# Fractional incidence coloring and star arboricity of graphs

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**Abstract**

This paper generalizes the results of Guiduli [B. Guiduli, On incidence coloring and star arboricity of graphs. Discrete Math. 163 (1997), 275–278] on the incidence coloring of graphs to the fractional incidence coloring. Tight asymptotic bounds analogous to Guiduli's results are given for the fractional incidence chromatic number of graphs. The fractional incidence chromatic number of circulant graphs is studied. Relationships between the $k$-tuple incidence chromatic number and the incidence chromatic number of the direct products and lexicographic products of graphs are established. Finally, for planar graphs $G$, it is shown that if $\Delta(G) \neq 6$, then $\chi_i(G) \leq \Delta(G) + 5$; if $\Delta(G) = 6$, then $\chi_i(G) \leq \Delta(G) + 6$; where $\chi_i(G)$ denotes the incidence chromatic number of $G$. This improves the bound $\chi_i(G) \leq \Delta(G) + 7$ for planar graphs given in [M. Hosseini Dolama, É. Sopena, X. Zhu, Incidence coloring of $k$-degenerated graphs. Discrete Math. 283 (2004), no. 1-3, 121–128].

## 1 Introduction

For a finite graph $G$, we use $V(G)$ and $E(G)$ to denote its vertex set and edge set. For a positive integer $k$, the set $\{1, ..., k\}$ is denoted by $[k]$. For a

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graph $G$, $\alpha(G)$ denotes the independence number of $G$, and $|G| = |V(G)|$. A *k-edge coloring* of a graph $G = (V, E)$ is a mapping $c$ from $E(G)$ to a color set $S$ with $|S| = k$, such that adjacent edges are assigned different colors. The *edge chromatic number*, or *chromatic index*, $\chi'(G)$ of $G$ is the smallest $k$ such that G admits a $k$-edge coloring. For the chromatic index $\chi'(G)$, we have the well known Vizing's theorem:

**Theorem 1.1** *(Vizing 1964) Every graph $G$ satisfies $\Delta \leq \chi'(G) \leq \Delta + 1$.*

Vizing's theorem divides the finite graphs into two classes according to their chromatic index; graphs satisfying $\chi' = \Delta$ are called *class 1*, those with $\chi' = \Delta + 1$ are called *class 2*.

An *incidence* in $G$ is a pair $(v, e)$ with $v \in V(G)$ and $e \in E(G)$, such that $v$ and $e$ are incident. We denote by $I(G)$ the set of all the incidences in $G$. For every vertex $v$, we denote by $I_v$ the set of incidences of the form $(v, vx)$, and by $A_v$ the set of incidences of the form $(y, yv)$. Two incidences $(v, e)$ and $(w, f)$ are adjacent if one of the following holds: (*i*) $v = w$, (*ii*) $e = f$, or (*iii*) $\{v, w\}$ is one of the edges $e$ or $f$. We note here that for any incidence $(v, vw)$, the set of all the incidences that are adjacent to $(v, vw)$ (including itself) is $I_v \cup A_v \cup I_w$.

A *k-incidence coloring* of a graph $G$ is a mapping $\sigma$ of $I(G)$ to a color set $C$ with $|C| = k$, such that adjacent incidences are assigned different colors. The *incidence chromatic number* (or *incidence coloring number*), $\chi_i(G)$ of $G$ is the smallest $k$ such that G admits a $k$-incidence coloring. The *incidence graph of a graph $G$*, denoted $Inc(G)$, has vertex set $I(G)$, and two vertices of $Inc(G)$ are adjacent in $Inc(G)$ if and only if the corresponding two incidences of $G$ are adjacent in $G$. Clearly, $|V(Inc(G))| = |I(G)| = 2|E(G)|$, and $\chi_i(G) = \chi(Inc(G))$.

Incidence colorings were introduced by Brualdi and Massey [4] in 1993. It is easy to see that for every graph G with at least one edge, $\chi_i(G) \geq \Delta(G) + 1$. Brualdi and Massey [4] proved the following results:

**Theorem 1.2** *([4]) For each $n \geq 2$, $\chi_i(K_n) = n$.*

**Theorem 1.3** *([4]) Let $T$ be a tree of order $\geq 2$ with maximum degree $\Delta$, then $\chi_i(T) = \Delta + 1$.*

**Theorem 1.4** *([4]) For every graph $G$, $\chi_i(G) \leq 2\Delta(G)$.*

In [9], Guiduli made the following observation: if we think of an incidence pair as a directed edge, directed toward the vertex, we are coloring the edges of the symmetrically directed graph $S(G)$ (here in $S(G)$ we replace each edge of $G$ by both directed edges). Then each color class is a directed star forest (edges are directed out of the center). Therefore the concept of incidence coloring is a particular case of directed star arboricity, introduced by Algor and Alon [1]. Recall that the *star arboricity* of a graph $G$ is the minimum number of star forests in $G$ whose union covers all edges of $G$. And a *star forest* is a forest whose connected components are stars. Following an example from [1], Guiduli proved that there exist graphs $G$ with $\chi_i(G) \geq \Delta(G) + \Omega(\log \Delta(G))$. He also proved, for every graph $G$, $\chi_i(G) \leq \Delta(G) + O(\log \Delta(G))$, which is the following theorem:

**Theorem 1.5** *(Guiduli [9]) Let $G$ be a graph with maximum degree $\Delta$, then*

$$\chi_i(G) \leq \Delta + 20 \log \Delta + 84.$$

With respect to the incidence chromatic number of special classes of graphs, in [6], Chen et al showed: For every Halin graph $G$ with $\Delta(G) \geq 5$, $\chi_i(G) = \Delta(G) + 1$. In [7], Dolama et al showed: (1) If $G$ is a k-degenerated graph, then $\chi_i(G) \leq \Delta(G) + 2k - 1$. (2) If $G$ is a $K_4$-minor free graph, then $\chi_i(G) \leq \Delta(G) + 2$, and this bound is tight. (3) If $G$ is a planar graph, then $\chi_i(G) \leq \Delta(G) + 7$. In [12], Maydanskiy showed that $\chi_i(G) \leq 5$ for all graphs $G$ with $\Delta(G) = 3$, which was conjectured by Chen et al. in [5]. In [10], Huang et al showed that square meshes, hexagonal meshes, and honeycomb meshes admit a $(\Delta + 1)$-incidence coloring.

This paper is organized as follows. In section 2, we will first introduce the definitions for the $k$-tuple coloring and fractional coloring of graphs, then we will generalize the results of Guiduli [9] on the incidence chromatic number of graphs to the fractional incidence chromatic number. Tight asymptotic bounds similar to [9] will be given for the fractional incidence chromatic number of graphs. Then the fractional incidence chromatic number of the circulant graph $G(n, S)$, where $S = [k]$, will be studied. In section 3, we will establish some relationships between the $k$-tuple incidence chromatic number and the incidence chromatic number of the direct products and lexicographic products of graphs. This gives us more reasons for studying the $k$-tuple coloring of the incidence graph, therefore the fractional incidence chromatic number of graphs. Finally in Section 4, we establish a relationship between the incidence coloring and star arboricity of graphs, and use it to show that for planar graphs $G$, if $\Delta(G) \neq 6$, then $\chi_i(G) \leq \Delta(G) + 5$; if $\Delta(G) = 6$, then $\chi_i(G) \leq \Delta(G) + 6$. This improves the bound $\chi_i(G) \leq \Delta(G) + 7$ for planar graphs given in [7].

## 2 Fractional incidence coloring of graphs

Let $G$ be a graph. One of the most natural generalizations of coloring consists of assigning each vertex, instead of one color, a set of $k$ colors, and requiring that adjacent vertices obtain disjoint sets of colors. Such an assignment is called a *k-tuple coloring*, or a *k-tuple n-coloring* if a total of $n$ colors is used. We always assume that $0 < k \le n$. Obviously, a 1-tuple $n$-coloring is just an usual $n$-coloring. The least $n$ for which $G$ has a $k$-tuple coloring is the *k-tuple chromatic number of G*, denoted by $\chi_k(G)$. Clearly $\chi_k(G) \le k\chi(G)$.

The *fractional chromatic number of G*, denoted $\chi_f(G)$, is the infimum of the fractions $\frac{n}{k}$ such that $G$ admits a $k$-tuple $n$-coloring. We note that the infimum in the definition can be replaced by the minimum. Trivially we have $\chi_f(G) \le \chi(G)$. The fractional chromatic number of graphs has been studied extensively. For more reading of this subject, the readers are referred to [14].

We call the $k$-tuple chromatic number of the incidence graph of $G$ the *k-tuple incidence chromatic number of G*, denoted $\chi_k(Inc(G))$. We call the fractional chromatic number of the incidence graph of $G$ the *fractional incidence chromatic number of G*, denoted $\chi_f(Inc(G))$. Figure 1 illustrates a 3-tuple 10-coloring of the incidence graph of the pentagon $C_5$. This fact shows the fractional incidence chromatic number of the pentagon $\chi_f(Inc(C_5)) \le \frac{10}{3}$. By Theorem 2.5 (in the following), we also have $\chi_f(Inc(C_5)) \ge \frac{10}{3}$. Therefore $\chi_f(Inc(C_5)) = \frac{10}{3}$. And it is not hard to see that $\chi_i(C_5) = 4$.

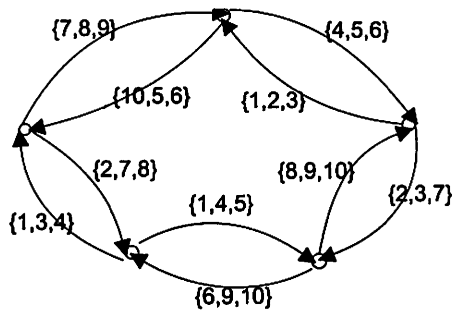


**Figure 1**

Note that from the definition and Theorem 1.5, we immediately have the following upper bounds of the fractional incidence coloring:

**Theorem 2.1** *Let G be a graph with maximum degree Δ, then*

$$\chi_f(Inc(G)) \le \Delta + 20\log\Delta + 84.$$

Next we will show that there exist graphs G with $\chi_f(Inc(G)) \ge \Delta(G) + \Omega(\log\Delta(G))$, therefore the upper bound of Theorem 2.1 is asymptotically tight. The proof for this follows an example from Algor and Alon in [1]. We will need the following well known proposition about the fractional chromatic number of a graph $G$.

**Proposition 2.2** *([14]) For any graph $G$, $\chi_f(G) \geq \frac{|V(G)|}{\alpha(G)}$.*

The following example was given in [1]. The Paley graph $G$ is defined as follows. Let $p$ be a prime with $p \equiv 1 \pmod 4$. Put $V(G) = \{0, 1, \cdots, p-1\}$. Two vertices $x$ and $y$ are adjacent in $G$ if and only if $x - y$ is a square in $GF(p)$. $G$ is $d = (p-1)/2$ regular. If $p > k^2 \cdot 2^{2k-2}$ and $A \subseteq V(G)$, $|A| = k$, then there is a $v \in V$ which is not adjacent to any member of $A$. So if $S \subseteq V$ is a dominating set (i.e. every vertex of $V - S$ is adjacent to a vertex of $S$) then $|S| > k$. Now we show that the graph $G$ satisfies $\chi_f(Inc(G)) \geq d + \Omega(\log d)$. The proof here is analogous to the proof in [1].

Let $H$ be a star forest in $G$ and let $S = \{v \in V : \deg_H(v) = 0$, or $v$ is the center of a star in $H\}$. Clearly $S$ is a dominating set so $|S| > k$. But $|E(H)| = p - |S| < p - k$. Note that by the observation that, in the incidence coloring, each color class is a directed star forest, we have $\alpha(Inc(G)) < p - k$. By Proposition 2.2 and $k \geq (\frac{1}{2} - o(1)) \log p$, we have

$$\begin{aligned}\chi_f(Inc(G)) &\geq \frac{|V(Inc(G))|}{\alpha(Inc(G))} = \frac{2|E(G)|}{\alpha(Inc(G))} \\ &> \frac{2|E(G)|}{p-k} = \frac{p(p-1)}{2(p-k)} \\ &\geq \frac{1}{2}(p + (\frac{1}{2} - o(1)) \log p) \geq d + (\frac{1}{4} - o(1)) \log d.\end{aligned}$$

In the following, we will study the fractional incidence chromatic number of the circulant graph $G(n, S)$, where $S = [k]$. Given a positive integer $n$ and a set $S \subseteq \{1, \cdots, \lfloor n/2 \rfloor\}$, the *circulant graph* $G(n, S)$ of order $n$ with generating set $S$ is defined as follows: $G(n, S)$ has vertex set $V(G) = \{0, 1, \cdots, n-1\}$ and edge set $E(G) = \{uv : |u - v|_n \in S\}$, where $|x|_n := \min\{|x|, n - |x|\}$ is the circular distance modulo $n$. We will use the following lemmas.

**Lemma 2.3** *([8, 11]) If $k' \leq n/2$, then $\chi_f(G(n, [k'])) = n / \lfloor \frac{n}{k'+1} \rfloor$, moreover, $\alpha(G(n, [k'])) = \lfloor \frac{n}{k'+1} \rfloor$.*

The square of a graph $G$, denoted by $G^2$, has the same vertex set as $G$, and $uv \in E(G^2)$ if and only if $\mathrm{dist}_G(u, v) \leq 2$, where $\mathrm{dist}_G(u, v)$ denotes the distance between $u$ and $v$ in $G$. The following lemma can be proved straightforward.

**Lemma 2.4** *For any graph $G$, $\chi_f(Inc(G)) \leq \chi_f(G^2)$.*

**Theorem 2.5** *If $k \leq n/4$, then $\frac{2kn}{n - \lceil \frac{n}{2k+1} \rceil} \leq \chi_f(Inc(G(n, [k]))) \leq \frac{n}{\lfloor \frac{n}{2k+1} \rfloor}$, i.e. $\frac{2kn}{\lfloor \frac{2kn}{2k+1} \rfloor} \leq \chi_f(Inc(G(n, [k]))) \leq \frac{2kn}{2k \lfloor \frac{n}{2k+1} \rfloor}$.*

**Proof.** Obviously $G^2(n,[k]) = G(n,[2k])$. Then applying Lemma 2.4 and Lemma 2.3, we have $\chi_f(Inc(G(n,[k])) \le \chi_f(G^2(n,[k]) = \chi_f(G(n,[2k]) = n/\lfloor\frac{n}{2k+1}\rfloor$.

On the other hand, each color class in the incidence coloring is a (directed) star forest, and note that $G(n,[k])$ is $2k$ regular, so each color class has at least $\lceil\frac{n}{2k+1}\rceil$ components, therefore at most $n - \lceil\frac{n}{2k+1}\rceil$ edges. Applying Proposition 2.2, we have $\chi_f(Inc(G(n,[k]) \ge \frac{2kn}{n-\lceil\frac{n}{2k+1}\rceil}$. ■

# 3 Incidence coloring of the direct and lexicographic products of graphs

Let $G$ and $H$ be graphs. Let $G \times H$ and $G[H]$ be the *direct product* and the *lexicographic product* of $G$ and $H$ respectively. The vertex set of $G \times H$ and $G[H]$ is $V(G) \times V(H) = \{(u,v) : u \in V(G) \ \wedge \ v \in V(H)\}$. Vertices $(u,v)$ and $(u',v')$ are adjacent in $G \times H$ i.e. $(u,v)(u',v') \in E(G \times H)$ whenever $uu' \in E(G)$ and $vv' \in E(H)$. Vertices $(u,v)$ and $(u',v')$ are adjacent in $G[H]$ i.e. $(u,v)(u',v') \in E(G[H])$ whenever $uu' \in E(G)$, or $u = u'$ and $vv' \in E(H)$. In this section, we will first establish a relationship between $\chi_i(G \times H)$ and $\chi_{\Delta(H)}(Inc(G))$. In the second part of this section, we will establish a relationship between $\chi_i(G[H])$ and $\chi_{|H|}(Inc(G))$.

## 3.1 Incidence coloring of the direct products of graphs

**Theorem 3.1** $\chi_i(G \times H) \le \min\{\chi_{\Delta(H)}(Inc(G)), \chi_{\Delta(G)}(Inc(H))\}$.

**Proof.** It is sufficient to show $\chi_i(G \times H) \le \chi_{\Delta(H)}(Inc(G))$. Suppose $\Delta(H) = k$, $\chi_k(Inc(G)) = s$. Suppose mapping $f : I(G) \to S^k$ ($k$-elements subsets of $S$) with $|S| = s$ witnesses $\chi_k(Inc(G)) = s$. We will use color set $S$ to properly color all the incidences of $G \times H$.

Let $\bar{i} = ((v,w), \{(v,w),(v_1,w_1)\})$ be an incidence of $G \times H$, then $vv_1 \in E(G)$ and $ww_1 \in E(H)$, therefore $i = (v, vv_1) \in I(G)$. Note that $f(i)$ is a $k$-color set of $S$. And for a given $w \in V(H)$, $|N(w)| \le k$. Therefore by using First-Fit, we can give a color assignment $h : I(G \times H) \to S$ such that $h(\bar{i}) \in f(i)$, and if $a, b \in N(w)$ and $a \ne b$, then $h(\bar{i_a}) \ne h(\bar{i_b})$, where $\bar{i_a} = ((v,w), \{(v,w),(v_1,a)\})$, $\bar{i_b} = ((v,w), \{(v,w),(v_1,b)\})$.

Next we will show that $h$ is a proper incidence coloring of $G \times H$, and thus prove the theorem. Note that for any incidence $\bar{i} = ((v,w), \{(v,w),(v_1,w_1)\})$, the set of all the incidences that are adjacent to $\bar{i}$ (including itself) is $I_{(v,w)} \cup A_{(v,w)} \cup I_{(v_1,w_1)}$ in $G \times H$.

**Case 1** First suppose $\bar{i_2} = ((v,w), \{(v,w),(v_2,w_2)\}) \in I_{(v,w)} \subseteq I(G \times H)$ and $\bar{i_2} \ne \bar{i}$. We show next $h(\bar{i_2}) \ne h(\bar{i})$. Note that since $\bar{i}, \bar{i_2} \in$

$I(G \times H)$, we have $i = (v, vv_1), i_2 = (v, vv_2) \in I_v \subseteq I(G)$, and $ww_1, ww_2 \in E(H)$. If $v_1 \neq v_2$, then $i \neq i_2$ and $i, i_2 \in I_v$ in $G$. Since $f : I(G) \to S^k$ is a proper $k$-tuple coloring for the incidence graph of $G$, we have $f(i) \cap f(i_2) = \emptyset$. Note that since $h(\bar{i}) \in f(i)$ and $h(\bar{i_2}) \in f(i_2)$, we have $h(\bar{i}) \neq h(\bar{i_2})$. Now if $v_1 = v_2$, since $\bar{i_2} \neq \bar{i}$, we have $w_1 \neq w_2$. Note that since $ww_1, ww_2 \in E(H)$, we have $w_1, w_2 \in N(w)$. Therefore by the definition of $h$, we have $h(\bar{i}), h(\bar{i_2}) \in f(i)$ and $h(\bar{i}) \neq h(\bar{i_2})$.

**Case 2** Suppose $\bar{i_3} = ((v_3, w_3), \{(v_3, w_3), (v, w)\}) \in A_{(v,w)} \subseteq I(G \times H)$, we show next $h(\bar{i_3}) \neq h(\bar{i})$. Note that since $\bar{i}, \bar{i_3} \in I(G \times H)$, we have $i = (v, vv_1) \in I_v \subseteq I(G)$, and $i_3 = (v_3, v_3v) \in A_v \subseteq I(G)$. Therefore $i$ and $i_3$ are two adjacent incidences of $G$. Since $f : I(G) \to S^k$ is a proper $k$-tuple coloring for the incidence graph of $G$, we have $f(i) \cap f(i_3) = \emptyset$. Since $h(\bar{i}) \in f(i)$ and $h(\bar{i_3}) \in f(i_3)$, we have $h(\bar{i}) \neq h(\bar{i_3})$.

**Case 3** Finally suppose $\bar{i_4} = ((v_1, w_1), \{(v_1, w_1), (v_4, w_4)\}) \in I_{(v_1,w_1)} \subseteq I(G \times H)$, we show next $h(\bar{i_4}) \neq h(\bar{i})$. Since $\bar{i}, \bar{i_4} \in I(G \times H)$, we have $i = (v, vv_1) \in I_v \subseteq I(G)$, and $i_4 = (v_1, v_1v_4) \in I_{v_1} \subseteq I(G)$. Therefore $i$ and $i_4$ are two adjacent incidences of $G$. Since $f : I(G) \to S^k$ is a proper $k$-tuple coloring for the incidence graph of $G$, we have $f(i) \cap f(i_4) = \emptyset$. Since $h(\bar{i}) \in f(i)$ and $h(\bar{i_4}) \in f(i_4)$, we have $h(\bar{i}) \neq h(\bar{i_4})$.

This finishes the proof of this theorem. ■

Since for any graph $G$, we have $\chi_k(G) \leq k\chi(G)$, immediately we have the following corollary:

**Corollary 3.2** $\chi_i(G \times H) \leq \min\{\chi_i(G)\Delta(H), \Delta(G)\chi_i(H)\}$.

By applying Corollary 3.2, we have the following corollaries about the incidence chromatic number of the direct products of some special graphs.

**Corollary 3.3** *For all* $n \geq 2$, $\chi_i(G \times K_n) \leq \chi_i(G)(n-1)$. *For all* $m \geq n \geq 2$, $\chi_i(K_m \times K_n) \leq m(n-1)$.

**Proof.** Note that by Theorem 1.2, for all $n \geq 2$, $\chi_i(K_n) = \Delta(K_n) + 1 = n$. ■

In Corollary 3.3, let $n = 2$, then for $m \geq 2$, we have $\chi_i(K_m \times K_2) \leq m$. Obviously this is best possible. For a complete bipartite graph $K_{m,n}$, where $m \geq n \geq 2$, Brualdi and Massey [4] proved $\chi_i(K_{m,n}) = m + 2$. Since $K_{m,m} = (K_m \times K_2) \cup M$, where $M$ is a perfect matching in $K_{m,m}$, immediately we have $\chi_i(K_{m,m}) \leq m + 2$. Note that this implies the upper bound of $\chi_i(K_{m,n})$.

**Corollary 3.4** *Let $T_1$, $T_2$ be trees of order $\geq 2$ with maximum degrees $\Delta_1 \leq \Delta_2$, then $\chi_i(T_1 \times T_2) \leq \Delta_1\Delta_2 + \Delta_1$.*

**Proof.** By Theorem 1.3, we have $\chi_i(T_1) = \Delta_1 + 1$. By Corollary 3.2, $\chi_i(T_1 \times T_2) \leq (\Delta_1 + 1)\Delta_2 = \Delta_1\Delta_2 + \Delta_1$. ■

## 3.2 Incidence coloring of the lexicographic products of graphs

In this part, we will prove the following theorem on the incidence coloring of the lexicographic products of graphs.

**Theorem 3.5** $\chi_i(G[H]) \leq \chi_{|H|}(Inc(G)) + \chi_i(H)$.

**Proof.** Suppose $\chi_{|H|}(Inc(G)) = s$, $\chi_i(H) = t$. Suppose mappings $f : I(G) \to S^{|H|}$ ($|H|$-elements subsets of $S$) with $|S| = s$ and $g : I(H) \to T$ with $|T| = t$ witness $\chi_{|H|}(Inc(G)) = s$ and $\chi_i(H) = t$. Without loss of generality, we suppose $S \cap T = \emptyset$. We will use color set $S \cup T = \{c : c \in S \text{ or } c \in T\}$ to properly color all the incidences of $G[H]$.

Let $\bar{i} = ((v, w), \{(v, w), (v_1, w_1)\})$ be an incidence of $G[H]$, then $\{(v, w), (v_1, w_1)\} \in E(G[H])$, then $vv_1 \in E(G)$, or $v = v_1$ and $ww_1 \in E(H)$, therefore $i = (v, vv_1) \in I(G)$, or $v = v_1$ and $j = (w, ww_1) \in I(H)$. Note that $f(i)$ is a $|H|$-color set of $S$. Define $h : I(G[H]) \to S \cup T$ by $h(\bar{i}) \in f(i)$ if $vv_1 \in E(G)$, $h(\bar{i}) = g(j)$ if $v = v_1$ and $ww_1 \in E(H)$. Moreover, for a given $(v, w)$, suppose $vv_1 \in E(G)$, and there are incidences $\bar{i_a}, \bar{i_b} \in Inc(G[H])$ of the form $\bar{i_a} = ((v, w), \{(v, w), (v_1, a)\})$, $\bar{i_b} = ((v, w), \{(v, w), (v_1, b)\})$, and $a \neq b$. Since $f(i)$ is a $|H|$-color set of $S$, by using First-Fit, we let color assignment $h : I(G[H]) \to S$ satisfies that $h(\bar{i_a}), h(\bar{i_b}) \in f(i)$, and $h(\bar{i_a}) \neq h(\bar{i_b})$.

Next we will show that $h$ is a proper incidence coloring for $G[H]$, and thus proves the theorem. Note that for any incidence $\bar{i} = ((v, w), \{(v, w), (v_1, w_1)\})$, the set of all the incidences that are adjacent to $\bar{i}$ (including itself) is $I_{(v,w)} \cup A_{(v,w)} \cup I_{(v_1,w_1)}$ in $G[H]$.

**Case 1** First suppose $\bar{i_2} = ((v, w), \{(v, w), (v_2, w_2)\}) \in I_{(v,w)} \subseteq I(G[H])$ and $\bar{i_2} \neq \bar{i}$, we show next $h(\bar{i_2}) \neq h(\bar{i})$.

**Subcase 1.1** If both $i = (v, vv_1), i_2 = (v, vv_2) \in I(G)$, then $i = (v, vv_1), i_2 = (v, vv_2) \in I_v$ in $G$. If $v_1 \neq v_2$, then $i \neq i_2$ and $i, i_2 \in I_v$. Since $f : I(G) \to S^{|H|}$ is a proper $|H|$-tuple coloring for the incidence graph of $G$, we have $f(i) \cap f(i_2) = \emptyset$. Since $h(\bar{i}) \in f(i)$ and $h(\bar{i_2}) \in f(i_2)$, we have $h(\bar{i}) \neq h(\bar{i_2})$. If $v_1 = v_2$, since $\bar{i_2} \neq \bar{i}$, we have $w_1 \neq w_2$. By the definition of $h$, we have $h(\bar{i}), h(\bar{i_2}) \in f(i)$ and $h(\bar{i}) \neq h(\bar{i_2})$.

**Subcase 1.2** If $v = v_1$ and $j = (w, ww_1) \in I(H)$, $v = v_2$ and $j_2 = (w, ww_2) \in I(H)$, then $j = (w, ww_1), j_2 = (w, ww_2) \in I_w$ in $H$. Since $g : I(H) \to T$ is a proper incidence coloring for $H$, we have $g(j) \neq g(j_2)$. By the definition of $h$, we have $h(\bar{i}) = g(j) \neq g(j_2) = h(\bar{i_2})$.

**Subcase 1.3** Suppose $i = (v, vv_1) \in I(G)$, $v = v_2$ and $j_2 = (w, ww_2) \in I(H)$ (the case will be similar for $v = v_1$ and $j = (w, ww_1) \in I(H)$, $i_2 = (v, vv_2) \in I(G)$.) By the definition of $h$, we have $h(\bar{i}) \in f(i) \subseteq S$ and $h(\bar{i_2}) = g(j_2) \in T$. Since $S \cap T = \emptyset$, we have $h(\bar{i}) \neq h(\bar{i_2})$.

**Case 2** Suppose $\bar{i_3} = ((v_3, w_3), \{(v_3, w_3), (v, w)\}) \in A_{(v,w)} \subseteq I(G[H])$, we show next $h(\bar{i_3}) \neq h(\bar{i})$.

**Subcase 2.1** If both $i = (v, vv_1), i_3 = (v_3, v_3v) \in I(G)$, then $i = (v, vv_1) \in I_v$ and $i_3 = (v_3, v_3v) \in A_v$ in $G$. Since $f : I(G) \to S^{|H|}$ is a proper $|H|$-tuple coloring for the incidence graph of $G$, we have $f(i) \cap f(i_3) = \emptyset$. Since $h(\bar{i}) \in f(i)$ and $h(\bar{i_3}) \in f(i_3)$, we have $h(\bar{i}) \neq h(\bar{i_3})$.

**Subcase 2.2** If $v = v_1$ and $j = (w, ww_1) \in I(H)$, $v = v_3$ and $j_3 = (w_3, w_3w) \in I(H)$, then $j = (w, ww_1) \in I_w, j_3 = (w_3, w_3w) \in A_w$ in $H$. Since $g : I(H) \to T$ is a proper incidence coloring for $H$, we have $g(j) \neq g(j_3)$. By the definition of $h$, we have $h(\bar{i}) = g(j) \neq g(j_3) = h(\bar{i_3})$.

**Subcase 2.3** Suppose $i = (v, vv_1) \in I(G)$, $v_3 = v$ and $j_3 = (w_3, w_3w) \in I(H)$ (the case will be similar for $v = v_1$ and $j = (w, ww_1) \in I(H)$, $i_3 = (v_3, v_3v) \in I(G)$.) By the definition of $h$, we have $h(\bar{i}) \in f(i) \subseteq S$ and $h(\bar{i_3}) = g(j_3) \in T$. Since $S \cap T = \emptyset$, we have $h(\bar{i}) \neq h(\bar{i_3})$.

**Case 3** Finally suppose $\bar{i_4} = ((v_1, w_1), \{(v_1, w_1), (v_4, w_4)\}) \in I_{(v_1,w_1)} \subseteq I(G[H])$, we show next $h(\bar{i_4}) \neq h(\bar{i})$.

**Subcase 3.1** If both $i = (v, vv_1), i_4 = (v_1, v_1v_4) \in I(G)$, then $i = (v, vv_1) \in I_v$ and $i_4 = (v_1, v_1v_4) \in I_{v_1}$ in $G$. Since $f : I(G) \to S^{|H|}$ is a proper $|H|$-tuple coloring for the incidence graph of $G$, we have $f(i) \cap f(i_4) = \emptyset$. Since $h(\bar{i}) \in f(i)$ and $h(\bar{i_4}) \in f(i_4)$, we have $h(\bar{i}) \neq h(\bar{i_4})$.

**Subcase 3.2** If $v = v_1$ and $j = (w, ww_1) \in I(H)$, $v_1 = v_4$ and $j_4 = (w_1, w_1w_4) \in I(H)$, then $j = (w, ww_1) \in I_w, j_4 = (w_1, w_1w_4) \in I_{w_1}$ in $H$. Since $g : I(H) \to T$ is a proper incidence coloring for $H$, we have $g(j) \neq g(j_4)$. By the definition of $h$, we have $h(\bar{i}) = g(j) \neq g(j_4) = h(\bar{i_4})$.

**Subcase 3.3** Suppose $i = (v, vv_1) \in I(G)$, $v_1 = v_4$ and $j_4 = (w_1, w_1w_4) \in I(H)$ (the case will be similar for $v = v_1$ and $j = (w, ww_1) \in I(H)$, $i_4 = (v_1, v_1v_4) \in I(G)$.) By the definition of $h$, we have $h(\bar{i}) \in f(i) \subseteq S$ and $h(\bar{i_4}) = g(j_4) \in T$. Since $S \cap T = \emptyset$, we have $h(\bar{i}) \neq h(\bar{i_4})$.

This finishes the proof of this theorem. ∎

We have the following corollaries about the incidence chromatic number of the lexicographic products of some special graphs.

**Corollary 3.6** *For all $n \geq 2$, $\chi_i(G[K_n]) \leq \chi_n(Inc(G)) + n$. For all $m \geq n \geq 2$, $\chi_i(K_m[K_n]) \leq \chi_n(Inc(K_m)) + n \leq mn + n$.*

**Corollary 3.7** *Let $T_1$, $T_2$ be trees of order $\geq 2$ with maximum degrees $\Delta_1, \Delta_2$, then $\chi_i(T_1[T_2]) \leq \chi_{|T_2|}(Inc(T_1)) + \chi_i(T_2) \leq (\Delta_1 + 1)|T_2| + \Delta_2 + 1$.*

# 4 Incidence coloring and star arboricity of graphs

In this section, we will establish a seemingly obvious relationship between the incidence coloring and the star arboricity of graphs, and use it to get the best known general upper bounds of the incidence chromatic number of planar graphs. For an undirected graph $G = (V, E)$, we use $st(G)$ to denote the star arboricity of $G$. An *acyclic coloring of $G$* is a proper coloring of the vertices of $G$ in which there are no two-colored cycles. The *acyclic chromatic number*, $\chi_a(G)$ of $G$ is the smallest $k$ such that G admits an acyclic $k$-coloring. For the relationship between $st(G)$ and $\chi_a(G)$, we have the following theorem (for a simple and interesting proof of this, refer to [2]).

**Theorem 4.1** *For any graph $G$, $st(G) \leq \chi_a(G)$.*

Borodin [3] proved that every planar graph admits an acyclic 5-coloring. Thus we have:

**Theorem 4.2** *If $G$ is a planar graph, then $st(G) \leq 5$.*

**Theorem 4.3** *For any graph $G$, $\chi_i(G) \leq st(G) + \chi'(G)$.*

**Proof.** Replace each edge of $G$ by both directed edges, we get the symmetrically directed graph $S(G)$. Then based on Guiduli's observation (refer to [9] or Theorem 1.5 of this paper) that each color class of the incidence graph of $G$ is a directed star forest of $G$, we can use $st(G)$ colors to color one direction of the two directed edges that comes with any edge of $G$, i.e. we can use $st(G)$ colors to color a "*copy*" of $E(G)$ in $S(G)$. Then we can use at most $\chi'(G)$ colors to color the left "*copy*" $E(G)$ in $S(G)$ (note that the left copy of $E(G)$ is actually directed in $S(G)$). This finished the proof of this theorem. ■

**Corollary 4.4** *If a graph $G$ is of class 1, then $\chi_i(G) \leq st(G) + \Delta(G)$; otherwise $\chi_i(G) \leq st(G) + \Delta(G) + 1$.*

**Proof.** This comes directly from Theorem 4.3 and Vizing's theorem 1.1. ■

For the edge chromatic number of planar graphs, Vizing [15] showed if $\Delta \geq 8$, then a planar graph is always Class 1. Zhang [16] and Sanders and Zhao [13] independently showed if $\Delta = 7$, then a planar graph is always Class 1. Now we can use the above results to get the best known general upper bounds of the incidence chromatic number of planar graphs.

**Theorem 4.5** *If $G$ is a planar graph and $\Delta(G) \neq 6$, then $\chi_i(G) \leq \Delta(G) + 5$; if $\Delta(G) = 6$, then $\chi_i(G) \leq \Delta(G) + 6$.*

**Proof.** If $\Delta(G) \geq 7$, applying Theorem 4.2, Corollary 4.4 and the known results about the edge chromatic number of planar graphs, we have $\chi_i(G) \leq \Delta(G) + 5$.

If $\Delta(G) \leq 6$, since $\chi'(G) \leq \Delta(G) + 1$, we have $\chi_i(G) \leq \Delta(G) + 6$. Moreover if $\Delta(G) \leq 5$, by Theorem 1.4, we have $\chi_i(G) \leq 2\Delta(G) \leq \Delta(G) + 5$. ■

The following seems to be an interesting open question to the author: if $\Delta(G) = 6$, will we have $\chi_i(G) \leq \Delta(G) + 5$?

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