On Transitive Cayley Graphs of Strong Semilattices of Rectangular Groups*

Yifei Hao[1]†, Xiaomei Yang[2], Niqianjun Jin[3]

1. Research Center for International Business and Economy,
Sichuan International Studies University,
Chongqing 400031, P.R. China

2. College of Maths, Southwest Jiaotong University,
Chengdu 610031, P.R. China

3. College of Economics and Management, Southwest University,
Chongqing 400715, P.R. China

Abstract

In this paper, we investigate the transitive Cayley graphs of strong semilattices of rectangular groups, and of normal bands, respectively. We show under which conditions they enjoy the property of automorphism vertex transitivity in analogy to Cayley graphs of groups.

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1 Introduction and Preliminaries

The definition of a Cayley graph was introduced by Arthur Cayley in 1878 to explain the concept of abstract groups which are described by a set of generators. Cayley graphs of groups have received serious attention, and many algebraic and combinatorial properties have been actively investigated (see, in particular, [4, 16]). Let S be a semigroup, and C a subset of S. The *Cayley graph* Cay(S, C) of S with respect to C is defined as the graph with vertex set S and edge set $E(Cay(S, C))$ consisting of those ordered pairs (x, y), where $xs = y$ for some $s \in C$. The set C is called the *connecting set* of Cay(S, C). Cayley graphs of semigorups are closely related to finite state automata and have many valuable applications, see the survey [14] and the monograph [10]. They are generalizations of Cayley graphs of groups. One of the earliest references on this

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†Corresponding author: yifei.hao@gmail.com

subject is [1]; see also [20] for another example of early work. The whole Section 2.4 of the book [10] is devoted to the Cayley graphs of semigroups. All vertex-transitive Cayley graphs produced by periodic semigroups are characterized in [12]. A combinatorial property of infinite semigroups defined in terms of Cayley graphs has been investigated (see [5, 13, 19]). In particular, the Cayley graphs of certain classes of semigroups have been studied, and combinatorial properties of these Cayley graphs have been described. In [11], Kelarev studied the Cayley graphs of inverse semigroups. In [2], a complete description of all vertex transitive Cayley graphs of bands was obtained. The undirected Cayley graphs of right groups are investigated in [6]. The basic structures of Cayley graphs of normal bands are investigated in [3]. The authors also investigated the Cayley graphs of symmetric inverse semigroups, Brandt semigroups and completely semigroups in [7], [8] and [17], respectively.

In this paper, the word "graph" means a finite directed graph without multiple edges, but possibly with loops. A semigroup S is said to be a *right (left) zero semigroup* if $xy = y$ $(xy = x)$ for all x, $y \in S$. Let G be a group, I a left zero semigroup, and set $S = I \times G$. Define multiplication on S by $(i,g)(j,h) = (i,gh)$ for any (i,g), $(j,h) \in S$. Then S is a semigroup which is called a *left group*. Correspondingly, if Λ is a right zero semigroup, we set $S = G \times \Lambda$ and define multiplication on S by $(g,\lambda)(h,\mu) = (gh,\mu)$ for any (g,λ), $(h,\mu) \in S$. Then S is a semigroup which is called a *right group*. We set $S = I \times G \times \Lambda$, and define multiplication on S by

$$(i,g,\lambda)(j,h,\mu) = (i,gh,\mu) \text{ for any } (i,g,\lambda),\ (j,h,\mu) \in S.$$

Then S is called a *rectangular group*. A semigroup S is said to be *completely simple* if it has no proper ideals and has a minimal idempotent with respect to the partial order $e \leq f \Leftrightarrow e = ef = fe$.

If $(Y, \leq)$ is a nonempty partially ordered set such that the meet $\alpha \wedge \beta$ of α and β exists for every α, $\beta \in Y$, we say that $(Y, \leq)$ is a *(lower) semilattice*. A semigroup S is said to be a *semilattice of (disjoint) semigroups* $(S_\alpha, \circ_\alpha)$, $\alpha \in Y$, if

1. Y is a semilattice,
2. $S = \bigcup_{\alpha \in Y} S_\alpha$,
3. $S_\alpha S_\beta \subseteq S_{\alpha\wedge\beta}$,

and a *strong semilattice of semigroups*, if in addition for all $\beta \geq \alpha$ in Y there exists a semigroup homomorphism $f_{\beta,\alpha}$: $S_\beta \longrightarrow S_\alpha$ called a defining homomorphism, with

4. for all $\alpha \in Y$, $f_{\alpha,\alpha} = id_{S_\alpha}$, the identity mapping,
5. for all α, β, $\gamma \in Y$ with $\alpha \leq \beta \leq \gamma$, we have $f_{\beta,\alpha} \circ f_{\gamma,\beta} = f_{\gamma,\alpha}$, where multiplication on $S = \bigcup_{\alpha \in Y} S_\alpha$ is defined for $x \in S_\alpha$ and $y \in S_\beta$ by

$$xy = f_{\alpha,\alpha\wedge\beta}(x) f_{\beta,\alpha\wedge\beta}(y).$$

If a semigroup S is a *rectangular band*, i.e., $xyx = x$, $xx = x$ for all x, $y \in S$, then S is isomorphic to a direct product of a left zero semigroup and a right zero semigroup ([9]). S is called a *normal band* if it is a strong semilattice of rectangular bands ([9]). A semigroup S is called a *Clifford semigroup* if it is a strong semilattice of groups.

If C is a nonempty subset of S, then denote by $\langle C\rangle$ the subsemigroup of S generated by C. The subsemigroup $\langle C\rangle$ consists of all elements of S that can be expressed as finite products of elements of C.

Let $X_1 \times X_2 \times \cdots \times X_n$ be a finite cartesian product of sets $X_1, X_2, \ldots, X_n$. We denote by $p_{i_1,i_2,\ldots,i_k}$ the usual projections of this product onto $X_{i_1} \times X_{i_2} \times \cdots \times X_{i_k}$, for any $\{i_1, i_2, \ldots, i_k\} \subseteq \{1, 2, \ldots, n\}$.

Let (V_1, E_1) and (V_2, E_2) be graphs. A mapping ϕ: $V_1 \longrightarrow V_2$ is called a *(graph) homomorphism* if $(u, v) \in E_1$ implies $(\phi(u), \phi(v)) \in E_2$, i.e. ϕ preserve arcs. We write ϕ: $(V_1, E_1) \longrightarrow (V_2, E_2)$. A graph homomorphism ϕ: $(V, E) \longrightarrow (V, E)$ is called a *(graph) endomorphism.* If ϕ: $(V_1, E_1) \longrightarrow (V_2, E_2)$ is a bijective graph homomorphism and ϕ^{-1} is also a graph homomorphism, then ϕ is called a *(graph) isomorphism.* A graph isomorphism ϕ: $(V, E) \longrightarrow (V, E)$ is called a *(graph) automorphism.*

A graph $D = (V, E)$ is said to be *$Aut(D)$-vertex-transitive* if, for any two vertices $x, y \in V$, there exists an automorphism $\phi \in Aut(D)$ such that $\phi(x) = y$. Now let S be a semigroup and $C \subseteq S$. Denote the automorphism group and the endomorphism monoid of the Cayley graph $Cay(S, C)$ by $Aut(S, C)$ and $End(S, C)$, respectively. An element $\phi \in End(S, C)$ will be called a *colour-preserving* endomorphism if $xa = y$ implies $\phi(x)a = \phi(y)$, for every x, $y \in C$ and $a \in C$. Denote by $ColEnd(S, C)$ and $ColAut(S, C)$ the sets of all colour-preserving endomorphisms and automorphisms of $Cay(S, C)$, respectively. Obviously, $ColAut(S, C) \subseteq Aut(S, C)$.

For terminology and notation not defined in this paper, We refer the reader to [4] and [9].

We introduce some well known results which will be used extensively in this paper.

Theorem 1.1. ([12]) *Let S be a semigroup, and let C be a subset of S which generates a subsemigroup $\langle C\rangle$ such that all principal right ideals of $\langle C\rangle$ are finite. Then, the Cayley graph $Cay(S, C)$ is $ColAut(S, C)$-vertex-transitive if and only if the following conditions hold:*

(1) *$Sc = S$, for all $c \in C$;*

(2) *$\langle C\rangle$ is a left group;*

(3) *$|s\langle C\rangle|$ is independent of the choice of $s \in S$.* □

Theorem 1.2. ([12]) *Let S be a semigroup, and let C be a subset of S which generates a subsemigroup $\langle C\rangle$ such that all principal right ideals of $\langle C\rangle$ are finite. Then, the Cayley graph $Cay(S, C)$ is $Aut(S, C)$-vertex-transitive if and only if the following conditions hold:*

(1) *$SC = S$;*

(2) *$\langle C\rangle$ is a completely simple semigroup;*

(3) *the Cayley graph $Cay(\langle C\rangle, C)$ is $Aut(\langle C\rangle, C)$-vertex-transitive;*

(4) *$|s\langle C\rangle|$ is independent of the choice of $s \in S$.* □

Lemma 1.3. ([12]) *Let S be a finite rectangular band, and C a subset of S. Then the Cayley graph $Cay(S,C)$ is $Aut(S,C)$-vertex-transitive if and only if $C \cap sS \neq \emptyset$ for all $s \in S$.* □

Lemma 1.4. ([18]) *Let Y be a finite semilattice, $S = \bigcup_{\alpha \in Y} S_\alpha$ a strong semilattice of finite semigroups, C a nonempty subset of S, and let $Cay(S,C)$ be $Aut(S,C)$-vertex-transitive. Then*

(1) *Y has a maximum m, and*

(2) *$C \subseteq S_m$.* □

Lemma 1.5. *Let $S = I \times G \times \Lambda$ be a finite rectangular group, where G is a group, I and Λ are left zero semigroup and right zero semigroup, respectively, and let A be a subset of S. Then $\langle A \rangle = I' \times \langle G' \rangle \times \Lambda'$ is a subsemigroup of S. In particular, $\langle A \rangle$ is a rectangular group, where $I' \subseteq I$, $G' \subseteq G$, $\Lambda' \subseteq \Lambda$.*

Proof. Let $I' = p_1(A)$, $G' = p_2(A)$ and $\Lambda' = p_3(A)$. Take any $x, y \in I' \times \langle G' \rangle \times \Lambda'$, where $x = (i, g_1 g_2 \cdots g_s, \lambda)$, $y = (i', g'_1 g'_2 \cdots g'_t, \lambda')$, g_i, $g'_j \in G'$; i, $i' \in I'$; λ, $\lambda' \in \Lambda'$. Since

$$xy = (i, g_1 g_2 \cdots g_s g'_1 g'_2 \cdots g'_t, \lambda') \in I' \times \langle G' \rangle \times \Lambda',$$

then $I' \times \langle G' \rangle \times \Lambda'$ is a rectangular group which is included in S. It is obvious that $A \subseteq I' \times \langle G' \rangle \times \Lambda'$, thus $\langle A \rangle \subseteq I' \times \langle G' \rangle \times \Lambda'$.

On the other hand, for any $z \in I' \times \langle G' \rangle \times \Lambda'$, let $z = (i, g_1 g_2 \cdots g_t, \lambda)$. We may choose $i' \in I'$, $g \in G'$ and $\lambda' \in \Lambda'$, such that (i, g, λ'), $(i'g, \lambda) \in A$. Since G is a finite group, there exists $q \in \mathbb{N}^+$, such that $g^q = e$, where e is the identity of G. Then

$$\begin{aligned} z &= (i, g_1 g_2 \cdots g_t, \lambda) \\ &= (i, g_1 g_2 \cdots g_t e, \lambda) \\ &= (i, g_1 g_2 \cdots g_t g^q, \lambda) \\ &= (i, g_1, \lambda_1)(i_2, g_2, \lambda_2) \cdots (i_t, g_t, \lambda_t)(i, g, \lambda')(i, g, \lambda') \cdots (i, g, \lambda')(i', g, \lambda) \in \langle A \rangle \end{aligned}$$

where $i_2, \ldots, i_t \in I'$, $\lambda_1, \ldots, \lambda_t \in \Lambda'$. Therefore, $I' \times \langle G' \rangle \times \Lambda' \subseteq \langle A \rangle$. □

2 Strong semilattice of rectangular groups

In this section, we investigate the $Aut(S,C)$-vertex-transitive Cayley graphs of strong semilattices of rectangular groups. We give the first main result in the following.

Theorem 2.1. *Let Y be a finite semilattice, $S = \bigcup_{\alpha \in Y} S_\alpha$ a strong semilattice of rectangular groups such that $S_\alpha = I_\alpha \times G_\alpha \times \Lambda_\alpha$ where G_α are groups, I_α and Λ_α are left zero semigroups and right zero semigroups, respectively, and let C be a nonempty subset of S, p_3 and $p_{2,3}$ are the third projection and $\{2,3\}$-projection, respectively. Then the Cayley graph $Cay(S,C)$ is $Aut(S,C)$-vertex-transitive if and only if the following conditions hold:*

(a) *Y has a maximum m;*
(b) *$C \subseteq S_m$;*
(c) *for all $\alpha \in Y$, $|p_3(f_{m,\alpha}(C))| = |\Lambda_\alpha|$ and*
(d) *$Cay(\langle C\rangle, C)$ is $Aut(\langle C\rangle, C)$-vertex-transitive;*
(e) *for any α, $\beta \in Y$, $|p_{2,3}(f_{m,\alpha}(\langle C\rangle))| = |p_{2,3}(f_{m,\beta}(\langle C\rangle))|$.*

Proof. Sufficiency. We prove the conditions (1)–(4) of Theorem 1.2 for S, then the results follow.

(1) Since $C \subseteq S_m$, we get

$$S_\alpha C = (I_\alpha \times G_\alpha \times \Lambda_\alpha) f_{m,\alpha}(C) = I_\alpha \times G_\alpha \times \Lambda_\alpha \text{ (by (c))} = S_\alpha \text{ for all } \alpha \in Y.$$

Therefore, $SC = (\bigcup_{\alpha\in Y} S_\alpha)C = \bigcup_{\alpha\in Y}(S_\alpha C) = \bigcup_{\alpha\in Y} S_\alpha = S$, since S_α are distinct.

(2) Since $C \subseteq S_m$, we obtain that $\langle C\rangle$ is a rectangular group and a subsemigroup of S_m by Lemma 1.5, in particular, a completely simple semigroup.

(3) This is (d).

(4) Let s, $s' \in S$ where $s = (i, g, \lambda)$ and $s' = (j, h, \mu)$. Suppose that $s \in S_\alpha$ and $s' \in S_\beta$ for some α, $\beta \in Y$. Then

$$\begin{aligned}|s\langle C\rangle| &= |(i, g, \lambda) f_{m,\alpha}(\langle C\rangle)| = |\{i\} \times p_{2,3}(f_{m,\alpha}(\langle C\rangle))| \\ &= |\{j\} \times p_{2,3}(f_{m,\beta}(\langle C\rangle))| \text{ (by (e))} = |(j, h, \mu) f_{m,\beta}(\langle C\rangle)| = |s'\langle C\rangle|.\end{aligned}$$

Therefore, the Cayley graph $Cay(S, C)$ is $Aut(S, C)$-vertex-transitive.

Necessity. From Lemma 1.4 it is clear that (a) and (b) are necessary. We will prove, by contradiction, that the Cayle graph $Cay(S, C)$ is not $Aut(S, C)$-vertex-transitive, if

(1) there exists $\beta \in Y$ such that $|p_3(f_{m,\beta}(C))| < |\Lambda_\beta|$, or
(2) there exist β, $\gamma \in Y$ such that $|p_{2,3}(f_{m,\beta}(\langle C\rangle))| = |p_{2,3}(f_{m,\gamma}(\langle C\rangle))|$, or
(3) the Cayley graph $Cay(\langle C\rangle, C)$ is not $Aut(\langle C\rangle, C)$-vertex-transitive.

(1) Suppose that there exists $\beta \in Y$ such that $|p_3(f_{m,\beta}(C))| < |\Lambda_\beta|$. Hence $S_\beta C = (I_\beta \times G_\beta \times \Lambda_\beta) f_{m,\beta}(C) = I_\beta \times G_\beta \times p_3(f_{m,\beta}(C)) \neq S_\beta$, and thus $SC = (\bigcup_{\alpha\in Y} S_\alpha)C = \bigcup_{\alpha\in Y}(S_\alpha C) \neq \bigcup_{\alpha\in Y} S_\alpha = S$. By Theorem 1.2, we obtain that the Cayley graph $Cay(\langle C\rangle, C)$ is not $Aut(\langle C\rangle, C)$-vertex-transitive.

(2) Let $|p_3(f_{m,\alpha}(C))| = |\Lambda_\alpha|$, for all $\alpha \in Y$, and suppose that there exist β, $\gamma \in Y$ such that $|p_{2,3}(f_{m,\beta}(\langle C\rangle))| \neq |p_{2,3}(f_{m,\gamma}(\langle C\rangle))|$. We take $s = (i, g, \lambda) \in S_\beta$ and $s' = (j, h, \mu) \in S_\gamma$. It follows that

$$\begin{aligned}|s\langle C\rangle| &= |(i, g, \lambda) f_{m,\beta}(\langle C\rangle)| = |\{i\} \times p_{2,3}(f_{m,\beta}(\langle C\rangle))| \\ &\neq |\{j\} \times p_{2,3}(f_{m,\gamma}(\langle C\rangle))| = |(j, h, \mu) f_{m,\gamma}(\langle C\rangle)| = |s'\langle C\rangle|.\end{aligned}$$

From Theorem 1.2, we have that the Cayley graph $Cay(\langle C\rangle, C)$ is not $Aut(\langle C\rangle, C)$-vertex-transitive.

(3) It is obvious by Theorem 1.2.

Now the proof is complete. □

Let $|Y| = 1$ in Theorem 2.1, we have a description of the $Aut(S, C)$-vertex-transitive of Cayley graphs of rectangular groups, which is introduced in [15].

Corollary 2.2. ([15]) *Let $S = I \times G \times \Lambda$ be a finite rectangular group, and C a subset of S, and let p_3 be the third projection. Then $Cay(S,C)$ is $Aut(S,C)$-vertex-transitive if and only if the following conditions hold:*

(1) $p_3(C) = \Lambda$;

(2) *the Cayley graph $Cay(\langle C\rangle, C)$ is $Aut(\langle C\rangle, C)$-vertex-transitive.* □

Taking $|I_\alpha| = 1$ for all $\alpha \in Y$, we have the description of $Aut(S,C)$-vertex-transitive Cayley graphs of the strong semilattice of right groups in [18].

Corollary 2.3. ([18]) *Let Y be a finite semilattice, $S = \bigcup_{\alpha\in Y} S_\alpha$ a strong semilattice of right groups such that $S_\alpha = G_\alpha \times \Lambda_\alpha$ where G_α and Λ_α are groups and right zero semigroups, respectively, and let C be a nonempty subset of S, p_2 is the second projection. Then the Cayley graph $Cay(S,C)$ is $Aut(S,C)$-vertex-transitive if and only if the following conditions hold:*

(a) *Y has a maximum m;*

(b) $C \subseteq S_m$;

(c) *for all $\alpha \in Y$, $|p_2(f_{m,\alpha}(C))| = |\Lambda_\alpha|$ and*

(d) *$Cay(\langle C\rangle, C)$ is $Aut(\langle C\rangle, C)$-vertex-transitive;*

(e) *for all α, $\beta \in Y$, $|f_{m,\alpha}(\langle C\rangle)| = |f_{m,\beta}(\langle C\rangle)|$.* □

In the following, we give the second main result of this paper, which is a complete description of $ColAut(S,C)$-vertex-transitive Cayley graphs of the strong semilattices of rectangular groups.

Theorem 2.4. *Let Y be a finite semilattice, $S = \bigcup_{\alpha\in Y} S_\alpha$ a strong semilattice of rectangular groups such that $S_\alpha = I_\alpha \times G_\alpha \times \Lambda_\alpha$ where G_α are groups, I_α and Λ_α are left zero semigroups and right zero semigroups, respectively, and C be a nonempty subset of S, p_2 is the second projection. Then the Cayley graph $Cay(S,C)$ is $ColAut(S,C)$-vertex-transitive if and only if the following conditions hold:*

(a) *Y has a maximum m;*

(b) $C \subseteq S_m$;

(c) *$|\Lambda_\alpha| = 1$ for all $\alpha \in Y$;*

(d) *for any α, $\beta \in Y$, $|p_2(f_{m,\alpha}(\langle C\rangle))| = |p_2(f_{m,\beta}(\langle C\rangle))|$.*

Proof. Sufficiency. We prove the conditions (1)–(3) of Theorem 1.1 for S, then the results follow.

(1) For any $c \in C$, $\alpha \in Y$,

$$S_\alpha c = (I_\alpha \times G_\alpha \times \Lambda_\alpha) f_{m,\alpha}(c) = I_\alpha \times G_\alpha \times p_3(f_{m,\alpha}(c)) = I_\alpha \times G_\alpha \times \Lambda_\alpha \text{ by (c).}$$

Therefore, $Sc = \bigcup_{\alpha\in Y} S_\alpha c = \bigcup_{\alpha\in Y}(S_\alpha c) = \bigcup_{\alpha\in Y} S_\alpha = S$.

(2) Since $C \subseteq S_m$ and $|p_3(C)| = 1$, then $\langle C\rangle = I'_m \times G'_m$ where $I'_m \subseteq I_m$ and G'_m is a subgroup of G_m. Therefore $\langle C\rangle$ is a left group by Lemma 1.5.

(3) For any s, $s' \in S$, we suppose that $s = (i,g,\lambda)$, $s' = (j,h,\mu)$, and $s \in S_\alpha$,

$s' \in S_\beta$, for some α, $\beta \in Y$. Then

$$\begin{aligned}|s\langle C\rangle| &= |(i,g,\lambda)f_{m,\alpha}(\langle C\rangle)| = |\{i\} \times gp_2(f_{m,\alpha}(\langle C\rangle)) \times p_3(f_{m,\alpha}(\langle C\rangle))| \\ &= |p_2(f_{m,\alpha}(\langle C\rangle))| \text{ (by (c))} = |p_2(f_{m,\beta}(\langle C\rangle))| \text{ (by (d))} \\ &= |\{j\} \times hp_2(f_{m,\beta}(\langle C\rangle)) \times p_3(f_{m,\beta}(\langle C\rangle))| = |(j,h,\mu)f_{m,\beta}(\langle C\rangle)| \\ &= |s'\langle C\rangle|.\end{aligned}$$

Necessity. From Lemma 1.4, we conclude that (a) and (b) are necessary. We suppose to the contrary that (1) $|\Lambda_\beta| > 1$ for some $\beta \in Y$, or (2) for some β, $\gamma \in Y$, $|p_2(f_{m,\alpha}(\langle C\rangle))| \neq |p_2(f_{m,\beta}(\langle C\rangle))|$.

(1) For $c \in C$,

$$\begin{aligned}S_\beta c = S_\beta f_{m,\beta}(c) &= I_\beta \times G_\beta \times p_3(f_{m,\beta}(c)) \\ &\neq I_\beta \times G_\beta \times \Lambda_\beta \text{ (by the hypothesis)} = S_\beta.\end{aligned}$$

Hence $Cay(S, C)$ is not $ColAut(S, C)$-vertex-transitive by (1) of Theorem 1.1.

(2) If $|\Lambda_\alpha| = 1$ for all $\alpha \in Y$, and there exist α, $\beta \in Y$, such that $|p_2(f_{m,\alpha}(\langle C\rangle))| \neq |p_2(f_{m,\beta}(\langle C\rangle))|$. We take s, $s' \in S$, and suppose that $s = (i,g,\lambda) \in S_\alpha$, $s' = (j,h,\mu) \in S_\beta$. Then

$$\begin{aligned}|s\langle C\rangle| &= |(i,g,\lambda)f_{m,\alpha}(\langle C\rangle)| = |p_2(f_{m,\alpha}(\langle C\rangle))| \\ &\neq |p_2(f_{m,\beta}(\langle C\rangle))| = |(j,h,\mu)(\langle C\rangle)| = |s'\langle C\rangle|.\end{aligned}$$

Hence $Cay(S, C)$ is not $ColAut(S, C)$-vertex-transitive by (3) of Theorem 1.1.□

From Theorem 2.4, we immediately have a characterization of $ColAut(S, C)$-vertex-transitive Cayley graphs of rectangular groups.

Corollary 2.5. *Let $S = I \times G \times \Lambda$ be a finite rectangular group, and let C be a subset of S, p_3 be the third projection. Then $Cay(S, C)$ is $ColAut(S, C)$-vertex-transitive if and only if the $|\Lambda| = 1$.* □

Combining Theorem 2.1 and Theorem 2.4, and taking $|\Lambda_\alpha| = 1$ for all $\alpha \in Y$, we obtain the following result which is a description of vertex transitivity of Cayley graphs of left groups in [18].

Corollary 2.6. ([18]) *Let Y be a finite semilattice, $S = \bigcup_{\alpha\in Y} S_\alpha$ a strong semilattice of left groups such that $S_\alpha = I_\alpha \times G_\alpha$ where G_α and I_α are groups and left zero semigroups, respectively, and C be a nonempty subset of S, p_2 is the second projection. Then the following conditions are equivalent.*

(a) *Y has a maximum m;*

(b) *$C \subseteq S_m$;*

(c) *for any α, $\beta \in Y$, $|p_2(f_{m,\alpha}(\langle C\rangle))| = |p_2(f_{m,\beta}(\langle C\rangle))|$;*

(d) *$Cay(S, C)$ is $ColAut(S, C)$-vertex-transitive;*

(e) *$Cay(S, C)$ is $Aut(S, C)$-vertex-transitive.* □

From Theorem 2.4, we also have a description of $ColAut(S, C)$-vertex-transitive Cayley graphs of the strong semilattice of right groups, by letting $|I_\alpha| = 1$ for all $\alpha \in Y$.

Corollary 2.7. *Let Y be a finite semilattice, $S = \bigcup_{\alpha \in Y} S_\alpha$ a strong semilattice of right groups such that $S_\alpha = G_\alpha \times \Lambda_\alpha$ where G_α and Λ_α are groups and right zero semigroups, respectively, and let C be a nonempty subset of S. Then Cayley graph $Cay(S, C)$ is $ColAut(S, C)$-vertex-transitive if and only if the following conditions hold,*

(a) *Y has a maximum m;*
(b) *$C \subseteq S_m$;*
(c) *S is a Clifford semigroup;*
(d) *for all α, $\beta \in Y$, $|f_{m,\alpha}(\langle C \rangle)| = |f_{m,\beta}(\langle C \rangle)|$.* □

3 Normal bands

In this section, we give a description of the $Aut(S, C)$-vertex-transitive and $ColAut(S, C)$-vertex-transitive Caley graphs of normal bands, (i.e., the strong semilattice of rectangular groups)respectively. Theorem 3.2 and Theorem 3.4 are direct consequences of the Theorem 2.1 and Theorem 2.4, respectively, when we take G_α to be trivial groups for all $\alpha \in Y$.

Lemma 3.1. *Let $S = I \times \Lambda$ be a finite rectangular band, where I and Λ are left zero semigroup and right zero semigroup, respectively, and let C be a subset of S. Then the Cayley graph $Cay(\langle C \rangle, C)$ is $Aut(S, C)$-vertex-transitive.*

Proof. For any $s = (i, j) \in \langle C \rangle$, $s\langle C \rangle = \{i\} \times p_2(\langle C \rangle)$. Since I is a left zero semigroup, then $p_1(\langle C \rangle) = p_1(C)$. It follows that $i \in p_1(C)$. Hence we can find $s' \in C$, such that $s' = (i, l) \in \{i\} \times p_2(\langle C \rangle)$. Therefore, the set $C \cap s\langle C \rangle$ is not empty, for all $s \in \langle C \rangle$. It follows that $Cay(\langle C \rangle, C)$ is $Aut(S, C)$-vertex-transitive by Lemma 1.3. □

Combining with Theorem 2.1 and Lemma 3.1, we have the following result, which characterizes the $Aut(S, C)$-vertex-transitive Cayley graphs of normal bands.

Theorem 3.2. *Let Y be a finite semilattice, $S = \bigcup_{\alpha \in Y} S_\alpha$ a strong semilattice of rectangular bands such that $S_\alpha = I_\alpha \times \Lambda_\alpha$ where I_α and Λ_α are left zero semigroups and right zero semigroups, respectively, and let C be a nonempty subset of S, p_2 the second projection. Then the Cayley graph $Cay(S, C)$ is $Aut(S, C)$-vertex-transitive if and only if the following conditions hold:*

(a) *Y has a maximum m;*
(b) *$C \subseteq S_m$;*
(c) *for all $\alpha \in Y$, $|p_2(f_{m,\alpha}(C))| = |\Lambda_\alpha|$;*
(d) *for any α, $\beta \in Y$, $|p_2(f_{m,\alpha}(\langle C \rangle))| = |p_2(f_{m,\beta}(\langle C \rangle))|$.* □

If $|Y| = 1$, as a direct consequence of Theorem 3.2, we have the following result which was obtained in [2].

Corollary 3.3. ([2]) *Let $S = I \times \Lambda$ be a rectangular band where I and Λ are left zero semigroup and right zero semigroup, respectively, and let C be a subset of*

S. Then the Cayley graph $Cay(S,C)$ is $Aut(S,C)$-vertex-transitive if and only if $|p_2(C)| = |\Lambda|$. □

From Theorem 2.4, we have a characterization of the $ColAut(S,C)$-vertex-transitive Cayley graph of normal bands in the following.

Theorem 3.4. *Let Y be a finite semilattice, $S = \bigcup_{\alpha \in Y} S_\alpha$ a strong semilattice of rectangular bands such that $S_\alpha = I_\alpha \times \Lambda_\alpha$ where I_α and Λ_α are left zero semigroups and right zero semigroups, respectively, and let C be a nonempty subset of S. Then the Cayley graph $Cay(S,C)$ is $ColAut(S,C)$-vertex-transitive if and only if the following conditions hold:*

(a) *Y has the maximum m;*
(b) *$C \subseteq S_m$;*
(c) *$|\Lambda_\alpha| = 1$ for all $\alpha \in Y$.* □

Let $|Y| = 1$, a description of $ColAut(S,C)$-vertex-transitive Cayley graphs of rectangular bands follows from Theorem 3.4 immediately.

Corollary 3.5. *Let $S = I \times \Lambda$ be a rectangular band where I and Λ are left zero semigroup and right zero semigroup, respectively, and C a subset of S. Then the Cayley graph $Cay(S,C)$ is $ColAut(S,C)$-vertex-transitive if and only if S is a left zero semigroup.* □

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References

[1] J.Dénes, Theory of Graphs, New York-Paris: Gordon and Breach, 1967, 93–101.

[2] S.H. Fan, Y.S. Zeng, On Cayley graphs of bands, Semigroup Forum 74 (2007) 99–105.

[3] X. Gao, W.W. Liu, Y.F. Luo, On Cayley graphs of normal bands, Ars Combinatoria, 100 (2011) 409–419.

[4] C. Godsil, G. Royle, Algebraic Graph Theory, Springer-Verlag, New York, 2001.

[5] Y.F. Hao, Y.F. Luo, Directed graphs and combinatorial properties of completely regular semigroups, Semigroup Forum 81 (2010) 524–530.

[6] Y.F. Hao, Y.F. Luo, On the Cayley graphs of left (right) groups, Southeast Asian Bulletin of Mathematics 34 (2010) 685–691.

[7] Y.F. Hao, X. Gao, Y.F. Luo, On Cayley graphs of symmetric inverse semigroups, Ars Combinatoria 100 (2011) 307–319.

[8] Y.F. Hao, X. Gao, Y.F. Luo, On the Cayley graphs of Brandt semigorups, Communications in Algebra, 39 (2011) 2847–2883.

[9] J.M. Howie, Fundamentals of Semigroup Theory, Clarendon Press, Oxford, 1995.

[10] A.V. Kelarev, Graph Algebras and Automata, Marcel Dekker, New York, 2003.

[11] A.V. Kelarev, On Cayley graphs of inverse semigroups, Semigroup Forum 72 (2006) 411–418.

[12] A.V. Kelarev, C.E. Praeger, On transitive Cayley graphs of groups and semigroups, European J. Combin. 24 (2003) 59–72.

[13] A.V. Kelarev, S.J. Quinn, A combinatorial property and Cayley graphs of semigroups, Semigroup Forum 66 (2003) 89–96.

[14] A.V. Kelarev, J. Ryan, J. Yearwood, Cayley graphs as classifiers for data mining: The influence of asymmetries, Discrete Math. 309 (2009) 5360–5369.

[15] B. Khosravi, M. Mahmoudi, On Cayley graphs of rectangular groups, Discrete Mathematics, 310 (2010) 804–811.

[16] C.H. Li, On isomorphisms of connected Cayley graphs, Discrete Math. 178 (1998) 109–122.

[17] Y.F. Luo, Y.F. Hao, G.T. Clarke, On the Cayley graphs of completely simple semigroups, Semigroup Forum, 82 (2011) 288 - 295.

[18] S. Panma, U. Knauer, Sr. Arworn, On the transitive Cayley graphs of strong semilattices of right(left) groups, Discrete Math. 309 (2009) 5393–5403.

[19] D. Yang, X. Gao, D-saturated property of the Cayley graphs of semigroups, Semigroup Forum 80 (2010) 174–180.

[20] B. Zelinka, Graphs of semigroups, Casopis. Pest. Mat. 106 (1981) 407–408.