A Note on Chromatic Uniqueness of Certain Complete Tripartite Graphs*

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Abstract

Let $P(G,\lambda)$ be the chromatic polynomial of a graph G. A graph G is chromatically unique if for any graph H, $P(H,\lambda) = P(G,\lambda)$ implies $H \cong G$. Some sufficient conditions guaranteeing that certain complete tripartite graph $K(l,n,r)$ is chromatically unique were obtained by many scholars. Especially, in 2003, H.W. Zou had given that if $n > \frac{1}{3}(m^2+k^2+mk+2\sqrt{m^2+k^2+mk}+m-k)$, where n, k and m, are non-negative integers, then $K(n-m,n,n+k)$ is chromatically unique (or simply χ-unique). In this paper, we give that for any positive integers n, m and k, let $G = K(n-m,n,n+k)$, where $m \geq 2$ and $k \geq 1$, if $n \geq \max\{\lceil \frac{1}{4}m^2+m+k\rceil, \lceil \frac{1}{4}m^2+\frac{3}{2}m+2k-\frac{11}{4}\rceil, \lceil mk+m-k+1\rceil\}$, then G is χ-unique. It is an improvement on H.W. Zou's result in the case $m \geq 2$ and $k \geq 1$.

Key Words: Complete tripartite graph; Chromatic polynomial; Chromatic uniqueness; Chromatically unique graph.

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1 Introduction

We consider only finite, undirected and simple graphs. Notation and terminology that are not defined here may be found in [1, 2].

Let G be a graph with vertex set $V(G)$ and edge set $E(G)$, order $p(G)$ and size $q(G)$. By $\overline{G}$ denotes the complement of G. We let $O_n = \overline{K_n}$, where K_n denotes the complete graph with n vertices. For disjoint graphs G and H, $G \vee H$ denotes the graphs whose vertex-set is $V(G) \cup V(H)$ and whose edge-set is $\{wv \in V(G) | w \in V(G), v \in V(H)\} \cup E(G) \cup E(H)$. By $K(l,n,r)$ we denote the complete tripartite graph with three parts of l, n, r vertices. Let S be a set of s edges of G. By $G - S$ we denote the graph by deleting all edges in S from G. Let $N_3(G)$ denotes the number of triangles in G. $\lceil \theta \rceil$

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denotes the smallest integer greater than or equal to θ. $\mathbf{N}$ is non-negative integers set.

Let $P(G,\lambda)$ be the chromatic polynomial of G. $m_r(G)$ denotes the number of distinct partitions of $V(G)$ into r color classes. Let $\lambda_{(r)} = \lambda(\lambda-1)...(\lambda-r+1)$, then we have $P(G,\lambda) = \sum_{r=1}^{p} m_r(G)\lambda_{(r)}$ (see [1]).

The notion of chromatic uniqueness was first introduced and studied by Chao and Whitehead in 1978 (see [5]). Koh and Teo, in their expository paper (see [7,8]), gave a survey of most of the work done before 1997. Two graphs H and G are said to be chromatically equivalent (in notation: $H \sim G$) if $P(H,\lambda) = P(G,\lambda)$. Let $\langle G\rangle = \{H|H \sim G\}$. A graph G is chromatically unique if $\langle G\rangle = \{G\}$. The polynomial $\sigma(G,\chi) = \sum_{r=1}^{p} m_r(G)\chi^r$ is called the σ–polynomial of G (see [3]). Clearly, $P(H,\chi) = P(G,\chi)$ iff $\sigma(G,\chi) = \sigma(H,\chi)$.

It has been shown in [4, 6, 9, 11, 13] that the following complete tripartite graphs are χ–unique graphs: $K(n,n,n+k)$ for $n \geq 2$ and $0 \leq k \leq 3$, $K(n-k,n,n+k)$ for $n \geq 5$ and $0 \leq k \leq 2$ (see [6]); $K(n_1,n_2,n_3)$ for $|n_i - n_j| \leq 1$ and $1 \leq i,j \leq 3$ (see [4]); $K(n-k,n,n)$ for $n \geq k+2 \geq 4$ (see [9]); $K(n-k,n,n)$ for $n > \frac{1}{3}k^2 + k$ (see [11,13]); $K(n,n,n+k)$ for $n > \frac{1}{3}(k^2+k)$ (see [11]); $K(n-k,n,n+k)$ for $n > k^2 + \frac{2\sqrt{3}}{3}k$ (see [11]).

In 2003 H.W.Zou had given the following χ–unique graphs (see [12]): $K(n-m,n,n+k)$ for any non-negative integers n,k and m with $n > \frac{1}{3}(m^2+k^2+mk+2\sqrt{m^2+k^2+mk}+m-k)$.

In this paper, we will show that the following complete tripartite graph is also χ–unique: $K(n-m,n,n+k)$ for $n \geq \max\{\lceil\frac{1}{4}m^2+m+k\rceil, \lceil\frac{1}{4}m^2+\frac{3}{2}m+2k-\frac{11}{4}\rceil, \lceil mk+m-k+1\rceil\}$, where k,m and n are any positive integers with $m \geq 2$ and $k \geq 1$. It is an improvement on H.W. Zou's result in the case $m \geq 2$ and $k \geq 1$.

2 Preliminaries

Lemma 2.1 (*C.P. Teo, K.M. Koh* [10]) *Let G and H be two graphs with $G \sim H$. Then $|V(G)| = |V(H)|$, $|E(G)| = |E(H)|$, $N_3(G) = N_3(H)$ and $m_r(G) = m_r(H)$ for $r = 1,2,\cdots,p(G)$.*

Lemma 2.2 (*C.P. Teo, K.M. Koh* [10]) *Let $c \geq d \geq 2$. Then $K(c,d)$ is χ–unique.*

Lemma 2.3 (*F. Brenti* [3]) *Let G and H be two disjoint graphs. Then*
$$\sigma(G \vee H,\tau) = \sigma(G,\tau)\sigma(H,\tau).$$
In particular, $\sigma(K(n_1,n_2,\cdots,n_t),\tau) = \prod_{i=1}^{t}\sigma(O_{n_i},\tau)$.

Lemma 2.4 (*H.W. Zou* [13]) *Let $G = K(n_1,n_2,n_3)$. Then*

(i) $m_3(G) = 1$ and $m_4(G) = \sum_{i=1}^{3} 2^{n_i-1} - 3$;

(ii) If $H \in \langle G\rangle$, there exists a complete tripartite graph $F = K(m_1,m_2,m_3)$ such that $H = F - S$ and $m_1+m_2+m_3 = n_1+n_2+n_3$, where S is a set of s edges of F and $s = q(F) - q(G)$.

Lemma 2.5 *(H.W. Zou [13]) Let $G = K(n_1, n_2, n_3)$ with $n_3 \geq n_2 \geq n_1 \geq 2$ and let $H = G - S$ for a set S of s edges of G. If $n_1 \geq s+1$, then $s \leq m_4(H) - m_4(G) \leq 2^s - 1$.*

Theorem 2.1 *(R.Y. Liu et al. [9]) For any integers $r \geq n \geq l \geq 2$, we have $\langle K(l,n,r)\rangle \subseteq \{K(x,y,z) - S | 1 \leq x \leq y \leq z, n \leq z \leq r, x+y+z = l+n+r, S \subset E(K(x,y,z)), |S| = xy+xz+yz-ln-lr-nr\}$. In particular, if $z = r$, then $K(l,n,r) \cong K(x,y,z)$.*

Theorem 2.2 *(H.W. Zou [12]) Let $G = K(l,n,r), l \leq n \leq r$ and $a = \{2[(l-n)^2 + (l-r)^2 + (n-r)^2]\}^{\frac{1}{2}}$. If $l+n+r > a + \frac{1}{4}a^2$, then G is χ-unique.*

Theorem 2.3 (*H.W. Zou* [12]) Let $K(l,n,r) = K(n-m,n,n+k)$, where m and k are non-negative integers. If $n > \frac{1}{3}(m^2+k^2+mk+2\sqrt{m^2+k^2+mk}+m-k)$, then $K(n-m,n,n+k)$ is χ-unique.

3 Main Results

Theorem 3.1 *For any positive integers m, k and n, where $m \geq 2$ and $k \geq 1$, let $G = K(n-m,n,n+k)$, if $n \geq \max\{\lceil \frac{1}{4}m^2 + m + k\rceil, \lceil \frac{1}{4}m^2 + \frac{3}{2}m + 2k - \frac{11}{4}\rceil, \lceil mk + m - k + 1\rceil\}$, then G is χ-unique.*

Proof: Let $H \in \langle G\rangle$. Then by Theorem 2.1, $H \in \{K(x,y,z) - S | 1 \leq x \leq y \leq z, n \leq z \leq n+k, |S| = s = xy + yz + xz - (n-m)n - (n-m)(n+k) - n(n+k), x+y+z = 3n+k-m\}$.

Case 1: If $z = n+k$, by Theorem 2.1, $H \cong G$.

Case 2: For $z = n$, we distinguish the following two subcases.

Subcase 2.1: $x \leq y = z = n$. Let $F = K(n+k-m,n,n)$, $H = F - S$ and $\beta(H) = m_4(H) - m_4(F)$. By Lemma 2.4, we have

$|S| = s = q(F) - q(G) = (n+k-m)n + (n+k-m)n + n^2 - (n-m)n - (n-m)(n+k) - n(n+k) = km > 0$,

$m_4(F) = 2^{n+k-m-1} + 2^{n-1} + 2^{n-1} - 3$,

$m_4(G) = 2^{n-m-1} + 2^{n-1} + 2^{n+k-1} - 3$.

By the conditions of the theorem and the Lemma 2.5, we have

$$s+1 = km+1 \leq n+k-m \text{ and } km \leq \beta(H) \leq 2^{km} - 1.$$

So

$m_4(G) - m_4(H) = (2^{n-m-1} + 2^{n-1} + 2^{n+k-1} - 3) - (2^{n+k-m-1} + 2^{n-1} + 2^{n-1} - 3 + \beta(H))$

$\geq 2^{n-m-1} + 2^{n+k-1} - 2^{n+k-m-1} - 2^{n-1} - 2^{km} + 1$

$\geq 2^{n-m-1} + 2^{n+k-1} - 2^{n+k-m} - 2^{n-1} + 1$.

Since $m \geq 2$ and $k \geq 1$, we have $\frac{1}{2} + 2^k 2^{m-1} - 2^k - 2^{m-1} > 0$, i.e., $(\frac{1}{2} + 2^k 2^{m-1} - 2^k - 2^{m-1})2^{-m} > 0$. Hence $2^{n-m-1} + 2^{n+k-1} - 2^{n+k-m} - 2^{n-1} > 0$, i.e., $m_4(G) - m_4(H) > 1$. This contradicts $m_4(G) = m_4(H)$.

Subcase 2.2: $z = n$ and $x \leq y \leq n-1$. Let $F = K(x,y,n)$, $H = F - S$. Let V_1, V_2, V_3 be the unique 3-independent partition of $K(x,y,n)$

such that $|V_1| = x$, $|V_2| = y$, $|V_3| = n$. By Lemma 2.1, $x + y = 2n + k - m$, $N_3(G) = N_3(H)$. Hence, we shall consider the number of triangles in G and H. Without loss of generality, let $S = \{e_1, e_2, \cdots, e_s\} \subset E(F)$. It is not hard to see that $N_3(e_i) \leq n$. Then

$$N_3(H) \geq N_3(F) - ns \tag{1}$$

and the equality holds only if $N_3(e_i) = n$ for all $e_i \in S$.

Let $\eta = N_3(F) - N_3(G)$. It is obvious that $N_3(F) = xyn$, $N_3(G) = n(n-m)(n+k)$ and $\eta = xyn - n(n-m)(n+k)$. So, we have

$$N_3(G) = N_3(F) - \eta. \tag{2}$$

Since $N_3(G) = N_3(H)$, from (1) and (2) it follows that $\eta \leq sn$.

Let $f(z) = \eta - sn$, recalling that $s = xy + xn + yn - n(n-m) - (n-m)(n+k) - n(n+k)$, we have $f(n) = \eta - sn = n^2[n+k+n-m-(x+y)] = 0$, i.e., $\eta = sn$. From (1) and (2), we have $N_3(G) = N_3(H) = N_3(F) - sn$ and $N_3(e_i) = n$ for all $e_i \in S$. Thus for every edge an end-vertex belongs to V_1, whereas the other end-vertex belongs V_2. Hence $\overline{H}$ contains K_n as its component. Set $\overline{H} = \overline{H_1} \bigcup K_n$. Then $H = H_1 \vee O_n$. From Lemma 2.3 and $\sigma(H, \tau) = \sigma(K(n-m, n, n+k), \tau)$, we have $\sigma(H_1 \vee O_n, \tau) = \sigma(O_{n-m} \vee O_n \vee O_{n+k}, \tau)$. So $\sigma(H_1, \tau) = \sigma(O_{n-m} \vee O_{n+k}, \tau) = \sigma(K(n-m, n+k), \tau)$. Hence, from Lemma 2.2 and the condition of the theorem, we have $H_1 = K(n+k, n-m)$. So $y = n + k$, which contradicts $y \leq n - 1$.

Case 3: For $z = n + k - 1$, let $H = K(n-k-m+u+1, n+k-u, n+k-1) - S$, where u is integer number. According to $n - k - m + u + 1 \leq n + k - u \leq n + k - 1$, we have $1 \leq u \leq \frac{1}{2}(m + 2k - 1)$. By Lemma 2.4, we have

$|S| = s = q(F) - q(G)$
$= -u^2 + (m + 2k - 1)u - k^2 - km + m + 2k - 1$
$= -[u - \frac{1}{2}(m + 2k - 1 - \sqrt{m^2 + 2m + 4k - 3})][u - \frac{1}{2}(m + 2k - 1 + \sqrt{m^2 + 2m + 4k - 3})]$.

Let $g(u) = n - k - m + u + 1 - (s + 1) = u^2 + (2 - m - 2k)u + n + km + k^2 - 3k - 2m + 1$, we shall consider sign of $g(u)$, we distinguish the following cases.

(i) When $\frac{1}{2}(m + 2k - 1 - \sqrt{m^2 + 2m + 4k - 3}) < 1$, we have $1 \leq u \leq \frac{1}{2}(m + 2k - 1)$ and $g_{min}(u) = g[\frac{1}{2}(m + 2k - 2)] = n - (\frac{1}{4}m^2 + m + k)$. It follows that $n - (\frac{1}{4}m^2 + m + k) \geq 0$. So $g(u) \geq 0$.

(ii) When $\frac{1}{2}(m + 2k - 1 - \sqrt{m^2 + 2m + 4k - 3}) \geq 1$, we have

$$\frac{1}{2}(m + 2k - 1 - \sqrt{m^2 + 2m + 4k - 3}) \leq u \leq \frac{1}{2}(m + 2k - 1).$$

Because of $m \geq 2$ and $k \geq 1$, so

$$\frac{1}{2}(m + 2k - 2) > \frac{1}{2}(m + 2k - 1 - \sqrt{m^2 + 2m + 4k - 3}).$$

By the condition of the theorem, it follows that

$$g_{min}(u) = g[\frac{1}{2}(m + 2k - 2)] = n - (\frac{1}{4}m^2 + m + k) \geq 0.$$

From (i) and (ii) it follows that $s+1 \le n-k-m+u+1$. So $g(u) \ge 0$.

From Lemma 2.5, we have $s \le m_4(H) - m_4(F) = \beta(H) \le 2^s - 1$ and $m_4(G) - m_4(H) = (2^{n-m-1} + 2^{n-1} + 2^{n+k-1} - 3) - (2^{n-k-m+u} + 2^{n+k-u-1} + 2^{n+k-2} - 3 + \beta(H))$
$\ge 2^{n-m-1} + 2^{n-1} + 2^{n+k-1} - 2^{n-k-m+u} - 2^{n+k-u-1} - 2^{n+k-2} - 2^{n-k-m+u} + 1$
$= 2^{n-m-1} + 2^{n-1} + 2^{n+k-2} - 2^{n+k-u-1} - 2^{n-k-m+u+1} + 1$.

Because of $u \le \frac{1}{2}(m+2k-1)$, we have $2^k + 2^{k+m} \ge 2^{u+2}$, i.e., $2^{k+u} + 2^{k+m+u} - 2^{2u+2} \ge 0$. Since $2^{m+2k+u-1} - 2^{m+2k} \ge 0$, we have $2^{k+u} + 2^{m+k+u} + 2^{m+2k+u-1} - 2^{m+2k} - 2^{2u+2} \ge 0$, i.e.,$2^{n-m-1} + 2^{n-1} + 2^{n+k-2} - 2^{n+k-u-1} - 2^{n-k-m+u+1} \ge 0$. Hence $m_4(G) - m_4(H) \ge 1$, this is impossible.

If $1 \le k \le 2$, from Case 1, Case 2 and Case 3 it shows that process of the proof has been completed. If $k \ge 3$, we shall consider the following Case 4.

Case 4: Let $z = n+k-t$ ($k \ge 3$ and $2 \le t \le k-1$), $F = K(n-k-m+u+t, n+k-u, n+k-t)$ and $H = F - S$, we can easily obtain that $t \le u \le \frac{1}{2}(m+2k-t)$. By lemma 2.4, we have

$|S| = s = q(F) - q(G)$
$= (n-k-m+u+t)(2n+2k-u-t) + (n+k-u)(n+k-t) - (n-m)(2n+k) - n(n+k)$
$= -u^2 + u(m+2k-t) + 2kt + mt - km - k^2 - t^2$.

Because of $2 \le t \le k-1$, we have $m^2 - 3t^2 + 4kt + 2mt > 0$. So

$s = -[u - \frac{1}{2}(m+2k-t-\sqrt{m^2-3t^2+4kt+2mt})][u - \frac{1}{2}(m+2k-t+\sqrt{m^2-3t^2+4kt+2mt})]$.

Let $h(u) = n-k-m+u+t-(s+1) = u^2 + u(t-m-2k+1) + t^2 + k^2 + km - 2kt - mt + n - k - m + t - 1$, we shall consider sign of $h(u)$, we distinguish the following cases.

(i) When $\frac{1}{2}(m+2k-t-\sqrt{m^2-3t^2+4kt+2mt}) < t$, we have $t \le u \le \frac{1}{2}(m+2k-t)$. Because of $m \ge 2$ and $k \ge 3$, we have $\frac{1}{2}(m+2k-t-1) > t$. So $h_{min}(u) = h[\frac{1}{2}(m+2k-t-1)] = \frac{1}{4}(3t^2 - m^2 + 2t - 2mt - 4kt + 4n - 2m - 5)$. Because of $2 \le t \le k-1$, we have $\frac{1}{4}(3t^2 - m^2 + 2t - 2mt - 4kt + 4n - 2m - 5) \ge \frac{1}{4}(4n - m^2 - 6m - 8k + 11)$. By the condition of the theorem, it follows that $\frac{1}{4}(4n - m^2 - 6m - 8k + 11) \ge 0$. So $h(u) \ge 0$.

(ii) When $\frac{1}{2}(m+2k-t-\sqrt{m^2-3t^2+4kt+2mt}) \ge t$, we have

$$\frac{1}{2}(m+2k-t-\sqrt{m^2-3t^2+4kt+2mt}) \le u \le \frac{1}{2}(m+2k-t).$$

Because of $m \ge 2$, $k \ge 3$ and $2 \le t \le k-1$, we have $\sqrt{m^2-3t^2+4kt+2mt} > 1$. So $\frac{1}{2}(m+2k-t-1) > \frac{1}{2}(m+2k-t-\sqrt{m^2-3t^2+4kt+2mt})$. By the condition of the theorem, it follows that $h_{min}(u) = h[\frac{1}{2}(m+2k-t-1)]$ $= \frac{1}{4}(3t^2 - m^2 + 2t - 2mt - 4kt + 4n - 2m - 5) \ge \frac{1}{4}(4n - m^2 - 6m - 8k + 11) \ge 0$. So $h(u) \ge 0$.

From (i) and (ii) it follows that $s+1 \le n-k-m+u+t$.

By Lemma 2.5, we have $s \le m_4(H) - m_4(F) = \beta(H) \le 2^s - 1$ and

$m_4(G)-m_4(H)$
$= (2^{n-m-1}+2^{n-1}+2^{n+k-1}-3)-(2^{n-k-m+u+t-1}+2^{n+k-u-1}+2^{n+k-t-1}-3+\beta(H))$
$\geq 2^{n-m-1}+2^{n-1}+2^{n+k-1}-2^{n-k-m+u+t-1}-2^{n+k-u-1}-2^{n+k-t-1}-2^{n-k-m+u+t-1}+1$
$=2^{n-m-1}+2^{n-1}+2^{n+k-1}-2^{n-k-m+u+t}-2^{n+k-u-1}-2^{n+k-t-1}+1$
$\geq 2^{n-m-1}+2^{n-1}+2^{n+k-1}-2^{n+k-u-1}-2^{n+k-t-1}-2^{n+k-u}+1$
$\geq 2^{n-m-1}+2^{n-1}+2^{n+k-1}-2^{n+k-t+1}+1.$
Because of $n+k-1\geq n+k-t+1$, we have $m_4(G)-m_4(H)\geq 1$, this is impossible. The proof is completed. □

Theorem 3.2 *Let* $\Omega=\{(m,k)|m\geq 2, k\geq 1, m,k\in \mathbf{N}\}$, $\xi=\frac{1}{3}(m^2+k^2+mk+2\sqrt{m^2+k^2+mk}+m-k)$, $A=\frac{1}{4}m^2+m+k, B=\frac{1}{4}m^2+\frac{3}{2}m+2k-\frac{11}{4}, C=mk+m-k+1$. *Suppose*

$$R=\{(m,k)|m\geq 2, k\geq 1, \lceil\xi\rceil=\max\{\lceil A\rceil,\lceil B\rceil,\lceil C\rceil\},$$
$$T=\{(m,k)|m\geq 2, k\geq 1, \lceil\xi\rceil>\max\{\lceil A\rceil,\lceil B\rceil,\lceil C\rceil\}.$$

Then $R\neq\emptyset, T\neq\emptyset, R\bigcap T=\emptyset$ *and* $R\bigcup T=\Omega$.

Proof: Since $\xi-A=\frac{1}{12}m^2+\frac{1}{3}k^2+\frac{1}{3}mk+\frac{2}{3}\sqrt{k^2+m^2+km}-\frac{4}{3}k-\frac{2}{3}m$, $\sqrt{k^2+m^2+km}>\frac{1}{2}k+m$, and $\frac{1}{3}k^2+\frac{1}{3}mk-k\geq 0$, we have

$$\frac{1}{12}m^2+\frac{1}{3}k^2+\frac{1}{3}mk+\frac{2}{3}\sqrt{k^2+m^2+km}-\frac{4}{3}k-\frac{2}{3}m$$
$$>\frac{1}{12}m^2+\frac{1}{3}k^2+\frac{1}{3}mk+\frac{1}{3}k+\frac{2}{3}m-\frac{4}{3}k-\frac{2}{3}m$$
$$=\frac{1}{12}m^2+\frac{1}{3}k^2+\frac{1}{3}mk-k\geq\frac{1}{12}m^2>0.$$

So we have the following fact

Fact 1: $\xi>A, i.e., \lceil\xi\rceil\geq\lceil A\rceil$.

Since $\xi-B=\frac{1}{12}m^2+\frac{1}{3}k^2+\frac{1}{3}mk+\frac{2}{3}\sqrt{k^2+m^2+km}-\frac{7}{6}m-\frac{7}{3}k+\frac{11}{4}$, and $\sqrt{k^2+m^2+km}>\frac{1}{2}k+m$, we have

$$\frac{1}{12}m^2+\frac{1}{3}k^2+\frac{1}{3}mk+\frac{2}{3}\sqrt{k^2+m^2+km}-\frac{7}{6}m-\frac{7}{3}k+\frac{11}{4}$$
$$>\frac{1}{12}m^2+\frac{1}{3}k^2+\frac{1}{3}mk+\frac{1}{3}k+\frac{2}{3}m-\frac{7}{6}m-\frac{7}{3}k+\frac{11}{4}$$
$$=\frac{1}{12}[(m-3)^2+4(k-3)^2+4mk-12].$$

If $mk\geq 3$, we have $\frac{1}{12}[(m-3)^2+4(k-3)^2+4mk-12]>0$. If $mk=2$, i.e., $m=2$ and $k=1$, we get $\frac{1}{12}[(m-3)^2+4(k-3)^2+4mk-12]=\frac{13}{12}>0$. So we have

Fact 2: $\xi>B, i.e., \lceil\xi\rceil\geq\lceil B\rceil$.

Because of $\xi-C=\frac{1}{3}m^2+\frac{1}{3}k^2+\frac{2}{3}k+\frac{2}{3}\sqrt{k^2+m^2+km}-\frac{2}{3}km-\frac{2}{3}m-1$ $=\frac{1}{3}(m-k)^2+\frac{2}{3}k+\frac{2}{3}\sqrt{k^2+m^2+km}-\frac{2}{3}m-1$, and $\sqrt{k^2+m^2+km}>m+\frac{1}{2}$, therefore $\frac{1}{3}(m-k)^2+\frac{2}{3}k+\frac{2}{3}\sqrt{k^2+m^2+km}-\frac{2}{3}m-1>\frac{1}{3}(m-k)^2+\frac{2}{3}k+\frac{2}{3}(m+\frac{1}{2})-\frac{2}{3}m-1>0$. So we have

Fact 3: $\xi>C, i.e., \lceil\xi\rceil\geq\lceil C\rceil$.

From **Fact 1**, **Fact 2** and **Fact 3**, we have that $\lceil\xi\rceil\geq\max\{\lceil A\rceil,\lceil B\rceil,\lceil C\rceil\}$. Thus $\Omega=R\bigcup T$. Obviously we have $R\bigcap T=\emptyset$. Since $(2,1)\in R$ and $(2,2)\in T$, we have $R,T\neq\emptyset$. This completes the proof. □

Remark. The condition of Theorem 2.3 is that $n > \xi$, i.e., $n \geq \lceil \xi \rceil$ (when ξ is an integer) or $n \geq \lceil \xi \rceil + 1$ (when ξ is not an integer). Theorem 3.1 is an improvement for Theorem 2.3 when $(m, k) \in T$ or $(m, k) \in R$ and ξ is an integer. For example, for graph $K(2,4,5)$, we have $\lceil \xi \rceil = 14, \max\{\lceil A \rceil, \lceil B \rceil, \lceil C \rceil\} = \max\{4,4,4\} = 4$. From Theorem 3.1, we know that $K(2,4,5)$ is χ–unique. But we did not deduce that by Theorem 2.3.

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