A sufficient condition for a toroidal graph to be 3-choosable[1]

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Abstract

In this paper, it is proved that a toroidal graph without cycles of length k for each $k \in \{4, 5, 7, 10\}$ is 3-choosable.

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1 Introduction

All graphs considered in this paper are finite, simple toroidal graphs. A graph G is toroidal (or planar) if G can be drawn on the torus (or on the plane) so that the edges meet only at the vertices of the graph. A face f is called a 2-cell if any simple closed curve inside f can be continuously contracted to a single point. An embedding of G is called a 2-cell embedding if all the faces are 2-cell. We assume that all graphs under consideration admit 2-cell embeddings on the torus.

$G = (V, E, F)$ denotes a toroidal graph, with V, E and F being the set of vertices, edges and faces of G, respectively. We use $b(f)$ to denote the boundary walk of a face f and write $b(f) = [v_1v_2v_3 \cdots v_n]$ if $v_1, v_2, v_3, \cdots, v_n$ are the vertices of $b(f)$ in a cyclic order. A face f is *incident* with all vertices and edges on $b(f)$. The *degree* of a face f of G, denoted also by $d_G(f)$, is the number of edges incident with it, where cut edges are counted twice. A vertex (face) of degree k is called a k-vertex (k-face). If $r \leq k$ or $1 \leq k \leq r$, then a k-vertex (k-face) is called an r^+- or r^--vertex (r^+- or r^--face), respectively. A k-cycle is a cycle of length k. The vertex set of a cycle C will also be denoted by C.

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A *color list* $L = \{L(v) : v \in V\}$ is a family of color sets assigned to each vertex of G. An *L-coloring* of G is an assignment of colors to each vertex $v \in V$ from $L(v)$ such that adjacent vertices receive distinct colors. For a positive integer k, we say that G is *k-choosable* if G admits an L-coloring for an arbitrary color list $L = \{L(v) : |L(v)| = k, v \in V\}$. The *choice number* of G, denoted by $\chi_l(G)$, is the minimum k such that G is k-choosable.

All 2-choosable graphs were characterized completely in [5]. Thomassen proved that plane graphs are 5-choosable [12] and plane graphs of girth at least 5 are 3-choosable [13]. Examples of plane graphs which are not 4-choosable were given by Voigt [14], and by Mirzakhani [8] independently. Voigt and Writh [15], and Gutner [6] independently, presented some plane graphs of girth 4 which are not 3-choosable. It is a hard problem to decide if a plane graph is 3-choosable, even for triangle-free plane graphs.

In 1976 Steinberg (See[11]) conjectured that every planar graph without 4- and 5-cycle is 3-colorable. In 1990, Erdös(See[11]) proposed to relax Steinberg's conjecture by asking if there exists an integer $k \geq 5$ such that every planar graph without i-cycle, where $4 \leq i \leq k$, is 3-colorable. Abbott and Zhou[1] showed that $k = 11$ is acceptable. In 1996, Borodin[2] and [3] improved that to $k = 10$ and 9 respectively. And to $k \leq 7$ by Borodin et al. in [4]. Xu[17] showed that every planar graph without 5-, 7-cycles and adjacent 3-cycles is 3-colorable, which implies that planar graphs without 4-, 5- and 7-cycles are 3-colorable.

However, the smallest value of k for either colorability or choosability has not yet been determined. Lam et al. [7] proved that for every plane graph G, if G is of girth 4 and contains no 5- and 6-cycles, then G is 3-choosable. In [18], Zhang et al. proved that every planar graph with girth 4 contains no cycles with length 8- and 9- is 3-choosable. Zhang et al. [20, 21] proved that every planar graph without cycles of length 4, 5, 6 and 9 or 4, 5, 7 and 9 is 3-choosable. Wang et al. [16] obtained a theorem on 3-choosability of planar graphs: planar graphs without cycles of length 4, i, j, 9 with $i < j$ and $i, j \in \{5, 6, 7, 8\}$ are 3-choosable. In [10], Montassier et al. proved that every planar graph either without 4- and 5-cycles and without triangles at distance less than 4, or without 4-, 5- and 6-cycles and without triangles at distance less than 3 is 3-choosable. Zhang et al.

[19] proved that every planar graph with neither 5-, 6-, and 7-cycles nor triangles of distance less than 3, or with neither 5-, 6-, and 8-cycles nor triangles of distance less than 2 is 3-choosable. In [9] Montassier proposed a conjecture that every planar graph without cycles of length 4, 5, 6, is 3-choosable.

In this paper, we consider the 3-choosability of graphs without cycles of length in $\{4, 5, 7, 10\}$.

Let $\mathcal{G}$ denote the set of toroidal graphs without 4-, 5-, 7- and 10-cycles. Following is our main theorem.

Theorem 1 *Every toroidal graph without 4-, 5-,7- and 10-cycles is 3-choosable.*

Two adjacent faces are *normally* adjacent if they have only two vertices in common (clearly, the two common vertices are adjacent), or is *abnormally* adjacent (that is, they have at least three vertices in common). A 3-face is often called a triangle.

For $x \in V(G) \cup F(G)$, we use $F_k(x)$ to denote the set of all k-faces that are incident or adjacent to x, and $V_k(x)$ to denote the set of all k-vertices that are incident or adjacent to x. If $|F_3(v)| = 1$ and $|d(v)| = k$ for any $v \in V(G)$, then v is called a *light* k-vertex, and it is called a *non-light* k-vertex otherwise.

For convenience, we use $N(f)$ and $V(f)$ to denote the set of faces adjacent to a face f and vertices incident with f respectively, and use $E(v)$ to denote the set of edges incident with a vertex v.

2 Preliminary Lemmas

Following lemmas will be needed for proving the main theorem. A graph G is called *minimal non-3-choosable* if G itself is not 3-choosable, but $G - v$ is 3-choosable for each vertex v of G.

Lemma 1 *[5] Every cycle of even length is 2-choosable.*

Lemma 2 *Let G be a minimal non-3-choosable graph. Then G does not contain any vertices of degree less than 3. That is $\delta(G) \geq 3$.*

Proof: Assume to the contrary that G contains a vertex v of degree less than 3. By the minimality of G, $G - v$ admits an L_0−coloring ϕ_0. In G, we can color v with a color in $L(v)$ different from the colors of its neighbours to extend ϕ_0 to an L-coloring of G. ∎

Lemma 3 *Let $G = (V, E)$ be a cycle $v_1v_2v_3 \cdots v_nv_1$ with exactly one chord v_1v_k $(3 \leq k \leq n-1)$. If $|L(v_1)|$=$|L(v_k)|= 3$ and $|L(v_i)|= 2$ where $i \neq 1$, k, then G is L-colorable.*

Proof: First we choose a color $c(v_1)$ for v_1 such that $c(v_1) \in L(v_1) \setminus L(v_n)$, and choose colors for v_2, v_3, $\cdots$, v_n successively, such that $c(v_i) \in L(v_i) \setminus \{c(v_{i-1})\}$ if $i \neq k$, and $c(v_i) \in L(v_i) \setminus \{c(v_{k-1}), c(v_1)\}$ otherwise. ∎

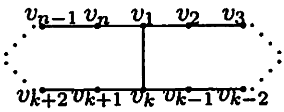

Figure 1

Lemma 4 *Let G be a minimal non-3-choosable graph. Then any $2n$-cycle C with at most one chord in G contains at least one 4^+-vertex.*

Proof: Suppose $d_G(v) = 3$ for all $v \in C$ by Lemma 2, and L is a color-list of G with $|L(v)|= 3$ for all $v \in V(G)$. If C has no chord. By assumption, there exists ϕ_0, an L_0-coloring of $G_0 = G - C$, where L_0 is the restriction of L to $V(G_0)$. Let $L' = \{L'(v_i) : 1 \leq i \leq 2n\}$ where $L'(v_i) = L(v_i) \setminus \{\phi_0(u) : u \in N_G(v_i) \setminus C\}$. It is clear that $|L'(v_i)|\geq 2$. Since even cycles are 2-choosable by Lemma 1, for examples, the cycle $v_1v_2 \cdots v_{2n}v_1$ is 2-choosable. So there exists an L'-coloring ϕ' on C. An L-coloring of G immediately follows by combining ϕ_0 and ϕ'. This contradiction implies that C contains at least one 4^+-vertex. If C has a chord, applying the similar method and combining it with Lemma 3, we get it. ∎

Lemma 5 *If $G \in \mathcal{G}$ and $\delta(G) \geq 3$, then all the followings hold.*

(I) No a 3-face is adjacent to a 3-face.
(II) No a 3-face is adjacent to a 6-face.
(III) No a 3-face is adjacent to a 9-face.
(IV) No a 6-face is adjacent to a 6-face.
(V) No a 8-face is adjacent to two 3-faces.

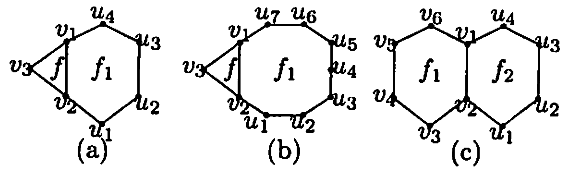

Figure 2

Proof of (I). Combine $\delta(G) \geq 3$ with the fact that G is a graph without 4-cycles, no a 3-face is adjacent to a 3-face.

Proof of (II). Let f be a 3-face with $b(f) = [v_1v_2v_3]$, and let f_1 be a 6-face with $b(f) = [v_1v_2u_1u_2u_3u_4]$. Suppose to the contrary that G has a 3-face f adjacent to a 6-face f_1. See Figure 2 (a). If f is *normally* adjacent to f_1, then G has a 7-cycle. So a 3-face must be *abnormally* adjacent to a 6-face. We claim that a 3-face can not be *abnormally* adjacent to a 6-face. To show this ,we need to prove that $v_3 \notin V(f_1)$. Clearly, $v_3 \neq u_1$ and u_4 since otherwise $\delta(G) \leq 2$. Next, $v_3 \neq u_2$ and u_3 since otherwise G would have a 4-cycle $v_2v_1v_3(v_3 = u_2)u_1v_2$ or $v_1v_2u_3(v_3 = u_3)u_4v_1$. Statement (II) is proved.

Proof of (III). Let f be a 3-face with $b(f) = [v_1v_2v_3]$, and let f_1 be a 9-face with $b(f) = [v_1v_2u_1u_2u_3u_4u_5u_6u_7]$. Suppose to the contrary that G has a 3-face f adjacent to a 9-fáce f_1. See Figure 2 (b). If f is *normal* adjacent to f_1, then G has a 10-cycle. So a 3-face must be *abnormally* adjacent to a 6-face. We claim that a 3-face can not be *abnormally* adjacent to a 9-face. To show this ,we need to prove that $v_3 \notin V(f_1)$. Clearly, $v_3 \neq u_1$, or we will get that $\delta(G) \leq 2$. Next, $v_3 \neq u_2$, u_3 and u_4 since G contains no 4-, 5- and 7-cycles. By symmetry, we also can get that $v_3 \notin \{5, 6, 7\}$. Statement (III) is proved.

Proof of (IV). Let f_1 with $b(f) = [v_1v_2v_3v_4v_5v_6]$ be an arbitrary 6-face that is adjacent to a 6-face f_2 with $b(f) = [v_1v_2u_1u_2u_4u_5]$ as shown in Figure 2 (c). If f_1 is *normally* adjacent to f_2, then G has a 10-cycle. So two 6-faces must be *abnormally* adjacent. Next we will show that $\{v_3, v_4, v_5, v_6\} \cap \{u_1, u_2, u_3, u_4\} = \phi$ to complete this proof. First, $v_3 \neq u_1$, or there will be a vertex v_2 of degree 2. Second, $v_3 \neq u_2$, $v_3 \neq u_3$ and $v_3 \neq u_4$ since G has no 4-, 5-cycles. So $v_3 \notin \{u_1, u_2, u_3, u_4\}$. Similarly, $v_4 \neq u_1$, $v_4 \neq u_2$, $v_4 \neq u_3$ and $v_3 \neq u_4$ since G has no 4-, 5-

cycles. So $v_4 \notin \{u_1, u_2, u_3, u_4\}$. By symmetry, we also can conclude that $v_5 \notin \{u_1, u_2, u_3, u_4\}$ and $v_6 \notin \{u_1, u_2, u_3, u_4\}$. Statement (IV) is proved.

Proof of (V). No two non-consecutive vertices on the boundary of a 8-face is adjacent, otherwise there will appear a cycle of length in $\{4, 5, 7\}$. So a 3-face must be *normal* adjacent to a 8-face. Statement (V) is proved because G contains no 10-cycles. ∎

3 Proof of Theorem 1

Proof: Suppose that the theorem is false. Then there exists a toroidal graph G in $\mathcal{G}$ such that G is *minimal non-3-choosable.*

We define a weight ω on $V \cup F$ by letting $\omega(x) = d_G(x) - 6$ if $x \in V$ and $\omega(x) = 2d_G(x) - 6$ if $x \in F$. By Euler's formula for toroidal graphs, $|V| + |F| - |E| = 0$, we have $\sum_{x \in V \cup F} \omega(x) = 0$. If we obtain a new nonnegative weight $\omega^*(x)$ for all $x \in V \cup F$ and some positive weight for some $x \in V \cup F$ by transferring weights from one element to another, then we have $0 = \sum_{x \in V \cup F} \omega(x) = \sum_{x \in V \cup F} \omega^*(x) > 0$. This contradiction will complete the proof.

During a discharging procedure, $\tau(x \to y)$ denotes the charge discharged from an element x to another element y.

Our transferring rules are as follows, in which, f is a 6^+-face incident with a vertex v to be charged to.

(R_1) (a) Every 6^+-face f gives 1 to each incident *non-light* 3-vertex.
(b) Every 8^+-face f gives $\frac{3}{2}$ to each incident *light* 3-vertex.

(R_2) (a) Every 6^+-face f gives $\frac{1}{2}$ to each incident *non-light* 4-vertex.
(b) If v is incident with one 3-face and three 6^+-face, then, $\tau(f \to v)$=1 if f is not adjacent to the 3-face; $\frac{1}{2}$, otherwise.
(c) If v is incident with two 3-faces, then the two 3-faces must be not adjacent, then, $\tau(f \to v) = 1$.

(R_3) Every 6^+-face f gives $\frac{1}{3}$ to each incident 5-vertex.

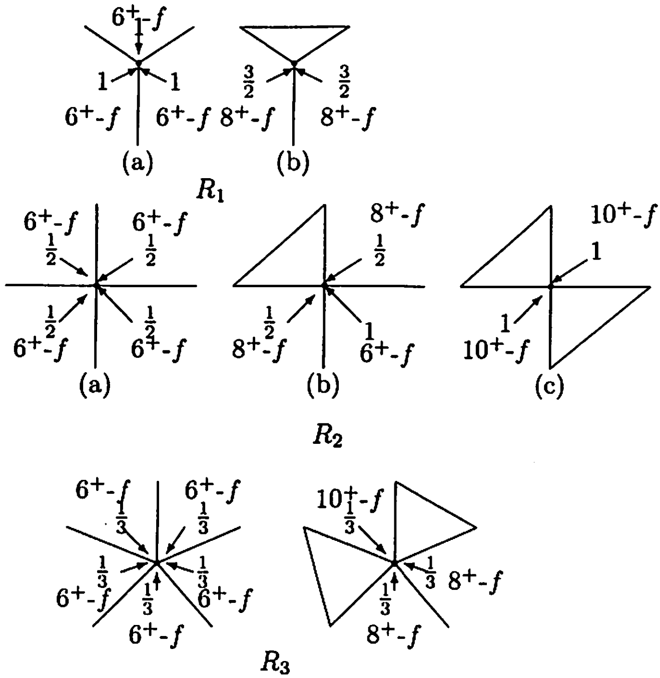

Figure 3

The rules are illustrated in Figure 3. Note that our discharging rules are designed to insure $\omega^*(v) \geq 0$ for all $v \in V(G)$.

Now let f be a face with $d(f) = h$. Then $h \in \{3, 6, 8, 9, 11^+\}$.

If $h = 3$, then $\omega^*(f) = \omega(f) = 0$ since no charge is discharged from or to f.

If $h = 6$, then by Lemma 4, $V(f)$ contains at least one 4^+-vertex. And by Lemma 5 (II), f is not incident to any *light* 3-vertices. And combining the discharging rules into consideration, the weight from a 6-face to any vertices is at most 1 and it appears only while the vertices in $V(f)$ are either *light* 4-vertices or *non-light* 3-vertices. For convenience, we denote p the number of *non-light* 3-vertices and q the number of *light* 4-vertices. It is easy to see that the worst case happens while $p \leq 5$, $q \geq 1$ and $p + q = 6$ So by R_1 (a) and R_2 (b),

$$\omega^*(f) \geq \omega(f) - p * 1 - q * 1 = 0, \tag{1}$$

Note that the worst case happens while the 6-f must contains a *light* 4-vertex v as shown in Figure 4.

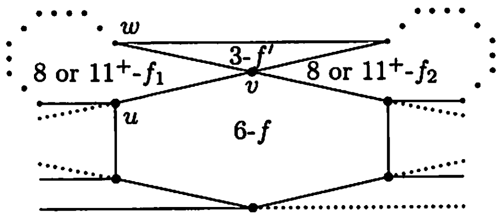

Figure 4

If $h = 8$, by Lemma 4, $V(f)$ contains at least one 4^+-vertex. And f is adjacent to at most one 3-face by Lemma 5 (V). So consider the worst case, we have two *light* 3-vertices and at least one *light* 4-vertex. So

$$\omega^*(f) \geq \omega(f) - 2 * \frac{3}{2} - 5 * 1 - 1 * 1 = 1 > 0. \tag{2}$$

If $h = 9$, by Lemma 4 (III), f is not adjacent to any 3-faces, so f is not incident to *light* 3-vertices. even if we transfer 1 to each incident vertex, we still have:

$$\omega^*(f) \geq \omega(f) - 9 * 1 > 0. \tag{3}$$

If $h = 11$, then consider the worst case and no two adjacent triangles, there must be a vertex which is not *light*. So by (R_1) and (R_2) we have:

$$\omega^*(f) \geq \omega(f) - 10 * \frac{3}{2} - 1 = 0. \tag{4}$$

If $h = 12$, then by Lemma 4, f is incident to at least one 4^+-vertex, so even if it is incident to eleven *light* 3-vertices, we have:

$$\omega^*(f) \geq \omega(f) - 11 * \frac{3}{2} - 1 = \frac{1}{2} > 0. \tag{5}$$

If $h \geq 13$, then even all the vertices on $b(f)$ are *light* 3-vertices. We have:

$$\omega^*(f) \geq \omega(f) - h * \frac{3}{2} = \frac{h - 12}{2} > 0. \tag{6}$$

Now, we get that $\omega^*(x) \geq 0$ for each $x \in V(G) \cup F(G)$. If follows that $0 = \sum_{x \in V \cup F} \omega(x) = \sum_{x \in V \cup F} \omega^*(x) \geq 0$. If $\omega^*(x)_{x \in V(G) \cup F(G)} > 0$, we are done. Assume that $\omega^*(x)_{x \in V(G) \cup F(G)} = 0$.

Claim 1 *Each face in G has degree 3, or 6, or 11.*

Proof: By (2), (3), (5), (6), G contains no faces of length in $\{8, 9, 12^+\}$. Combine with the fact that G is a graph in $\mathcal{G}$. G only has faces of degree in $\{3, 6, 11\}$.

Claim 2 *G contains no i-face for i in* $\{6, 11\}$.

Proof: If G contains a 6-face, then by (1) we must have the configuration as shown in Figure 4, that is, the 6-face is incident to at least one *light* 4-vertex v. Moreover, the two faces f_1 and f_2 incident to the *light* 4-vertex v and adjacent to the only triangle f' must be 8- or 11^+-faces. By Claim 1 it must be two 11-faces. But in this case, we must have another configuration, that is, a 11-face is adjacent to a 6-face and a 3-face on its consecutive edges and the two faces have the common *light* 4-vertex, without loss of generality, we consider the 11-face f_1, because of Lemma 5 (II), the incident vertex u must be a *non-light* 3-vertex on $b(f_1)$. Therefore the *light* 3-vertex incident to the 11-face is at most 9. So we have $\omega^*(f) \geq \omega(f) - 9 * \frac{3}{2} - 2 = \frac{1}{2} > 0$.

Claim 1 and Claim 2 ensure that G contain only 3-faces. Then G=K_3. A contradiction. ∎

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