3$. Consequently, we have $c(\tau_{10}) \geq 4$ according to

Corollary 2.3. Thus, $c(\tau_{10}) = 4$.

Like τ_7, τ_{10} has also two classes maximal subtrees, each of which is obtained by deleting a pendant vertex with distance 1 or 3 from v_0 and which are homeomorphic to τ'_{10} in Figure 3(e) or τ''_{10} in Figure 3(f). The labbelings of τ'_{10} and τ''_{10} in Figure 3(e) and (f) show that $c(\tau'_{10}) \leq 3$ and $c(\tau''_{10}) \leq 3$. So, the cutwidth of any proper subtree of τ_{10} is at most three, that is to say, τ_{10} is 4-cutwidth critical.

The 4-cutwidth that τ_{11} and τ_{12} are 4-cutwidth critical can be proved by the same method. The proof is complete. □

Theorem 3.5. All 4-cutwidth critical trees are $\tau_1, \tau_2, \ldots, \tau_{12}$.

Proof. We have shown that τ_i $(i = 1, 2, \ldots, 12)$ are 4-cutwidth critical by Lemma 3.1–3.4. Let T be a 4-cutwidth critical tree and $d_T(\cdot)$ denote the degree of vertex. By definition, $d_T(v) \geq 3$ except pendant vertices. So $\Delta(T) \geq 3$. If $\Delta(T) \geq 7$, then $\tau_1 \subseteq T$ and thus $T = \tau_1$ by the minimality of T. In addition, we know that T is not homeomorphic a caterpillar with $\Delta(T) \leq 6$ by Lemma 2.1.

We now consider the case of $\Delta(T) \leq 6$. Due to Corollary 2.3, let v_0 be the nonpendant vertex such that for any two neighbors v_i, v_j, $c(T(v_0; v_i, v_j)) \geq 3$, and let T_i be the largest subtree of T containing v_0 and its neighbor v_i but not containing any other neighbors v_j $(j \neq i)$. Since T is 4-cutwidth critical, $c(T(v_0; v_i, v_j)) \neq 4$ (otherwise yielding a contradiction to the minimality).

Case 1: T includes at least 3 subtrees T_i whose cutwidth is $c(T_i) = 3$. Then 3-cutwidth critical trees H_1 or H_2 (see Figure 1) must be included in each T_i. Hence T must be one of $\{\tau_3, \tau_4, \tau_5, \tau_6\}$ by minimality. In particular, this case must appear when $d_T(v_0) = 3$.

Case 2: $d_T(v_0) = 4$. Note that $c(T(v_0; v_i, v_j)) = 3$ for any two neighbors v_i, v_j of v_0 $(1 \leq i < j \leq 4)$, and the degree of v_0 is two in subtree $T(v_0; v_i, v_j)$. If one neighbor of v_0, say v_1, is a pendant vertex of T, then the other subtrees T_2, T_3, T_4 must have cutwidth 3, thus the subtree T_1 (namely the edge v_0v_1) can be deleted, contradicting that T is critical. So, we may assume that all neighbors v_1, v_2, v_3, v_4 of v_0 are not pendant. Due to that T is critical, among all subtrees $T(v_0; v_i, v_j)$ $(1 \leq i < j < 4)$, there must be one being minimal (if the degree two vertex v_0 is ignored, then it is critical). Therefore, at least one subtree $T(v_0; v_i, v_j)$ is an H_2 with v_0 as a subdivision vertex; and the subtree T_i and T_j in the remaining part may contain an H_1. By the minimality, T is one of $\{\tau_7, \tau_8, \tau_9\}$.

Case 3: $d_T(v_0) = 5$. If all neighbors v_i of v_0 have $d_T(v_i) \geq 3$ $(i = 1, 2, 3, 4, 5)$, then τ_2 is included in T and thus $T = \tau_2$ by the minimality. If only one neighbor of v_0, say v_1, is pendant, then by $c(T_1 \bigcup T_i \bigcup T_j) = 3$ $(2 \leq i < j \leq 5)$, the edge v_0v_1 can be deleted without effect on $c(T) = 4$, but contradicting that T is critical. By $c(T(v_0; v_i, v_j)) = 3$, it is impossible that v_0 has three or more pendant neighbors. So, we may assume that there are two neighbors of v_0 being pendant. By the fact that $c(T(v_0; v_i, v_j)) = 3$ for any two neighbors v_i, v_j of v_0 $(1 \leq i < j \leq 5)$ and that T is critical, it can be seen that there must be a subtree $T(v_0; v_i, v_j)$ being an H_2 containing those two pendant neighbors of v_0. And the subtree T_i or T_j in the remaining parts may contain an H_1. Therefore, T is one of $\{\tau_{10}, \tau_{11}, \tau_{12}\}$ by the minimality.

Case 4: $d_T(v_0) = 6$. By using the fact that $c(T(v_0; v_i, v_j)) = 3$ for any i and j $(1 \leq i < j \leq 6)$, it can be deduced that T must contain a subtree in Case 2 or Case 3, which contradicts that T is critical. This establishes the proof. □

Corollary 3.6. A tree T has cutwidth at most 3 if and only if it does not contain any subtree homeomorphic to one of $\{\tau_1, \tau_2, ..., \tau_{12}\}$. □

4 Concluding Remarks

The foregoing discussion characterizes the set of 4-cutwidth critical trees. As to the critical trees with cutwidth k $(k \geq 5)$, we have obtained some results. For example, star $K_{1,2k-1}$ is a critical tree with cutwidth k.

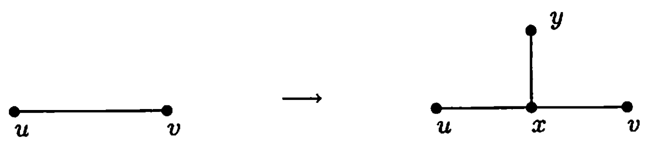


Figure 4. Definition of T_1'

If tree T_1' is obtained from star $K_{1,2k-3}$ by replacing every edge uv of it with tree shown in Figure 4, where $d_{K_{1,2k-3}}(u) = 2k-3$, $d_{K_{1,2k-3}}(v) = 1$, x and y are new vertices and y is a new pendant vertex and tree T_2' is formed by identifying a pendant vertex in three copies of the star $K_{1,2k-3}$, i.e., T_2' has a vertex v_0 of degree 3, three vertices v_1, v_2 and v_3 of degree $(2k-3)$ which are adjacent to v_0, $(6k-12)$ pendant vertices. Then trees T_1' and T_2' are all k-cutwidth critical.

Let $T_3(k)$ denote the critical tree with $d_T(v) = 3$ except the pendant vertices and with cutwidth k. For instance, $T_3(1) = K_2$, $T_3(2) = K_{1,3}$, $T_3(3) = H_2$ and $T_3(4) = \tau_4$. For $k > 1$, the k-cutwidth critical tree $T_3(k)$ can be formed by identifying a pendant vertex in three copies of $(k-1)$-cutwidth critical tree $T_3(k-1)$.

From the theorem 2.4, we find a method of constructing a class k-cutwidth critical tree from $(k-1)$-cutwidth critical trees as follows: For $k > 0$, let T_1, T_2 and T_3 be arbitrary three $(k-1)$-cutwidth critical trees (not necessarily distinct). If tree T is formed by identifying a pendant vertex in T_1, T_2 and T_3, then tree T is a k-cutwidth critical tree. However, this method can not construct all k-cutwidth critical trees.

A further task is to characterize the set of 4-cutwidth nontree critical graphs which includes K_4 and all 5-critical graphs. More general properties of critical trees are expected.

References

[1] S. Arnborg, A. Proskurowski and D. G. Corneil, Forbidden minors characterization of partial 3-trees, Discrete Math., 80(1990) 1–19.

[2] F. R. K. Chung and P. D. Seymour, Graphs with small bandwidth and cutwidth, Discrete Math., 75(1985) 268–277.

[3] M. J. Chung, F. Makedon, I. H. Sudborough and J. Turner, Polynomial time algorithms for the min cut problem on degree restricted trees, SIAM J.Comput., 14(1985) 158–177.

[4] J. Diaz, J. Petit and M. Serna, A survey of graph layout problems, ACM Computing Surveys, 34(2002) 313–356.

[5] M. R. Garey and D. S. Johnson, Computers and Intractability: A Guide to the Theory of NP-Completeness, W. H. Freeman & Company, San Francisco, 1979.

[6] N. G. Kinnersley and M. A. Langston, Obstruction set isolation for the gate matrix layout problem, Discrete Appl Math., 54(1994)169–213.

[7] E. Korach and N. Solel, Treewidth, pathwidth and cutwidth, Discrete Appl. Math., 43(1993) 97–101.

[8] Y. Lin and A. Yang, On 3-cutwidth critical graphs, Discrete Math., 275(2004) 339–346.

[9] Y. Lin, X. Li and A. Yang, A degree sequence method for the cutwidth problem of graphs, Appl. Math. J. Chinese Univ., Ser. B, 17(2)(2002) 125–134.

[10] Y. Lin, The cutwidth of trees with diameter at most 4, Appl. Math. J. Chinese Univ., Ser. *B*, 18(3)(2003) 361–369.

[11] H. Liu and J. Yuan, Cutwidth problem on graphs, Appl. Math. J. Chinese Univ., Ser. A, 10(3)(1995) 339–348.

[12] N. Robertson and P.D. Seymour, Graph minors XX. Wagner's conjecture, J. Combinatorial Theory, Series B, Vol. 92(2)(2004) 325-357.

[13] J. Rolin, O. Sykora and I. Vrto, Optimal cutwidth of meshes, Lecture Notes in Computer Science 1017, Springer Verlag 1995, 252–264.

[14] A. Satyanarayana and L. Tung, A charaterization of partial 3-trees, Networks, 20(1990) 299–322.

[15] A. Takahashi, S. Ueno and Y. Kajitani, Minimal acyclic forbidden minors for the family of graphs with bounded pathwidth, Discrete math., 127(1994) 293–304.

[16] M. Yannakakis, A polynomial algorithm for the min-cut arrangement of trees, J. ACM, 32(1985) 950–989.