Graph Eigenvalues Under a Graph Transformation *

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Abstract: For a graph X and a digraph D, we define the β transformation of X and the α transformation of D denoted by X^β and D^α respectively. D^α is defined as the bipartite graph with vertex set $V(D) \times \{0,1\}$ and edge set $\{\{(v_i, 0), (v_j, 1)\} | v_i v_j \in A(D)\}$. X^β is defined as the bipartite graph with vertex set $V(X) \times \{0,1\}$ and edge set $\{\{(v_i, 0), (v_j, 1)\} | v_i v_j \in A(\overrightarrow{X})\}$ where $\overrightarrow{X}$ is the associated digraph of X. In this paper, we give the relation between the eigenvalues of the digraph D and the graph D^α when the adjacency matrix of D is normal. Especially, we obtain the eigenvalues of D^α when D is some special Cayley digraph.
Keywords:: digraph; eigenvalue; normal matrix.

1 Preliminaries

For a graph X without loops and multiple edges, let $X = (V(X), E(X))$. The adjacency matrix of X is the integer matrix with rows and columus indexed by the vertices of X, such that the uv-entry of A(X) is equal to the number of edges joining u and v. A(x) is obvious symmetric 01-matrix. For a digraph D without loops and multiple arcs, let $D = (V(D), A(D))$. The adjacency matrix A(D) of digraph D is the integer matrix with rows and columns indexed by the vertices of D, such that uv-entry of A(D) is equal to the number of arcs from u to v. In general A(D) is not symmetric.

we define a graph transformation α of D as follows.

$$\alpha : D \longrightarrow D^\alpha$$

D^α is defined as the bipartite graph with vertex set $V(D) \times \{0,1\}$ and edge set $\{\{(v_i, 0), (v_j, 1)\} | v_i v_j \in A(D)\}$.

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To illustrate the definition, we give an example(Show below).

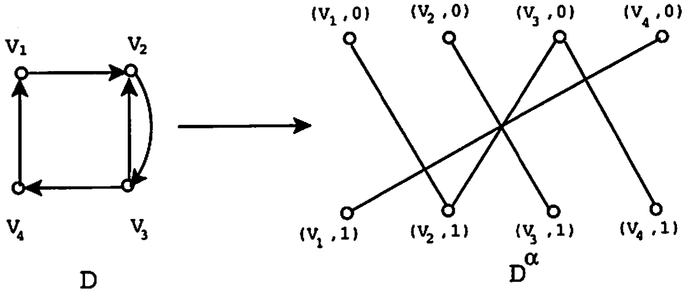

Figure.1

we define transformation β of X .

$$\beta : X \longrightarrow X^\beta$$

X^β is defined as the bipartite graph with vertex set $V(X) \times \{0,1\}$ and edge set $\{\{(v_i,0),(v_j,1)\}|v_iv_j \in A(\overrightarrow{X})\}$ where $\overrightarrow{X}$ is the associated digraph[2] of X.

To illustrate the definition, we give an example(Show below).

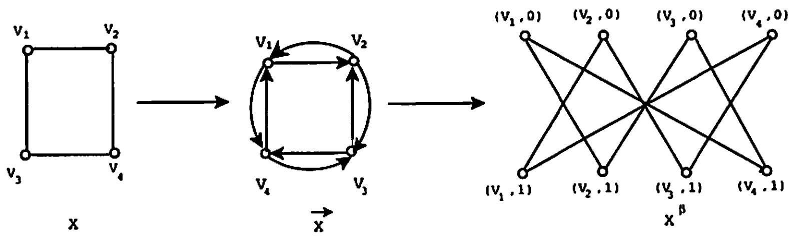

Figure.2

For a finite group G and a subset T of G, the Cayley digraph $D(G,T)$ is a directed graph with vertex set G and arc set $\{(x,tx)|x \in G, t \in T\}$. When $T = T^{-1}$, $D(G,T)$ corresponds to an undirected graph $C(G,T)$, which is called a Cayley graph. To study semi-symmetric graphs, Xu defined the Bi-Cayley graph[5]. For a finite group G and a subset T of G, the Bi-Cayley graph $X = BC(G,T)$ of G with respect to T is defined as the bipartite graph with vertex set $G \times \{0,1\}$ and edge set $\{\{(g,0),(tg,1)\}|g \in G, t \in T\}$. It is easy to see that Bi-Cayley graph BC(G,T) is $D^\alpha(G,T)(or\ C^\beta(G,T))$ of the Cayley digraph $D(G,T)$ (or graph $C(G,T)$) . Let A be the adjacency matrix of the digraph D(or graph X). Then the adjacency matrix B of $D^\alpha(orX^\beta)$ is

$$\begin{pmatrix} 0 & A \\ A^T & 0 \end{pmatrix}$$

Lemma 1.1. [4] *Suppose M, N, P, Q are $n \times n$ matrices and $|M| \neq 0$, $MP = PM$, then*

$$\begin{vmatrix} M & N \\ P & Q \end{vmatrix} = |MQ - PN|$$

Thus $|\lambda I - B| = \begin{vmatrix} \lambda I & -A \\ -A^T & \lambda I \end{vmatrix} = |\lambda^2 I - A^T A|$ according to Lemma 1.1. In this paper, we consider the relation between the eigenvalues of the digraph(or graph X) and the graph $D^\alpha (or X^\beta)$. By the above formula, it suffices to consider the relation between the eigenvalues of A and $A^T A$. In the following, we cite some known results which will be used in the next section.

A matrix $A \in C^{n \times n}$ is said to be normal if $A^* A = AA^*$, where A^* denotes the complex conjugate of the transpose of A.

Lemma 1.2. [4] *Let A be a normal matrix and λ_1, λ_2, $\cdots$, λ_n be its eigenvalues. Then there exists an unitary matrix U such that*

$$U^* A U = \begin{pmatrix} \lambda_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \lambda_n \end{pmatrix}.$$

Lemma 1.3. [7] *Let G be a finite group and and T be the union of a family of conjugacy classes of G. Let further $\{\chi_1, \cdots, \chi_s\}$ be the set of all irreducible complex characters of G and define*

$$\lambda_j := \frac{1}{\chi_j(1)} \sum_{t \in T} \chi_j(t)$$

for all $j \in \{1, \cdots, s\}$.
Then $\{\lambda_1, \cdots, \lambda_s\}$ is the set of all values of the spectrum of the Cayley digraph$D(G,T)$. Moreover, if m_j is the multiplicity of λ_j, then

$$m_j = \sum_{\substack{\lambda_k = \lambda_j \\ 1 \leq k \leq s}} \chi_k(1)^2$$

2 Main Results

Theorem 2.1. *Let D be a digraph of order n and let A be the adjacency matrix of D. Let $\{\lambda_1, \cdots, \lambda_n\}$ be the eigenvalues of A. If A is normal, then the eigenvalues of the adjacency matrix of D^α are $\pm|\lambda_1|$, $\pm|\lambda_2|$, $\cdots$, $\pm|\lambda_n|$.*

Proof. Let B be the adjacency matrix of D^α. Then $B = \begin{pmatrix} 0 & A \\ A^T & 0 \end{pmatrix}$, thus

$$|\lambda I - B| = \begin{vmatrix} \lambda I & -A \\ -A^T & \lambda I \end{vmatrix} = |\lambda^2 I - A^T A|. \tag{1}$$

Since A is normal, by Lemma 1.2, there exists an unitary matrix U satisfying

$$U^* A U = \begin{pmatrix} \lambda_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \lambda_n \end{pmatrix}. \tag{2}$$

Taking transpose of the two sides of (2) gives the following

$$U^* A^T U = \begin{pmatrix} \overline{\lambda_1} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \overline{\lambda_n} \end{pmatrix}.$$

Thus

$$U^* A^T A U = U^* A^T U U^* A U = \begin{pmatrix} \overline{\lambda_1} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \overline{\lambda_n} \end{pmatrix} \begin{pmatrix} \lambda_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \lambda_n \end{pmatrix}.$$

$$= \begin{pmatrix} |\lambda_1|^2 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & |\lambda_n|^2 \end{pmatrix}.$$

Therefore the eigenvalues of $A^T A$ are $|\lambda_1|^2, \cdots, |\lambda_n|^2$. By (1) we see that the eigenvalues of B are $\pm|\lambda_1|, \pm|\lambda_2|, \cdots, \pm|\lambda_n|$. □

By Theorem 2.1, we can obtain the following result.

Corollary 2.2. *Let X be a graph of order n. Let $\{\lambda_1, \cdots, \lambda_n\}$ be the eigenvalues of the adjacency matrix of X. Then the eigenvalues of the adjacency matrix of X^β are $\pm\lambda_1, \pm\lambda_2, \cdots, \pm\lambda_n$.*

Lemma 2.3. *Let G be a finite group and T be the union of a family of conjugacy classes of G. Then the adjacency matrix of the Cayley digraph $D(G,T)$ is normal.*

Proof. Let $A = (a_{ij})_{n\times n}$ be the adjacency matrix of $D(G,T)$. Then $A^* = A^T$, where A^T denotes the transpose of A.
Consider $(AA^T)_{ij}$ and $(A^T A)_{ij}$:

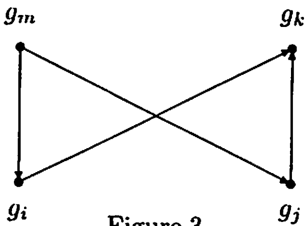

Figure.3

$$(A^T A)_{ij} = \sum_{k=1}^{n} a_{ki} a_{kj} \tag{3}$$

$$(AA^T)_{ij} = \sum_{k=1}^{n} a_{ik} a_{jk} \tag{4}$$

If $a_{ik}a_{jk} = 1$, then there exist two arcs from g_i and g_j to g_k in D(G,T), respectively (Figure.3). Thus, there exist $t_i, t_j \in T$ such that

$$\begin{cases} g_k = t_i g_i \\ g_k = t_j g_j \end{cases}$$

Then $t_i g_i = t_j g_j$. $g_j g_i^{-1} = t_j^{-1} t_i = t_j^{-1} t_i t_j t_j^{-1}$. Let $t' = t_j^{-1} t_i t_j$, then

$$g_j g_i^{-1} = t' t_j^{-1}$$

If $a_{mi}a_{mj} = 1$, then there exits two arcs from g_m to g_i and g_j in $D(G,T)$, respectively. Thus, there exist $t_i^{'}, t_j^{'} \in T$ such that

$$\begin{cases} g_i = t_i^{'} g_m \\ g_j = t_j^{'} g_m \end{cases} \tag{5}$$

Then $t_i^{'-1} g_i = t_j^{'-1} g_j$. Therefore

$$t_j^{'} t_i^{'-1} = g_j g_i^{-1}$$

Let

$$\begin{cases} t_i^{'} = t_j \\ t_j^{'} = t_j^{-1} t_i t_j \end{cases} \tag{6}$$

If $a_{il}a_{jl} = 1$ for $l \neq k$, then there exist $t_i^{''}, t_j^{''} \in T$ such that

$$\begin{cases} t_i^{''} g_i = g_l \\ t_j^{''} g_j = g_l \end{cases}$$

Since $l \neq k$, we have $g_l \neq g_k$, and hence $t_i^{''} \neq t_i$ and $t_j^{''} \neq t_j$. Therefore $t_j^{''} \neq t_i^{'}$. By (5) we know there exists $g \in G$ such that there exists two arcs from g to g_i and g_j. By (5) and (6) we have $g_i = t_i^{''} g$. Then $g = t_i^{''-1} g_i$. Because $t_j^{''} \neq t_i^{'}$, $g_m \neq g$. We have thus proved that different non-zero terms in the right hand side of (3) correspond to different non-zero terms in that of (4). The result follows. □

Theorem 2.4. *Let G be a group and T be the union of a family of conjugacy classes of G. Let $\lambda_1, \lambda_2, \cdots, \lambda_n$ be the eigenvalues of the adjacency matrix of the Cayley digraph $D(G,T)$. Then the eigenvalues of the adjacency matrix of $BC(G,T)$ are $\pm|\lambda_1|, \pm|\lambda_2|, \cdots, \pm|\lambda_n|$.*

Proof. The result follows directly from Theorem 2.1 and Lemma 2.3 . □

Corollary 2.5. *Let G be a finite group and T be the the union of a family of conjugacy classes of G. Let further $\{\chi_1, \cdots, \chi_s\}$ be the set of all irreducible complex characters of G and define*

$$\lambda_j := |\frac{1}{\chi_j(1)} \sum_{t \in T} \chi_j(t)|, \quad \lambda_{s+j} := -|\frac{1}{\chi_j(1)} \sum_{t \in T} \chi_j(t)|$$

for all $j \in \{1, \cdots, s\}$.
Then $\{\lambda_1, \cdots, \lambda_{2s}\}$ is the set of all values of the spectrum of $BC(G,T)$. Moreover, if m_j is the mulplicity of λ_j, λ_{j+s}, then

$$m_j = \sum_{\substack{\lambda_k = \lambda_j \\ 1 \leq k \leq s}} \chi_k(1)^2$$

Proof. The result follows directly from Lemma 1.3 and Theorem 2.4. □

3 Examples

In this section, we apply the results in section 2 to give some interesting examples.

Example 1: The Kneser graph $K_{v:r}$ is the graph with the r-subsets of a fixed v-set as its vertices with twor-subsets adjacent if they are disjoint. It is known that the eigenvalues of the Kneser graph $K_{v:r}$ are the integers $(-1)^i \begin{pmatrix} v-r-i \\ r-i \end{pmatrix}$ (see[1]), $i = 0, 1, \cdots, r$. By Corollary 2.2 we can get the eigenvalues of graph $K^{\beta}_{v:r}$ are $\pm(-1)^i \begin{pmatrix} v-r-i \\ r-i \end{pmatrix}$, $i = 0, 1, \cdots, r$.

Example 2: Recall the character tables of circulant group C_3(see [2]):

	1	r	r^2
χ_1	1	1	1
χ_2	1	w	w^2
χ_3	1	w^2	w

where $w = e^{2\pi i/n}$

Because every conjugacy class of C_3 contains exactly one element, let $T = \{r\}$. By Lemma 1.3 we can get the eigenvalues of $C(C_3, T)$:

$$\begin{cases} \lambda_1 = 1 \\ \lambda_2 = w \\ \lambda_3 = w^2 \end{cases}$$

The multiplicities are $\{1, 1, 1\}$. Therefore by Theorem 2.4 the eigenvalues of Bi-Cayley graphs $BC(C_3, T)$ are 1 and -1, but the multiplicity is $\{3, 3\}$ respectively.

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