A characterization of the set of the planes of PG(4, q) which meet a non–singular quadric in a conic

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Abstract. In [2] Stefano Innamorati and Mauro Zannetti gave a characterization of the planes secant to a non-singular quadric in PG(4, q). Their result is based on a particular hypothesis (we call it "polynomial") that, as the same authors wrote at the end of the paper, could not exclude possible sporadic cases. In this paper we improve their result by giving a characterization without the "polynomial" hypothesis. So possible sporadic cases are definitely excluded.

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1. Introduction and motivation.

Let PG(4, q) be the projective space of dimension 4 and order q, with $q = p^h$ a prime power. In [1] D.K. Butler proved the following:

Result 1. Let K be a set of planes in PG(4, q) such that:
- every point lies on q^4 or q^4+q^2 planes of K;
- every line lies on 0 or q^2 planes of K;
- every 3–space contains at least one plane of K.

Then K is the set of planes meeting a non–singular quadric in a conic.

As usual, a *star* (resp. *hyperstar*) of planes is the set of all the planes through a same line (resp. through a same point). If we denote by m and n two integers, with $0 \le m < n$, then a set K of planes of PG(4, q) is said to be of *type* (m, n) with respect to stars (resp. hyperstars) of planes, if each star (resp. hyperstar) contains either m or n planes of K and all such stars (resp. hyperstar) do exist (see [4] and [5]). A set K of type (m, n) is also called a *two character set*.
In [2] S. Innamorati and M. Zannetti said that K is a ***two polynomial character set*** if both $m = m(q)$ and $n = n(q)$ are polynomial function of q. Furthermore, under this hypothesis, they proved (by Result 1) the following:

Result 2. Let K be a set of planes in PG(4, q) such that:
- $|K| = q^4(q^2+1)$;
- K has exactly $q^5(q^2+1)(q^5-q+1)/2$ pairs of planes which meet in exactly one point;
- K is a two **polynomial** character set with respect to stars of planes;
- K is a two **polynomial** character set with respect to hyperstars of planes;
- K is an intersection-set (i.e. every 3–space contains at least one plane of K).

Then K is the set of planes meeting a non–singular quadric in a conic.

<u>The motivation of this paper lies in the **polynomial** character set "hypothesis"</u>.

Indeed the authors firstly suppose that K is a set of type (m, n) with respect to stars of planes (resp. to hyperstars of planes). By the standard equations they obtain that $n = N/D$, where $N = N(m, q)$ and $D = D(m, q)$ are integers depending on m and q. By supposing that $m = m(q)$ is polynomial functions of q, they have that both $N = N(m(q), q)$ and $D = D(m(q), q)$ are polynomial functions of q too. So $n = N(m(q), q)/D(m(q), q)$. Finally, by supposing that the integer n is also a polynomial function of q, they need that $R(m(q), q)/D(m(q), q)$ is an integer for any q, where $R(m(q), q)$ is the remainder of the division $N(m(q), q)/D(m(q), q)$. So the authors have to require that $R(m(q), q) = 0$ for any q, i.e. $R(m(q), q) \equiv 0$. Let us note that, by supposing that both m and n are polynomial function of q, <u>the authors do not consider all the possible cases</u> where $R(m, q)/D(m, q)$ could be an integer different from zero. As a matter of fact, the autors themselves realize it and, at the end of their paper, they write: "*By requiring the existence of an appropriate set of planes enjoying the same properties for all q, sporadic cases are not considered*".

In this paper we improve Result 2. Indeed we prove (by Result 1) the following:

Theorem. *Let K be a set of planes in PG(4, q) such that:*
- $|K| = k = q^4(q^2+1)$;
- *K has exactly* $\tau = q^5(q^2+1)(q^5-q+1)/2$ *pairs of planes which meet in exactly one point;*
- *K is of type (m, n) with respect to stars of planes;*
- *K is of type (a, b) with respect to hyperstars of planes;*
- *every 3–space contains at least one plane of K.*

Then K is the set of planes meeting a non–singular quadric in a conic.

Let us note that we <u>eliminate</u> the **polynomial** character set "hypothesis" from the statement. In such a way we can definitely <u>exclude</u> possible sporadic cases.

2. The proof of the Theorem

Firstly, let us suppose that K is of type (m, n) with respect to stars of planes. If we denote by $t_j > 0$ the number of stars of planes meeting K in exactly j planes, then by counting in double way the total number of stars, the total number of incident planes-stars pairs (α, S) with $\alpha \in K \cap S$, and the total number of triples (α, β, S) with $\alpha, \beta \in K \cap S$, we have the following *standard equations* (see [3]):

$$\begin{cases} t_m + t_n = (q^2+1)(q^4+q^3+q^2+q+1) & (2.1.1) \\ mt_m + nt_n = k(q^2+q+1) = q^4(q^2+1)(q^2+q+1) & (2.1.2) \\ m(m-1)t_m + n(n-1)t_n = k(k-1) - 2\tau = q^4(q^2+1)(q^2-1)(q^2+q+1) & (2.1.3) \end{cases}$$

From (2.1.1) and (2.1.2), we get

$$\begin{cases} (n-m)t_m = (q^2+1)(nA - Bq^4) & (2.2.1) \\ (n-m)t_n = (q^2+1)(Bq^4 - mA) & (2.2.2) \end{cases}$$

where $A = q^4+q^3+q^2+q+1$ and $B = q^2+q+1$.

Since $(n-m)>0$, $t_m>0$ and $t_n>0$, equations (2.2) imply that $m < q^4B/A < n$. So

$$0 \le m \le q^2-1 < q^2 \le n \le q^2+q+1 \tag{2.3}$$

By the following combination $[n^2(2.1.1) - (2.1.2) - (2.1.3)]$ we obtain

$$(n+m)(n-m)t_m = (q^2+1)(An^2 - Bq^6).$$

Furthermore, in view of (2.2.1), we have

$$mnA = q^4(m+n-q^2)B \tag{2.4}$$

So $nm \equiv 0 \pmod{q^4}$. Since $m < n < q^3$, both $m \equiv 0 \pmod p$ and $n \equiv 0 \pmod p$.
If $m > 0$, then there are four positive integers a, b, s and t such that
- $m = ap^s$ with $a \not\equiv 0 \pmod p$ and $s \le 2h-1$;
- $n = bp^t$ with $b \not\equiv 0 \pmod p$ and $(t+s) \ge 4h$.

From $t \ge 4h-s \ge 2h+1$ we get $n \ge p^{2h+1} = pq^2 \ge 2q^2 > q^2+q+1$, an absurd.
Hence $m = 0$ **necessarily**. By equation (2.4) we immediately get $n = q^2$.
• So we proved than K is of type $(0, q^2)$ with respect to stars of planes.

Now let us suppose that K is of type (a, b) with respect to hyperstars of planes. If we denote by $t_j > 0$ the number of hyperstars of planes meeting K in exactly j planes, then by counting in double way the total number of hyperstars and the total number of incident plane-hyperstar pairs (π, h) with $\pi \in K \cap h$, we have the following *standard equations* (see [3]):

$$\begin{cases} t_a + t_b = q^4+q^3+q^2+q+1 \\ at_a + bt_b = k(q^2+q+1) = q^4(q^2+1)(q^2+q+1) \end{cases} \tag{2.5}$$

As in [2], we call *a-point* (resp. *b-point*) the centre of a hyperstar intersecting K in a (resp. b) planes. Since K is a set of type $(0, q^2)$ with respect to stars of planes, we call *0–line* (resp. *q^2–line*) the centre of a star which intersects K in 0 (resp. q^2) planes. Moreover, if P is a point of PG(4, q), then by $x(P)$ we denote that the number of q^2–lines through P. By counting in double way the number of pairs (r, π) where $P \in r \subset \alpha$, r is a q^2–line and π is a plane belonging to K, we have $q^2x(P) = (q+1)a$ if P is an a–point, while $q^2x(P) = (q+1)b$ if P is a b–point. Since $\mathrm{GCD}(q^2, q+1) = 1$, let us note that $a \equiv 0 \pmod{q^2}$ and $b \equiv 0 \pmod{q^2}$. Putting $a = \alpha q^2$ and $b = \beta q^2$ we have that $x(P) = \alpha(q+1)$ (resp. $x(P) = \beta(q+1)$) is the number of q^2–lines through P where P is an a–point (resp. P is a b-point). Furthermore, since $b \le (q^2+1)(q^2+q+1)$ we have that $\beta \le q^2+q+2$.

Again as in [2], by counting the number $\tau = q^5(q^2 + 1)(q^5 - q + 1)/2$ of pairs of planes which meet in exactly one point we obtain:

(2.6) $$\left[\binom{a}{2} - \alpha(q+1)\binom{q^2}{2}\right]t_a + \left[\binom{b}{2} - \beta(q+1)\binom{q^2}{2}\right]t_b = \tau$$

By (2.6) and the second equation of (2.5) we obtain

(2.7) $$a^2t_a + b^2t_b = q^8(q^2 + 1)(q^2 + q + 2)$$

Since $a = \alpha q^2$ and $b = \beta q^2$ from (2.5) and (2.7) we get the following equations

(2.8) $$\begin{cases} t_a + t_b = A & (2.8.1) \\ \alpha t_a + \beta t_b = q^2(q^2+1)B & (2.8.2) \\ \alpha^2 t_a + \beta^2 t_b = q^4(q^2+1)(B+1) & (2.8.3) \end{cases}$$

From the first two equations of (2.8), we obtain

(2.9) $$\begin{cases} (\beta - \alpha)t_a = \beta A - q^2(q^2+1)B & (2.9.1) \\ (\beta - \alpha)t_b = q^2(q^2+1)B - \alpha A & (2.9.2) \end{cases}$$

Since $(\beta - \alpha) > 0$, $t_a > 0$ and $t_b > 0$, equations (2.9) imply $\alpha < q^2(q^2+1)B/A < \beta$. So

(2.10) $$0 \le \alpha \le q^2 < q^2+1 \le \beta \le q^2+q+2$$

By the following combination $[\beta^2(2.8.1) - (2.8.3)]$ we obtain

$$(\beta + \alpha)(\beta - \alpha)t_a = \beta^2 A - q^4(q^2 + 1)B.$$

Furthermore, in view of (2.9.1), we have

(2.11) $$\alpha\beta A = q^2(q^2 + 1)[\alpha B + \beta B - (B + 1)q^2]$$

So $\alpha\beta \equiv 0 \pmod{q^2}$. If $\alpha \not\equiv 0 \pmod{q^2}$ and $\beta \not\equiv 0 \pmod{q^2}$, then there are four positive integers m, n, s and t such that:

- $\alpha = mp^s$ with $m \not\equiv 0 \pmod{p}$ and $1 \le s \le (2h-1)$
- $\beta = np^t$ with $n \not\equiv 0 \pmod{p}$ and $(2h-s) \le t \le (2h-1)$

Denoting by r the minimum integer between s and t, then from (2.11) we get

$$mnAp^{s+t} = p^{2h+r}(q^2+1)[mBp^{s-r} + nBp^{t-r} - (B+1)p^{2h-r}]$$

Since $mnA \neq 0 \pmod p$ we have that $s + t \geq 2h + r$.
If $r = s$ then $t \geq 2h$, while if $r = t$ then $s \geq 2h$. In each case we have an absurd.
So $\alpha \equiv 0 \pmod{q^2}$ or $\beta \equiv 0 \pmod{q^2}$. If $\beta \equiv 0 \pmod{q^2}$, then, since $\beta \geq q^2+1$, we have that $\beta \geq 2q^2$. By (2.10) we obtain $(q+1)(q-2) \leq 0$ which is an absurd for each $q \geq 3$. For $q = 2$ we get $\beta \geq 8$ and so, by (2.10), $\beta = 8$. Then, by (2.11), we have that $9\alpha = 40$ which is an absurd. So $\alpha \equiv 0 \pmod{q^2}$. If $\alpha = 0$, then, by (2.11), we get $\beta = q^2(B+1)/B$ which is not an integer, an absurd. So $\alpha > 0$. In view of (2.10) we have that $\alpha = q^2$ **necessarily**. By (2.11), we get $\beta = (q^2+1)$.
Hence, $a = \alpha q^2 = q^4$ and $b = \beta q^2 = q^2(q^2+1)$.
• So we proved that K is of type (q^4, q^4+q^2) with respect to hyperstars of planes.

Finally, we proved that K is of type $(0, q^2)$ with respect to stars and of type (q^4, q^4+q^2) with respect to hyperstars. Hence, by Result 1, K is the set of the planes meeting a non-singular quadric in a conic. So the proof is completed. ■

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