On the Metric Dimension of Generalized Petersen Graphs

Imran Javaid , Shabbir Ahmad, M. Naeem Azhar

Center for Advanced Studies in Pure and Applied Mathematics,
Bahauddin Zakariya University Multan, Pakistan.
{imranjavaid45, shabiraleem}@gmail.com, mna303@hotmail.com

Abstract

A family of connected graphs $\mathcal{G}$ is said to be a family with constant metric dimension if its metric dimension is finite and does not depend upon the choice of G in $\mathcal{G}$. In this paper, we study the metric dimension of the generalized Petersen graphs $P(n, m)$ for $n = 2m + 1$ and $m \geq 1$ and give partial answer of the question raised in [9]: Is $P(n, m)$ for $n \geq 7$ and $3 \leq m \leq \lfloor \frac{n-1}{2} \rfloor$, a family of graphs with constant metric dimension? We prove that the generalized Petersen graphs $P(n, m)$ with $n = 2m + 1$ have metric dimension 3 for every $m \geq 2$.

Keywords: resolving set, metric dimension, generalized Petersen graphs.

1 Introduction

Let G be a connected graph and distance between two distinct vertices v and w in G, denoted by $d(v, w)$, is the length of a shortest path between them. For an ordered subset $W = \{w_1, w_2, \ldots, w_k\}$ of vertices and a vertex v in G, the code of v with respect to W is the ordered k-tuple $c_W(v) = (d(v, w_1), d(v, w_2), \ldots, d(v, w_k))$. The set W is a *resolving set*[7] (or locating set[13]) for G if every two vertices of G have distinct codes. The *metric dimension* of G is the minimum cardinality of a resolving set for G, denoted by $\beta(G)$. A resolving set containing a minimum number of vertices is called a *basis* for G [1].

The concepts of resolving set and metric dimension have previously appeared in the literature (see [1],[3],[4],[7],[10],[12],[14]). Slater referred to the metric dimension of a graph as its location number and initiated the study of this invariant by its application to the placement of minimum number of Sonar/Loran detecting devices in a network so that the position of every vertex in the network can be uniquely described in terms of its distances to the devices in the set [14]. These concepts also have some applications in chemistry for representing chemical compounds ([3],[10]) and to the problems of pattern recognition and image processing, some of which involve the use of hierarchical data structures [12]. It was noted in [6] that the problem of finding the metric dimension is NP-hard. Khuller et al. [11] gave a construction showing that the metric dimension of a graph is NP-hard. Their interest in this invariant was motivated by the navigation of robots in a graph space.

From the definition, it can be observed that the property of a given set W of vertices of a graph G to be a resolving set of G can be verified by investigating the vertices of $V(G) \setminus W$ since every vertex $w \in W$ is the only vertex of G whose distance from w is 0. If $d(x,t) \neq d(y,t)$, we shall say that vertex t distinguishes vertices x and y.

For each odd integer $n = 2m+1 \geq 3$, the generalized Petersen graph $P(n,m)$ is a graph with vertex set $\mathcal{O} \cup \mathcal{I}$ where $\mathcal{O} = \{O_i \mid 0 \leq i \leq n-1\}$ and $\mathcal{I} = \{I_i \mid 0 \leq i \leq n-1\}$, and edge set $E_1 \cup E_2 \cup E_3$, where $E_1 = \{O_iO_{i+1} \mid 0 \leq i \leq n-1\}$, $E_2 = \{I_iI_{i+m} \mid 0 \leq i \leq n-1\}$ and $E_3 = \{O_iI_i \mid 0 \leq i \leq n-1\}$. Here and throughout the paper, the subscripts are to be taken as integers modulo n. Description of the graph $P(2m+1,m)$ and some of its properties may be found in [15], where it was introduced for the first time. An idea of the structure of these graphs may be obtained from the diagram for $P(5,2)$ given in Figure 1. These graphs are quite symmetrical,

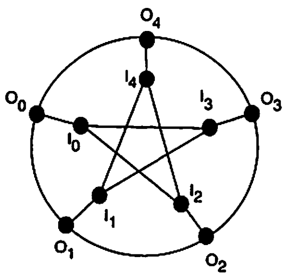

Figure 1: The Petersen graph $P(5,2)$

admitting the automorphism $O_i \mapsto O_{i+1}$, $I_i \mapsto I_{i+1} (0 \leq i \leq n-1)$ which generates a subgroup of automorphisms with vertex orbits $\mathcal{O}, \mathcal{I}$ and edge orbits E_1, E_2, E_3. Moreover, it follows from a result of Frucht et al. [5] that $P(n,m)$ is vertex-transitive only in the case $n = 3$ (the 3-sided prism) and $n = 5$ (the Petersen graph) shown in Figure 1. Generalized Petersen graphs $P(n,m)$ form an important class of 3-regular graphs with $2n$ vertices and $3n$ edges. For our purpose, we call the vertices $O_0, O_1, \ldots, O_{n-1}$, *outer vertices* and the vertices $I_0, I_1, \ldots, I_{n-1}$, *inner vertices.*

2 Main Results

Let $\mathcal{F}=(G_n)_{n\geq 1}$ be a family of graph G_n of order $\varphi(n)$ for which $lim_{n\to\infty} \varphi(n) = \infty$. If there exist a constant $M > 0$ such that $\beta(G_n) \leq M$ for every $n \geq 1$ then we shall say that $\mathcal{F}$ has bounded metric dimension otherwise $\mathcal{F}$ has unbounded metric dimension. If all graphs in $\mathcal{F}$ have same metric dimension (which does not depend on n) then $\mathcal{F}$ is called a family with constant metric dimension [9].

Since in applications, elements of resolving sets are referred to as censors or detecting devices, so it is natural to look for graphs with constant metric dimension or graphs whose metric dimension does not increase with increase in the number of vertices of the graphs. With this motivation, Javaid et al. [9] considered the generalized Petersen graphs $P(n,2)$ and proved that it is a family of graphs with constant metric dimension by showing that $\beta(P(n,2)) = 3$ for every $n \geq 5$, and asked the following question:

Question:[9] Is $P(n,m)$ for $n \geq 7$ and $3 \leq m \leq \lfloor \frac{n-1}{2} \rfloor$, a family of graphs with constant metric dimension?

Imran et al. [8] gave partial answer of this question by considering the generalized Petersen graphs $P(n,3)$ and proved the following theorem:

Theorem 1 *[8] For a generalized Petersen graph $P(n,3)$,*
(a) $\beta(P(n,3)) = 4$ for $n \equiv 0 (mod\ 6)$ and $n \geq 24$,
(b) $\beta(P(n,3)) = 3$ for $n \equiv 1 (mod\ 6)$ and $n \geq 25$,
(c) $\beta(P(n,3)) \leq 5$ for $n \equiv 2 (mod\ 6)$ and $n \geq 8$,
(d) $\beta(P(n,3)) \leq 4$ for $n \equiv 3,4,5 (mod\ 6)$ and $n \geq 17$.

In this paper, we consider a more general family of generalized Petersen graphs $P(n,m)$ with $n = 2m+1$ and $m \geq 1$, and partially answer the question mentioned above by proving the following result:

Theorem 2 *Let $P(n,m)$ be generalized Petersen graphs with $n = 2m+1$ and $m \geq 1$. Then* $\beta(P(n,m)) = \begin{cases} 2 & \text{if } m = 1, \\ 3 & \text{if } m > 1. \end{cases}$

We prove this theorem by proving following five lemmas. Throughout the proofs of lemma 1-4 each integer entry in codes tables is the distance between the vertices of column 1, column 5 (if exist) and the vertices in row 1. Each row represents the code of a vertex lying in column 1 and column 5 (if exist) of that row.

Lemma 1 *Let $P(n,m)$ be generalized Petersen graphs with $n = 2m+1$ and $m \equiv 0 (mod\ 4)$, then $\beta(P(n,m)) \leq 3$.*

Proof. For $m = 4$, it is easy to see that $W = \{I_0, I_4, O_5\}$ is a minimal resolving set for $P(9,4)$. For $m \geq 8$ and $m \equiv 0 \pmod 4$, and for the chosen index i such that $0 \leq i \leq n-1$, we shall show that $W = \{I_i, I_{i+2k+2}, O_{i+m}\}$ where $k = \frac{m}{4}$ is a resolving set for $P(n,m)$. For $m = 8$, the codes of the vertices in $V(P(17,8)) \setminus W$ with respect to $W = \{I_0, I_6, O_8\}$ are in Tables 1 and 2.

Table 1: Codes of the outer vertices of $P(17,8)$

$d(.,.)$	I_0	I_6	O_8	$d(.,.)$	I_0	I_6	O_8
O_0	1	4	3	O_1	2	5	4
O_2	3	5	5	O_3	4	4	5
O_4	5	3	4	O_5	5	2	3
O_6	4	1	2	O_7	3	2	1
O_9	2	4	1	O_{10}	3	5	2
O_{11}	4	5	3	O_{12}	5	4	4
O_{13}	5	3	5	O_{14}	4	2	5
O_{15}	3	2	4	O_{16}	2	3	3

Table 2: Codes of the inner vertices of $P(17,8)$

$d(.,.)$	I_0	I_6	O_8	$d(.,.)$	I_0	I_6	O_8
I_1	2	6	3	I_2	4	6	4
I_3	5	5	5	I_4	6	4	5
I_5	6	2	4	I_7	3	2	2
I_8	1	4	1	I_9	1	5	2
I_{10}	3	6	3	I_{11}	5	6	4
I_{12}	6	5	5	I_{13}	6	3	5
I_{14}	5	1	4	I_{15}	4	1	3
I_{16}	2	3	2				

It can be seen that all the vertices in $V(P(17,8)) \setminus W$ have distinct codes with respect to W. Now for $m > 8$ the codes of the vertices of $V(P(n,m)) \setminus W$ are: $c_W(I_{i+1}) = (2, 2k+2, 3)$, $c_W(I_{i+2m}) = (2, 2k, 2)$, and in Tables 3 and 4.

Table 3: Codes of the outer vertices of $P(n,m)$

$d(.,.)$	I_i	I_{i+2k+2}	O_{i+m}
$O_{i+j} : 0 \leq j \leq 1$	$j+1$	$2k+j$	$j+3$
$O_{i+j+2} : 0 \leq j \leq 2k-4$	$j+3$	$2k-j+1$	$j+5$
$O_{i+2k+j-1} : 0 \leq j \leq 1$	$2k+j$	$4-j$	$2k-j+1$
$O_{i+2k+j+1} : 0 \leq j \leq 1$	$2k-j+1$	$2-j$	$2k-j-1$
$O_{i+m-j-1} : 0 \leq j \leq 2k-4$	$j+3$	$2k-j-2$	$j+1$
$O_{i+m+j+1} : 0 \leq j \leq 1$	$j+2$	$2k+j$	$j+1$
$O_{i+m+j+3} : 0 \leq j \leq 2k-3$	$j+4$	$2k-j+1$	$j+3$
$O_{i+m+j+2k+1} : 0 \leq j \leq 1$	$2k-j+1$	$3-j$	$2k+1$
$O_{i+2m-j} : 0 \leq j \leq 2k-3$	$j+2$	$2k-j-1$	$j+3$

Table 4: Codes of the inner vertices of $P(n,m)$

$d(.,.)$	I_i	I_{i+2k+2}	O_{i+m}
$I_{i+j+2} : 0 \leq j \leq 2k-3$	$j+4$	$2k-j+2$	$j+4$
$I_{i+2k+j} : 0 \leq j \leq 1$	$2k+2$	$4-2j$	$2k-j+1$
$I_{i+2k+j+3} : 0 \leq j \leq 1$	$2k-j$	$2j+2$	$2k-j-2$
$I_{i+2k+j+5} : 0 \leq j \leq 2k-7$	$2k-j-2$	$j+5$	$2k-j-4$
$I_{i+m-j} : 0 \leq j \leq 1$	$2j+1$	$2k-j$	$j+1$
$I_{i+m+j+1} : 0 \leq j \leq 1$	$2j+1$	$2k+j+1$	$j+2$
$I_{i+m+j+3} : 0 \leq j \leq 2k-3$	$j+5$	$2k-j+2$	$j+4$
$I_{i+m+2k+j+1} : 0 \leq j \leq 1$	$2k-j+2$	$3-2j$	$2k-j+1$
$I_{i+m+2k+j+3} : 0 \leq j \leq 1$	$2k-j$	$2j+1$	$2k-j-1$
$I_{i+2m-j-1} : 0 \leq j \leq 2k-6$	$j+4$	$2k-j-1$	$j+3$

From these tables, one can see that all the vertices of $P(n,m)$ lying in column 1 of Table 3 and Table 4 have distinct codes with respect to W. Thus W is resolving set for $P(n,m)$. Hence $\beta(P(n,m)) \leq 3$ for all $m \geq 8$ and $m \equiv 0 \pmod 4$.

Lemma 2 *Let $P(n,m)$ be generalized Petersen graphs with $n = 2m+1$ and $m \equiv 1 (mod\ 4)$, then $\beta(P(3,1)) = 2$ and $\beta(P(n,m)) \leq 3$ for $m \geq 5$.*

Proof. For $m = 1$, $\beta(P(3,1)) = 2$ since $P(3,1) \cong K_2 \square C_3$ and it was shown that $\beta(K_2 \square C_3)$=2 [2]. For $m = 5$, it is easy to see that $W = \{I_2, I_4, O_5\}$ is a minimal resolving set for $P(11,5)$. For $m > 5$ and $m \equiv 1 \pmod 4$, and for the chosen index i such that $0 \leq i \leq n-1$, we shall show that $W = \{I_{i+2}, I_{i+2k+2}, O_{i+m}\}$ where $k = \frac{m-1}{4}$ is a resolving set for $P(n,m)$.

The codes of the vertices in $V(P(n,m)) \backslash W$ with respect to W are: $c_W(I_{i+3}) = (2, 2k+1, 5)$, $c_W(I_{i+2k+1}) = (2k+1, 2, 2k+1)$, $c_W(I_{i+2k+3}) = (2k+3, 2, 2k-1)$ and in Tables 5 and 6. It can be noted that all the vertices of $P(n,m)$ lying in column 1 of Table 5 and Table 6 have distinct codes with respect to W. Thus W is resolving set for $P(n,m)$.

Table 5: Coding of the outer vertices of $P(n,m)$

$d(.,.)$	I_{i+2}	I_{i+2k+2}	O_{i+m}
$O_{i+j} : 0 \leq j \leq 1$	$3-j$	$2k+j+1$	$j+3$
$O_{i+j+2} : 0 \leq j \leq 2k-3$	$j+1$	$2k-j+1$	$j+5$
$O_{i+2k+j} : 0 \leq j \leq 2$	$2k+j-1$	$3-j$	$2k-j+1$
$O_{i+m-j-1} : 0 \leq j \leq 2k-3$	$j+5$	$2k-j-1$	$j+1$
$O_{i+m+j+1} : 0 \leq j \leq 1$	$3-j$	$2k+j+1$	$j+1$
$O_{i+m+j+3} : 0 \leq j \leq 2k-1$	$j+2$	$2k-j+1$	$j+3$
$O_{i+2m-j} : 0 \leq j \leq 2k-2$	$j+4$	$2k-j$	$j+3$

Table 6: Codes of the inner vertices of $P(n,m)$

$d(.,.)$	I_{i+2}	I_{i+2k+2}	O_{i+m}
$I_{i+j} : 0 \leq j \leq 1$	$4-2j$	$2k+j+2$	$j+2$
$I_{i+j+4} : 0 \leq j \leq 2k-4$	$j+4$	$2k-j$	$j+6$
$I_{i+m-j} : 0 \leq j \leq 2k-3$	$j+5$	$2k-j+1$	$j+1$
$I_{i+m+j+1} : 0 \leq j \leq 1$	$3-2j$	$2k+j+2$	$j+2$
$I_{i+m+j+3} : 0 \leq j \leq 1$	$2j+1$	$2k-j+2$	$j+4$
$I_{i+m+j+5} : 0 \leq j \leq 2k-5$	$j+5$	$2k-j$	$j+6$
$I_{i+m+2k+j+1} : 0 \leq j \leq 1$	$2k+j+1$	$3-2j$	$2k-j+2$
$I_{i+m+2k+j+3} : 0 \leq j \leq 1$	$2k-j+3$	$2j+1$	$2k-j$
$I_{i+2m-j} : 0 \leq j \leq 2k-4$	$j+5$	$2k-j+1$	$j+2$

Hence $\beta(P(n,m)) \leq 3$.

Lemma 3 *Let $P(n,m)$ be generalized Petersen graphs with $n = 2m+1$ and $m \equiv 2 (mod\ 4)$, then $\beta(P(n,m)) \leq 3$ for any $m \geq 2$.*

Proof. In [9], it was shown that $\beta(P(5,2)) = 3$. For $m = 6$, the codes of the vertices in $V(P(13,6)) \setminus W$ with respect to $W = \{I_3, I_4, O_7\}$ are in Tables 7 and 8. These tables show that all the vertices in $V(P(13,6)) \setminus W$ have distinct codes with respect to W. For $m \geq 10$ and $m \equiv 2 \pmod 4$, and for the chosen index i such that $0 \leq i \leq n-1$, we shall show that $W = \{I_{i+3}, I_{i+2k+2}, O_{i+m+1}\}$ where $k = \frac{m-2}{4}$ is a resolving set for $P(n,m)$.

Table 7: Codes of the outer vertices of $P(13,6)$

$d(.,.)$	I_3	I_4	O_7	$d(.,.)$	I_3	I_4	O_7
O_0	4	4	3	O_1	3	4	3
O_2	2	3	4	O_3	1	2	4
O_4	2	1	3	O_5	3	2	2
O_6	4	3	1	O_8	3	4	1
O_9	2	3	2	O_{10}	2	2	3
O_{11}	3	2	4	O_{12}	4	3	4

Table 8: Codes of the inner vertices of $P(13,6)$

$d(.,.)$	I_3	I_4	O_7	$d(.,.)$	I_3	I_4	O_7
I_0	5	5	2	I_1	4	5	2
I_2	2	4	3	I_5	4	2	3
I_6	5	4	2	I_7	5	5	1
I_8	3	5	2	I_9	1	3	3
I_{10}	1	1	4	I_{11}	3	1	4
I_{12}	5	3	3				

When $m = 10$, then $W = \{I_3, I_6, O_{11}\}$ resolves all vertices in $P(21,10)$ as the codes of the vertices are in Tables 9 and 10.

Table 9: Codes of the outer vertices of $P(21,10)$

$d(.,.)$	I_3	I_6	O_{11}	$d(.,.)$	I_3	I_6	O_{11}
O_0	4	6	3	O_1	3	6	3
O_2	2	5	4	O_3	1	4	5
O_4	2	3	6	O_5	3	2	6
O_6	4	1	5	O_7	5	2	4
O_8	6	3	3	O_9	6	4	2
O_{10}	5	5	1	O_{12}	3	6	1
O_{13}	2	5	2	O_{14}	2	4	3
O_{15}	3	3	4	O_{16}	4	2	5
O_{17}	5	2	6	O_{18}	6	3	6
O_{19}	6	4	5	O_{20}	5	5	4

Table 10: Codes of the inner vertices of $P(21,10)$

$d(.,.)$	I_3	I_6	O_{11}	$d(.,.)$	I_3	I_6	O_{11}
I_0	5	7	2	I_1	4	7	2
I_2	2	6	3	I_4	2	4	5
I_5	4	2	6	I_7	6	2	5
I_8	7	4	4	I_9	7	5	3
I_{10}	6	6	2	I_{11}	5	7	1
I_{12}	3	7	2	I_{13}	1	6	3
I_{14}	1	5	4	I_{15}	3	3	5
I_{16}	5	1	6	I_{17}	6	1	6
I_{18}	7	3	5	I_{19}	7	5	4
I_{20}	6	6	3				

From these tables, it is evident that all the vertices of $V(P(21,10)) \setminus W$ have distinct codes with respect to W. Now codes of the vertices of $V(P(n,m)) \setminus W$ with $m > 10$ and $m \equiv 2 (mod\ 4)$ are: $c_W(O_i) = (4, 2k+2, 3)$, $c_W(I_i) = (5, 2k+3, 2)$, $c_W(I_{i+4}) = (2, 2k, 5)$, $c_W(I_{i+2k+1}) = (2k, 2, 2k+2)$, $c_W(I_{i+2k+3}) = (2k+2, 2, 2k+1)$, and in Tables 11 and 12. Note that all the vertices of $P(n,m)$ lying in column 1 of Table 11 and Table 12 have distinct codes with respect to W. Thus W is a resolving set for $P(n,m)$.

Table 11: Codes of the outer vertices of $P(n, m)$

$d(.,.)$	I_{i+3}	I_{i+2k+2}	O_{i+m+1}
$O_{i+j+1} : 0 \leq j \leq 1$	$3-j$	$2k-j+2$	$j+3$
$O_{i+j+3} : 0 \leq j \leq 2k-3$	$j+1$	$2k-j$	$j+5$
$O_{i+2k+j+1} : 0 \leq j \leq 1$	$2k+j-1$	$2-j$	$2k-j+2$
$O_{i+2k+j+3} : 0 \leq j \leq 1$	$2k+j+1$	$j+2$	$2k-j$
$O_{i+m-j} : 0 \leq j \leq 2k-3$	$j+5$	$2k-j+1$	$j+1$
$O_{i+m+j+2} : 0 \leq j \leq 1$	$3-j$	$2k-j+2$	$j+1$
$O_{i+m+j+4} : 0 \leq j \leq 2k-2$	$j+2$	$2k-j$	$j+3$
$O_{i+m+j+2k+3} : 0 \leq j \leq 1$	$2k+j+1$	$j+2$	$2k+2$
$O_{i+2m-j} : 0 \leq j \leq 2k-3$	$j+5$	$2k-j+1$	$j+4$

Table 12: Codes of the inner vertices of $P(n, m)$

$d(.,.)$	I_{i+3}	I_{i+2k+2}	O_{i+m+1}
$I_{i+j+1} : 0 \leq j \leq 1$	$4-2j$	$2k-j+3$	$j+2$
$I_{i+j+5} : 0 \leq j \leq 2k-5$	$j+4$	$2k-j-1$	$j+6$
$I_{i+2k+j+4} : 0 \leq j \leq 1$	$2k+3$	$j+4$	$2k-j$
$I_{i+m-j+1} : 0 \leq j \leq 2k-3$	$j+5$	$2k-j+3$	$j+1$
$I_{i+m+j+2} : 0 \leq j \leq 1$	$3-2j$	$2k-j+3$	$j+2$
$I_{i+m+j+4} : 0 \leq j \leq 1$	$2j+1$	$2k-j+1$	$j+4$
$I_{i+m+6} : 0 \leq j \leq 2k-6$	$j+5$	$2k-j-1$	$j+6$
$I_{i+m+2k+j+1} : 0 \leq j \leq 1$	$2k+j$	$3-2j$	$2k+j+1$
$I_{i+m+2k+j+3} : 0 \leq j \leq 1$	$2k+j+2$	$2j+1$	$2k-j+2$
$I_{i+2m-j} : 0 \leq j \leq 2k-3$	$j+6$	$2k-j+2$	$j+3$

Hence $\beta(P(n, m)) \leq 3$ for all $m \geq 6$ and $m \equiv 2(mod\ 4)$.

Lemma 4 *Let $P(n, m)$ be generalized Petersen graphs with $n = 2m + 1$ and $m \equiv 3(mod\ 4)$, then $\beta(P(n, m)) \leq 3$.*

Proof. For $m = 3$, it is easy to see that $W = \{I_6, I_2, O_2\}$ is a minimal resolving set for $P(7, 3)$. For $m \geq 7$ and $m \equiv 3 \pmod 4$, and for chosen index i such that $0 \leq i \leq n-1$,

Table 13: Codes of the outer vertices of $P(15, 7)$

$d(.,.)$	I_{14}	I_4	O_6	$d(.,.)$	I_{14}	I_4	O_6
O_0	2	5	4	O_1	3	4	5
O_2	4	3	4	O_3	5	2	3
O_4	4	1	2	O_5	3	2	1
O_7	2	4	1	O_8	3	5	2
O_9	4	4	3	O_{10}	5	3	4
O_{11}	4	2	5	O_{12}	3	2	4
O_{13}	2	3	3	O_{14}	1	4	3

Table 14: Codes of the inner vertices of $P(15, 7)$

$d(.,.)$	I_{14}	I_4	O_6	$d(.,.)$	I_{14}	I_4	O_6
I_0	2	6	3	I_1	4	5	4
I_2	5	4	5	I_3	6	2	4
I_5	3	2	2	I_6	1	4	1
I_7	1	5	2	I_8	3	6	3
I_9	5	5	4	I_{10}	6	3	5
I_{11}	5	1	4	I_{12}	4	1	3
I_{13}	2	3	2				

we shall show that $W = \{I_{i+2m}, I_{i+2k+2}, O_{i+m-1}\}$ where $k = \frac{m-3}{4}$ is a resolving set for $P(n, m)$. For $m = 7$, codes of the vertices in $V(P(15, 7)) \setminus W$ with respect to $W = \{I_{14}, I_4, O_6\}$ are in Tables 13 and 14, showing that all the vertices in $V(P(15, 7)) \setminus W$ have distinct codes with respect to W.

Now for $m > 7$ the codes of the vertices of $V(P(n, m)) \setminus W$ are: $c_W(O_{i+m+2k+2}) = (2k+2, 2, 2k+3)$, $c_W(O_{i+2m}) = (1, 2k+2, 3)$, $c_W(I_i) = (2, 2k+4, 3)$, $c_W(I_{i+2k+1}) = (2k+4, 2, 2k+2)$, $c_W(I_{i+2k+3}) = (2k+2, 2, 2k)$, $c_W(I_{i+2m-1}) = (2, 2k+2, 2)$, and in Tables 15 and 16.

Table 15: Codes of the outer vertices of $P(n, m)$

$d(.,.)$	I_{i+2m}	I_{i+2k+2}	O_{i+m-1}
$O_{i+j} : 0 \leq j \leq 2k-1$	$j+2$	$2k-j+3$	$j+4$
$O_{i+2k+j} : 0 \leq j \leq 1$	$2k+j+2$	$3-j$	$2k-j+2$
$O_{i+2k+j+2} : 0 \leq j \leq 2k-1$	$2k-j+2$	$j+1$	$2k-j$
$O_{i+m+j} : 0 \leq j \leq 1$	$j+2$	$2k+j+2$	$j+1$
$O_{i+m+j+2} : 0 \leq j \leq 2k-1$	$j+4$	$2k-j+2$	$j+3$
$O_{i+2m-j-1} : 0 \leq j \leq 2k-1$	$j+2$	$2k-j+1$	$j+3$

Table 16: Codes of the inner vertices of $P(n, m)$

$d(.,.)$	I_{i+2m}	I_{i+2k+2}	O_{i+m-1}
$I_{i+j+1} : 0 \leq j \leq 2k-1$	$j+4$	$2k-j+3$	$j+4$
$I_{i+2k+j+4} : 0 \leq j \leq 2k-4$	$2k-j+1$	$j+4$	$2k-j-1$
$I_{i+m-j-1} : 0 \leq j \leq 1$	$2j+1$	$2k-j+2$	$j+1$
$I_{i+m+j} : 0 \leq j \leq 1$	$2j+1$	$2k+j+3$	$j+2$
$I_{i+m+j+2} : 0 \leq j \leq 2k-2$	$j+5$	$2k-j+3$	$j+4$
$I_{i+m+2k+j+1} : 0 \leq j \leq 1$	$2k-j+4$	$3-2j$	$2k-j+3$
$I_{i+m+2k+j+3} : 0 \leq j \leq 1$	$2k-j+2$	$2j+1$	$2k-j+1$
$I_{i+2m-j-2} : 0 \leq j \leq 2k-4$	$j+4$	$2k-j+1$	$j+3$

Note that all the vertices of $P(n, m)$ lying in column 1 of Table 15 and Table 16 have distinct codes with respect to W. Thus W is resolving set for $P(n, m)$. Hence $\beta(P(n, m)) \leq 3$ for all $m \geq 7$ and $m \equiv 3(mod\ 4)$.

In Lemma 1-4, we have seen that $\beta(P(n,m)) \leq 3$ for $n = 2m+1$ and $m \geq 1$. Proof of Theorem 2 will be complete if we show that $\beta(P(n,m)) \geq 3$. We prove this in the following lemma:

Lemma 5 *Let $P(n,m)$ be generalized Petersen graphs with $n = 2m+1$, then $\beta(P(n,m)) \geq 3$.*

Proof. We suppose contrarily that $\beta(P(n,m)) = 2$, and only two vertices form a metric basis W for $P(n,m)$. Then we have the following three cases:
Case 1: For fixed i, let $W = \{O_i, O_{i+j}\}$, we have $cw(I_{i+j+1}) = cw(I_{i+m+j+1})$, when $1 \leq j \leq m-1$; $cw(I_{i+m+1}) = cw(I_{i+2m})$, when $j = m$; $cw(I_{i+1}) = cw(I_{i+m})$, when $j = m+1$; $cw(I_{i+j-1}) = cw(I_{i+j-m-1})$, when $m+2 \leq j \leq 2m$, a contradiction.
Case 2: For fixed i, let $W = \{I_i, I_{i+j}\}$, we have $cw(O_{i+j+1}) = cw(O_{i+m+j+1})$, when $1 \leq j \leq m-1$; $cw(O_{i+m}) = cw(I_{i+2m})$, when $j = m$; $cw(O_{i+m+1}) = cw(I_{i+1})$, when $j = m+1$; $cw(O_{i+j-1}) = cw(O_{i+j-m-1})$, when $m+2 \leq j \leq 2m$, a contradiction.
Case 3. When one vertex is in the outer cycle and the other is in the inner cycle, then we have two subcases.
Subcase 1: For fixed i, let $W = \{O_i, I_{i+j}\}$.
(a).
(i). When $m \equiv 0(mod\ 4)$, and for $k = 1$, we have $cw(O_{i+5}) = cw(O_{i+6})$, when $j = 1,3,8$; $cw(O_{i+4}) = cw(O_{i+5})$, when $j = 0,2,7$; $cw(O_{i+2}) = cw(O_{i+7})$, when $j = 4,5,6$.
(ii). When $m \equiv 1(mod\ 4)$, and for $k = 1$, we have $cw(O_{i+3}) = cw(I_{i+9})$, when $j = 0,3,5,8$; $cw(O_{i+3}) = cw(O_{i+5})$, when $j = 1,4,7$; $cw(O_{i+3}) = cw(O_{i+8})$, when $j = 2,9,10$; $cw(O_{i+3}) = cw(I_{i+4})$, when $j = 6$.
(iii). When $m \equiv 2(mod\ 4)$, and for $k = 1$, we have $cw(O_{i+j+3}) = cw(I_{i+j+2})$, $j = 0,1,2,3$; $cw(O_{i+j-3}) = cw(I_{i+j-2})$, $j = 10,11,12$; $cw(O_{i+3}) = cw(O_{i+10})$, when $j = 4,5,6,7,8,9$.
(iv). When $m \equiv 3(mod\ 4)$, and for $k = 1$, we have $cw(O_{i+j+3}) = cw(I_{i+j+2})$, when $j = 0,1,2,3,4$; $cw(O_{i+j+4}) = cw(I_{i+j+5})$, when $j = 5,6,7$; $cw(O_{i+j-4}) = cw(I_{i+j-5})$, when $j = 8,9,10$; $cw(O_{i+j-3}) = cw(I_{i+j-2})$, when $j = 11,12,13,14$.
(b). When $k \geq 2$, we have $cw(I_{i+j+2}) = cw(O_{i+j+3})$, when $0 \leq j \leq m-3$; $cw(I_{i+j-m+2}) = cw(O_{i+j-m+3}))$, when $m-2 \leq j \leq m+3$; $cw(I_{i+j-2}) = cw(O_{i+j-3})$, when $m+4 \leq j \leq 2m$, a contradiction.
Subcase 2: For fixed i, let $W = \{I_i, O_{i+j}\}$.
(a).
(i). When $m \equiv 0(mod\ 4)$, and for $k = 1$, we have $cw(O_{i+1}) = cw(O_{i+8})$, when $j = 0,4,5$; $cw(O_{i+6}) = cw(O_{i+7})$, when $j = 1,2,3$; $cw(O_{i+2}) = cw(O_{i+3})$, when $j = 6,7,8$.
(ii). When $m \equiv 1(mod\ 4)$ and for $k = 1$, we have $cw(I_{i+5}) = cw(I_{i+6})$, when $j = 0$; $cw(O_{i+8}) = cw(I_{i+9})$, when $j = 5$; $cw(O_{i+3}) = cw(I_{i+2})$, when $j = 6$; $cw(O_{i+6}) = cw(I_{i+10})$, when $j = 1,2,3,4$; $cwO_{i+5}) = cw(I_{i+1})$, when $j = 7,8,9,10$.
(iii). When $m \equiv 2(mod\ 4)$, and for $k = 1$, we have $cw(O_{i+3}) = cw(I_{i+2})$, when

$j = 0,1,2,11,12$; $cw(O_{i+10}) = cw(I_{i+2})$, when $j = 4,5,6,7,8$; $cw(O_{i+6}) = cw(I_{i+1})$, when $j = 3,9,10$.
(iv). When $m \equiv 3(mod\ 4)$, and for $k = 1$, then we have $cw(O_{i+3}){=}cw(I_{i+2})$, when $j = 0,1,2,11,12,13,14$; $cw(O_{i+6}) = cw(I_{i+9})$, when $j = 3,8$; $cw(O_{i+3}) = cw(O_{i+5})$, when $j = 4$; $cw(O_{i+2}) = cw(O_{i+13})$, when $j = 5,10$; $cw(O_{i+9}) = cw(O_{i+13})$, when $j = 6$; $cw(O_{i+10}) = cw(I_{i+13})$, when $j = 7$; $cw(O_{i+7}) = cw(I_{i+1})$, when $j = 9$.
(b). When $k \geq 2$ and $m \equiv 0,2(mod\ 4)$, then we have $cw(O_{i+2k}) = cw(I_{i+2k-1})$, when $0 \leq j \leq 2k-1$; $cw(I_{i+2k+1}) = cw(I_{i+m+2k+1})$, when $j = 2k$; $cw(I_{i+2(k+1)}) = cw(I_{i+m+2(k+1)})$, when $j = 2k+1$; $cw(O_{i+2k+1}) = cw(I_{i+2(k+1)})$, when $2k+2 \leq j \leq m$; $cw(O_{i+m+2k}) = cw(I_{i+m+2k-1})$, when $m+1 \leq j \leq m+2k-1$; $cw(I_{i+2k-1}) = cw(I_{i+m+2k-1})$, when $j = m+2k$; $cw(I_{i+2k}) = cw(I_{i+m+2k})$, when $j = m+2k+1$; $cw(O_{i+m+2k+1}) = cw(I_{i+m+2(k+1)})$, when $m+2k+2 \leq j \leq 2m$, a contradiction.
(c). When $k \geq 2$ and $m \equiv 1,3(mod\ 4)$, then we have $cw(O_{i+2k+1}) = cw(I_{i+2k})$, when $0 \leq j \leq 2k$; $cw(O_{i+2k}) = cw(O_{i+2(k+1)})$, when $j = 2k+1$; $cw(O_{i+2k+1}) = cw(I_{i+2(k+1)})$, when $2k+2 \leq j \leq m$; $cw(O_{i+m+2k+1}) = cw(I_{i+m+2k})$, when $m+1 \leq j \leq m+2k$; $cw(O_{i+m+2k}) = cw(O_{i+m+2(k+1)})$, when $j = m+2k+1$; $cw(O_{i+m+2k+1}) = cw(I_{i+m+2(k+1)})$, when $m+2k+1 \leq j \leq 2m$, a contradiction.
Thus, we conclude that to work with a resolving set consisting of two vertices of $P(n,m)$ is not possible. Hence, $\beta(P(2m+1,m)) \geq 3$.

Acknowledgements
The authors are grateful to the anonymous referee whose valuable suggestions resulted in producing an improved paper. This research is supported by the Higher Education Commission of Pakistan.

References

[1] P. Buczkowski, G. Chartrand, C. Poisson and P. Zhang, On k-dimensional graphs and their bases, *Period. Math. Hungar.* 46(1)(2003) 9-15.

[2] J. Caceres, C. Hernando, M. Mora, I. M. Pelayoe, M. L. Puertas, C. Seara and D. R. Wood, On the metric dimension of cartesian product of graphs, *SIAM J. of Disc. Math.* 21(2)(2007) 423-441.

[3] G. Chartrand, L. Eroh, M. A. Johnson and O. R. Oellermann, Resolvability in graphs and the metric dimension of a graph, *Disc. Appl. Math.* 105(1-3)(2000) 99-113.

[4] G. Chartrand and P. Zhang, The theory and applications of resolvability in graphs, A survey, *Congr. Numer.* 160(2003) 47-68.

[5] R. Frucht, J.E. Graver, M.E. Watkins, The groups of the generalized Petersen graphs, Proc. Cambridge Philos. Soc. 70(1971)211-218.

[6] M. R. Garey and D. S. Johnson, Computers and Intractability. A Guide to the Theory of NP-Completeness, A Series of Books in the Mathematical Sciences, W. H. Freeman, California, 1979.

[7] F. Harary and R. A. Melter, On the metric dimension of a graph, *Ars Combin.* 2(1976) 191-195.

[8] M. Imran, A. Q. Baig, M. K. Shafiq, On metric dimension of generalized Petersen graphs $P(n, 3)$, To appear.

[9] I. Javaid, M. T. Rahim and Kashif Ali, Families of regular graphs with constant metric dimension, *Util. Math.* 75(2008) 21-33.

[10] M. A. Johnson, Structure-activity maps for visualizing the graph variables arising in drug design, *J. Biopharm. Statist* 3(1993) 203-236.

[11] S. Khuller, B. Raghavachari and A. Rosenfeld, Landmarks in graphs, *Disc. Appl. Math.* 70(1996) 217-229.

[12] R. A. Melter, I. Tomescu, Metric bases in digital geometry, *Computer Vision, Graphics, and Image Processing* 25(1984) 113-121.

[13] P. J. Slater, Leaves of trees, Congr. Numer. 14(1975) 549-559.

[14] P. J. Slater, Dominating and reference sets in a graph, *J. Math. Phys. Sci.* 22(4)(1988) 445-455.

[15] M. E. Watkins, A theorem on Tait coloring with an application to the generalized Petersen graphs, J. Combin. Theory 6(1969)152-164.