Pancyclicity, Panconnectivity, and Panpositionability for General Graphs and Bipartite Graphs

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Abstract

A graph G is *pancyclic* if it contains a cycle of every length from 3 to $|V(G)|$ inclusive. A graph G is *panconnected* if there exists a path of length l joining

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any two different vertices x and y with $d_G(x,y) \leq l \leq |V(G)|-1$, where $d_G(x,y)$ denotes the distance between x and y in G. A hamiltonian graph G is *panpositionable* if for any two different vertices x and y of G and any integer k with $d_G(x,y) \leq k \leq |V(G)|/2$, there exists a hamiltonian cycle C of G with $d_C(x,y) = k$, where $d_C(x,y)$ denotes the distance between x and y in a hamiltonian cycle C of G. It is obvious that panconnected graphs are pancyclic, and panpositionable graphs are pancyclic. The above properties can be studied in bipartite graphs after some modification. A graph $H = (V_0 \cup V_1, E)$ is *bipartite* if $V(H) = V_0 \cup V_1$ and $E(H)$ is a subset of $\{(u,v) \mid u \in V_0, v \in V_1\}$. A graph is *bipancyclic* if it contains a cycle of every even length from 4 to $2 \cdot \lfloor \frac{|V(H)|}{2} \rfloor$ inclusive. A graph H is *bipanconnected* if there exists a path of length l joining any two different vertices x and y with $d_H(x,y) \leq l \leq |V(H)|-1$, where $d_H(x,y)$ denotes the distance between x and y in H and $l - d_H(x,y)$ is even. A hamiltonian graph H is *bipanpositionable* if for any two different vertices x and y of H and for any integer k with $d_H(x,y) \leq k < |V(H)|/2$, there exists a hamiltonian cycle C of H with $d_C(x,y) = k$, where $d_C(x,y)$ denotes the distance between x and y in a hamiltonian cycle C of H and $k - d_H(x,y)$ is even. It can be shown that bipanconnected graphs are bipancyclic, and bipanpositionable graphs are bipancyclic. In this paper, we present some examples of pancyclic graphs that are neither panconnected nor panpositionable, some examples of panconnected graphs that are not panpositionable, and some examples of graphs that are panconnected and panpositionable, for nonbipartite graphs. Corresponding examples for bipartite graphs are discussed. The existence of panpositionable (or bipanpositionable, resp.) graphs that are not panconnected (or bipanconnected, resp.) is still an open problem.

Keywords: Hamiltonian, pancyclic, panconnected, panpositionable.

1 Introduction

In this paper, for the graph definitions and notations we follow [2]. $G = (V, E)$ is a *graph* if V is a finite set and E is a subset of $\{(u, v) \mid (u, v)$ is an unordered pair of $V\}$. We say that V is the *vertex set* and E is the *edge set* of G. Two vertices u and v are *adjacent* if $(u, v) \in E$. A *path* P is represented by $\langle v_0, v_1, \cdots, v_k\rangle$. We use P^{-1} to denote the path $\langle v_k, v_{k-1}, v_{k-2}, \cdots, v_1, v_0\rangle$. We also write the path $\langle v_0, v_1, v_2, \cdots, v_k\rangle$ as $\langle v_0, v_1, \cdots, v_i, Q, v_j, v_{j+1}, \cdots, v_k\rangle$, where Q is the path $\langle v_i, v_{i+1}, \cdots, v_j\rangle$. A path is called a *hamiltonian path* if its vertices are distinct and span V. A *cycle* is a path of at least three vertices such that the first vertex is the same as the last vertex. A cycle is called a *hamiltonian cycle* if its vertices are distinct except for the first vertex and the last vertex and if they span V. A *hamiltonian graph* is a graph with a hamiltonian cycle.

Let $x, y \in V(G)$. We use $d_G(x, y)$ to denote the distance between x and y in G, and $d_C(x, y)$ the distance between x and y in a hamiltonian cycle C of G. A graph is *pancyclic* if it contains a cycle of every length from 3 to $|V(G)|$ inclusive. The concept of pancyclic graphs is proposed by Bondy [3]. A graph G is *panconnected* if there exists a path of length l joining any two different vertices x and y with $d_G(x, y) \leq l \leq |V(G)| - 1$. The concept of panconnected graphs is proposed by Alavi and Williamson [1]. A hamiltonian graph G is *panpositionable* if for any two different vertices x and y of G and any integer k with $d_G(x, y) \leq k \leq |V(G)|/2$, there exists a hamiltonian cycle C of G with $d_C(x, y) = k$. The concept of panpositionable graphs is studied by Kao et. al [4]. Let G be a panconnected graph and

u, v be two adjacent vertices of G. Then there exists a path P_i between u and v such that the length of P_i is i for $1 \leq i \leq |V(G)| - 1$. It is obvious that $P_1 \bigcup P_i$ is a cycle of length $i + 1$ for $2 \leq i \leq |V(G)| - 1$. Thus, there exists a cycle of length l with $3 \leq l \leq |V(G)|$ in G. Therefore, every panconnected graph is pancyclic. Let G be a panpositionable graph and x, y be two adjacent vertices of G. Then there exists a hamiltonian cycle of the form $\langle x, P_i, y, Q_i, x\rangle$ such that the length of P_i is i and the length of Q_i is $|V(G)| - i$ for $1 \leq i \leq |V(G)|/2$. Obviously, $\langle x, P_1, y, P_i^{-1}, x\rangle$ is a cycle of length $i + 1$ and $\langle x, P_1, y, Q_i, x\rangle$ is a cycle of length $|V(G)| - i + 1$ for $2 \leq i \leq |V(G)|/2$. Thus, G contains a cycle of length l for $3 \leq l \leq |V(G)|$. Therefore, every panpositionable graph is pancyclic.

The similar concepts can be studied in bipartite graphs. A graph $H = (V_0 \cup V_1, E)$ is *bipartite* if $V(H) = V_0 \cup V_1$ and $E(H)$ is a subset of $\{(u, v) \mid u \in V_0, v \in V_1\}$. Since there is no odd cycle in any bipartite graph, any bipartite graph is not pancyclic. For this reason, the concept of bipancyclicity is proposed [6]. A bipartite graph is *bipancyclic* if it contains a cycle of every even length from 4 to $|V(H)|$ inclusive. It is proved that the hypercube is bipancyclic [5, 7]. It is obvious that there exists no path of odd length between any two distinct vertices of the same partition in a bipartite graph with at least 3 vertices. Thus any bipartite graph is not panconnected. For this reason, we say a bipartite graph is *bipanconnected* if there exists a path of length l joining any two different vertices x and y with $d_H(x, y) \leq l \leq |V(H)| - 1$ and $(l - d_H(x, y))$ is even. It is proved that the hypercube is bipanconnected [5]. The concept of bipanpositionable graphs is proposed in [4]. A hamiltonian bipartite graph H is *bipanpositionable* if for any two different vertices x and y of H and for any integer k with $d_H(x, y) \leq k \leq |V(H)|/2$ and $(k - d_H(x, y))$ is even, there exists a hamiltonian cycle C of H such that $d_C(x, y) = k$. Obviously, the complete bipartite graph $K_{n,n}$ with $n \geq 2$ is bipanpositionable. With the argument similar to the previous paragraph, we know that every bipanconnected graph is bipancyclic, and every bipanpositionable graph is bipancyclic.

In this paper, we present some examples of pancyclic graphs that are neither panconnected nor panpositionable, some examples of panconnected graphs that are not panpositionable, and some examples of graphs that are panconnected and panpositionable. Similarly, for bipartite graphs, we present some examples of bipancyclic graphs that are neither bipanconnected nor bipanpositionable, some examples of bipanconnected graphs that are not bipanpositionable, and some examples of graphs that are bipanconnected and bipanpositionable. The existence of panpositionable (or bipanpositionable, resp.) graphs that are not panconnected (or bipanconnected, resp.) is still an open problem.

2 General Graphs: Pancyclicity, Panconnectivity, and Panpositionability

2.1 Pancyclic graphs that are neither panconnected nor panpositionable

In this section, we show two examples of pancyclic graphs that are neither panconnected nor panpositionable, namely, the graphs $\bar{G}_s$ as $s \geq 2$ and the graph T defined below. The set of $\{\bar{G}_s \mid s \geq 2\}$ are nonplanar examples, while the graph T is a planar graph.

Let G_s be the bipartite graph $K_{s,s}$ for $s \geq 2$. More specifically, $G_s = (B \cup W, E)$, where $B = \{0, 1, 2, \cdots, (s-1)\}$, $W = \{0', 1', 2', \cdots, (s-1)'\}$ and $E = \{(i, j') \mid 0 \leq i, j \leq s-1\}$. Define $\bar{G}_s = G_s \cup \{(0, 1)\}$. Obviously, $\bar{G}_s$ is a nonbipartite graph.

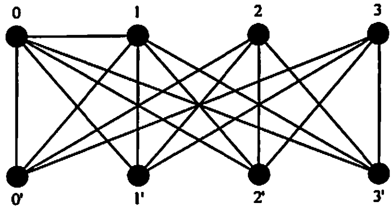

Figure 1: $\bar{G}_4 = K_{4,4} \cup \{(0,1)\}$.

Figure 1 is an illustration of $\bar{G}_4$

Theorem 1. *$\bar{G}_s$ is a pancyclic graph that is neither panconnected nor panpositionable.*

Proof. To show that $\bar{G}_s$ is a pancyclic graph, we show that $\bar{G}_s$ contains a cycle of length l, denoted by C_l below, for $3 \leq l \leq 2s$.

Case 1. l is odd.

l	C_l in G_s
3	$\langle 0, 1', 1, 0 \rangle$
5	$\langle 0, 1', 2, 2', 1, 0 \rangle$
$7 \leq l \leq 2s-1$	$\langle 0, 1', 2, 2', \cdots, \frac{l-1}{2}, (\frac{l-1}{2})', 1, 0 \rangle$

Case 2. l is even.

l	C_l in G_s
4	$\langle 0, 0', 1, 1', 0 \rangle$
6	$\langle 0, 0', 1, 1', 2, 2', 0 \rangle$
$8 \leq l \leq 2s$	$\langle 0, 0', 1, 1', 2, 2', \cdots, \frac{l}{2}-1, (\frac{l}{2}-1)', 0 \rangle$

Let $u = 2$ and $v = 2'$ of $\bar{G}_s$. Obviously, $d_{\bar{G}_s}(u.v) = 1$. Since $N(u) \cap N(v) = \emptyset$, it is impossible to have a path with length 2 between u and v. Thus $\bar{G}_s$ is not panconnected. For the

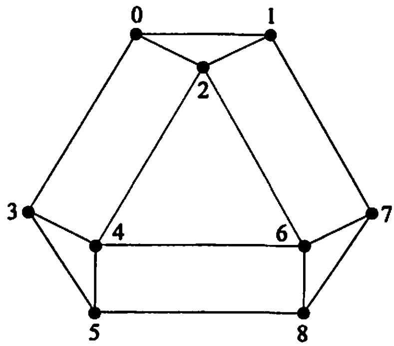

Figure 2: The graph T.

same reason, $\bar{G}_s$ is not panpositionable. This proves the theorem. □

Let T be the graph shown in Figure 2. More precisey, $V(T) = \{i \mid 0 \leq i \leq 8\}$ and $E(T) = \{(0,1), (0,2), (1,2), (3,4), (3,5), (4,5), (6,7), (6,8), (7,8), (0,3), (2,4), (4,6), (5,8), (1,7), (2,6)\}$. We have the following theorem.

Theorem 2. *The graph T is pancyclic, but neither panconnected nor panpositionable.*

Proof. To show that T is a pancyclic graph, we prove that T contains cycles with length l for $3 \leq l \leq 9$. The corresponding cycles are listed in the following.

l	the cycle C in T
3	$\langle 0,1,2,0\rangle$
4	$\langle 0,3,4,2,0\rangle$
5	$\langle 0,3,4,2,1,0\rangle$
6	$\langle 0,3,5,4,2,1,0\rangle$
7	$\langle 0,3,5,8,6,4,2,0\rangle$
8	$\langle 0,3,5,8,7,6,4,2,0\rangle$
9	$\langle 0,3,5,8,7,6,4,2,1,0\rangle$

Let $u = 0$ and $v = 3$ of T. Obviously, $d_T(u, v) = 1$. Since $N(u) \cap N(v) = \emptyset$, it is impossible to have a path with length 2 between u and v. Thus, T is not panconnected. For the same reason, T cannot be panpositionable. This proves the theorem. □

2.2 Panconnected graphs

Assume that $n, s_1, s_2, \cdots, s_r$ are integers with $1 \leq s_1 < s_2 < \cdots < s_r \leq \frac{n}{2}$. The circulant graph $G(n; s_1, s_2, \cdots, s_r)$ is the graph with the vertex set $\{0, 1, \cdots, n-1\}$. Two vertices i and j are adjacent if and only if $i - j = \pm s_k \pmod{n}$ for some k where $1 \leq k \leq r$.

Theorem 3. *Let n be an integer with $n \geq 5$. $G(n; 1, 2)$ is a panconnected graph.*

Proof. Let $G = G(n; 1, 2)$. To show that G is panconnected, we prove that there exists a path of length l for $d_G(x, y) \leq l \leq n-1$ between any two distinct vertices x, y of G. Without loss of generality, let $x = 0$ and $y \leq \lfloor \frac{n}{2} \rfloor$. We define some paths in the following:

$$\begin{aligned} I(i,j) &= \langle i, i+1, i+2, \cdots, j-1, j\rangle; \\ S(i) &= \langle i, i+2\rangle; \\ S^t(i) &= \langle i, i+2, i+4, \cdots, i+2(t-1), i+2t\rangle; \\ S^{-t}(i) &= \langle i, i-2, i-4, \cdots, i-2(t-1), i-2t\rangle. \end{aligned}$$

There are two cases.

Case 1. $l \leq y$. Let $k = 2(l - d_G(0,y))$. The corresponding path is $\langle 0, I(0,k), S^{\lfloor \frac{y-k}{2} \rfloor}(k), y\rangle$.

Case 2. $l > y$. Let $k = \lceil \frac{l-y}{2} \rceil$. The corresponding path is constructed below.

$l-y$	the path between 0 and y with length l
even	$\langle 0, I(0,y-1), S^k(y-1), y+2k-1, y+2k, S^{-k}(y+2k), y\rangle$
odd	$\langle 0, I(0,y-1), S^k(y-1), y+2k-1, y+2k-2, S^{-(k-1)}(y+2k-2), y\rangle$

The theorem is proved. □

The following theorem is proved in [4].

Theorem 4. *$G(n;1,2)$ is panpositionable hamiltonian if and only if $n \in \{5,6,7,8,9,11\}$.*

With Theorem 3 and Theorem 4, we have the following results.

Theorem 5. *$G(n;1,2)$ is panconnected but not panpositionable if and only if $n = 10$ or $n \geq 12$.*

Theorem 6. *$G(n;1,2)$ is panconnected and panpositionable if and only if $n \in \{5,6,7,8,9,11\}$.*

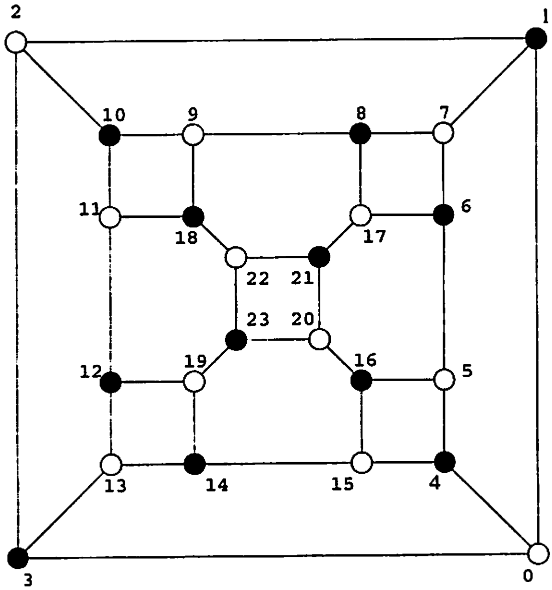

Figure 3: The graph B.

3 Bipartite Graphs: Bipancyclicity, Bipanconnectivity, and Bipanpositionability

3.1 Bipancyclic graphs that are neither bipanconnected nor bipanpositionable

Let B be the graph shown in Figure 3. More precisely, $V(B) = \{i \mid 0 \le i \le 23\}$ and $E(B) = \{(i, i+1) \mid 0 \le i \le 2, 4 \le i \le 14, 20 \le i \le 22\} \cup \{(0,3), (4,15), (20,23), (0,4), (1,7), (2,10), (3,13), (5,16), (15,16), (6,17), (8,17), (9,18), (11,18), (12,19), (14,19), (16,20), (17,21), (18,22), (19,23)\}$.

We have the following theorem.

Theorem 7. *The graph B is bipancyclic that is neither bipanconnected nor bipanpositionable.*

Proof. To show that B is bipancyclic, we prove that B contains cycles of length l for any even integer l with $4 \leq l \leq 24$. The corresponding cycles are listed below.

l	the cycle with length l
4	$\langle 6,7,8,17,6 \rangle$
6	$\langle 0,1,7,6,5,4,0 \rangle$
8	$\langle 6,7,8,17,21,20,16,5,6 \rangle$
10	$\langle 6,7,8,17,21,20,16,15,4,5,6 \rangle$
12	$\langle 6,7,8,9,10,11,12,13,14,15,4,5,6 \rangle$
14	$\langle 6,17,21,20,23,22,18,11,12,13,14,15,4,5,6 \rangle$
16	$\langle 6,7,8,17,21,20,23,22,18,11,12,13,14,15,4,5,6 \rangle$
18	$\langle 6,7,8,17,21,20,23,22,18,9,10,11,12,13,14,15,4,5,6 \rangle$
20	$\langle 6,7,8,17,21,22,18,9,10,11,12,13,14,19,23,20,16,15,4, 5,6 \rangle$
22	$\langle 0,1,7,6,17,8,9,10,11,18,22,21,20,23,19,12,13,14,15, 16,5,4,0 \rangle$
24	$\langle 0,1,2,3,13,12,11,10,9,18,22,23,19,14,15,16,20,21,17, 8,7,6,5,4,0 \rangle$

Let $u = 1$ and $v = 7$ of B. Obviously, $d_B(u, v) = 1$, $N(u) = \{0, 2, 7\}$ and $N(v) = \{1, 6, 8\}$. Let $A_0 = N(0) \cup N(2) \cup N(7)$ and $A_2 = N(1) \cup N(6) \cup N(8)$. If there exists a path P between u and v such that the length of P is 3, then $A_0 \cap A_1 \neq \emptyset$. However, $A_0 = \{1, 3, 4, 6, 8, 10\}$, $A_1 = \{0, 2, 5, 7, 9, 17\}$, and $A_0 \cap A_1 = \emptyset$. Therefore, B is not bipanconnected. For the same reason, B is not bipanpositionable. This proves the theorem. □

3.2 Bipanconnected graphs that is not bipanpositionable

Theorem 8. *Let m be an integer with $m \geq 2$. $C_{2m} \times K_2$ is a bipanconnected graph.*

Proof. Let $G = C_{2m} \times K_2$. Then $V(G) = \{i, i' \mid 0 \leq i < 2m\}$ and $E(G) = \{(i, i+1 \pmod{2m}) \mid 0 \leq i < 2m\} \cup \{(i', (i+1 \pmod{2m})') \mid 0 \leq i < 2m\} \cup \{(i, i') \mid 0 \leq i < 2m\}$. To show that G is bipanconnected, we prove that there exists a path of length l for $d_G(u,v) \leq l \leq 2m-1$ and $l - d_G(u,v)$ is even between any two distinct vertices u, v of G. Since G is symmetric, we only consider two cases: (1)$u = 0, v = y$ with $y \leq m$; (2) $u = 0, v = y'$ with $y \leq m$. We define some paths in the following:

$$
\begin{aligned}
I(i,j) &= \langle i, i+1, i+2, \ldots, j-1, j \rangle; \\
I^{-1}(i,j) &= \langle i, i-1, i-2, \ldots, j+1, j \rangle; \\
I(i',j') &= \langle i', (i+1)', (i+2)', \ldots, (j-1)', j' \rangle; \\
I^{-1}(i',j') &= \langle i', (i-1)', (i-2)', \ldots, (j+1)', j' \rangle; \\
Q(i) &= \langle i, i', (i+1)', i+1, i+2 \rangle; \\
Q^t(i) &= \langle i, Q(i), i+2, Q(i+2), i+4, \ldots, i+2t-2, \\
&\quad\; Q(i+2t-2), i+2t \rangle.
\end{aligned}
$$

Case 1. $u = 0, v = y$ with $y \leq m$.

(1.1) $l = y$. The corresponding path is $\langle 0, I(0,y), y \rangle$.

(1.2) $y < l \leq 4m - y$. Let $k = \frac{l-y}{2}$. The corresponding path is $\langle 0, I(0, y-1), y-1, (y-1)', I((y-1)', (y+k-1)'), (y+k-1)', y+k-1, I^{-1}(y+k-1, y), y \rangle$.

(1.3) $4m - y < l \leq 4m - 1$. Let $k = \frac{l-4m+y}{2}$. The corresponding path is $\langle 0, Q^k(0, 2k), 2k, (2k)'$,
$I((2k)', (2m-1)'), (2m-1)', 2m-1, I^{-1}(2m-1, y), y \rangle$.

Case 2. $u = 0, v = y'$ with $y \leq m$.

(2.1) $y < l \leq 4m - y$. Let $k = \frac{l+y-1}{2}$. The corresponding path is $\langle 0, I(0,k), k, k', I^{-1}(k', y'), y' \rangle$.

(2.2) $4m-y < l \leq 4m-1$. Let $k = \lceil \frac{l-4m+y}{2} \rceil$. The corresponding path is $\langle 0, Q^k(0, 2k), 2k,$
$I(2k, 2m-1), 2m-1, (2m-1)', I^{-1}((2m-1)', y'), y' \rangle$.

This proves the theorem. □

Theorem 9. *Let m be an integer with $m \geq 3$. $C_{2m} \times K_2$ is bipanconnected but not bipanpositionable.*

Proof. Let $G = C_{2m} \times K_2$. $V(G)$ and $E(G)$ is the same as in Theorem 8. With Theorem 8, G is bipanconnected. To prove that G is not bipanpositionable, we show that there is no hamiltonian cycle, C, such that $d_C(0,2) = 2m$. Suppose that G has a hamiltonian cycle C with $d_C(0,2) = 2m$, then only one of the following cases holds: (1)$\langle 2m-1, 0, 1 \rangle \in C$, (2)$\langle 2m-1, 0, 0' \rangle \in C$ or (3)$\langle 0', 0, 1 \rangle \in C$.

Case 1. If $\langle 2m-1, 0, 1 \rangle \in C$, then $(1,2) \notin C$. Otherwise, $d_C(0,2) = 2 < 2m$. Thus $(1,1') \in C$ and $\langle 3, 2, 2' \rangle \in C$. Then $(1', 2') \notin C$, otherwise $d_C(0,2) = 4 < 2m$. Hence $Q = \langle 3, 2, 2', 3' \rangle \in C$. Thus C contains the path $P_0 = \langle 2m-1, 0, 1, 1', 0', (2m-1)' \rangle$. Now $(2m-1, (2m-1)') \notin C$, otherwise $P_0 \cup (2m-1, (2m-1)')$ is a cycle. Thus C contains a path $P_1 = \langle 2m-2, P_0, (2m-2)' \rangle$. With the similar argument, $(2m-k, (2m-k)') \notin C$ for $1 \leq k \leq 2m-3$, otherwise $P_{k-1} \cup (2m-k, (2m-k)')$ is a cycle of G. Then C contains a path P_{2m-4}. Therefore, $C = P_{2m-4} \cup Q$ is a hamiltonian cycle of G and $d_C(0,2) = 2m-2$ However, $2m-2 < 2m$. This is a contradiction.

Case 2. If $\langle 2m-1, 0, 0' \rangle \in C$, then $(0,1) \notin C$ and $\langle 2, 1, 1' \rangle \in$

C. Thus $(0',1') \notin C$. Otherwise, $d_C(0,2) = 4 < 2m$. Then C contains the paths $P_0 = \langle 2m-1, 0, 0', (2m-1)' \rangle$ and $Q = \langle 2, 1, 1', 2' \rangle$. With the same argument as in Case 1, $(i, i') \notin C$ for $2 \le i \le 2m-1$. Therefore, $C = \langle 2m-1, 0, 0', (2m-1)', (2m-2)', (2m-3)', \ldots, 1', 1, 2, 3 \ldots, 2m-1 \rangle$. However, $d_C(0,2) = 2m-2 < 2m$. This is a contradiction.

Case 3. If $\langle 0', 0, 1 \rangle \in C$, then $(1,2) \notin C$. Otherwise, $d_C(0,2) = 2 < 2m$. Then $(1,1') \in C$ and $\langle 2', 2, 3 \rangle \in C$. Now $(1', 0') \notin C$, otherwise $\langle 1', 0', 0, 1, 1' \rangle$ is a cycle. Thus $(1', 2') \in C$. Then C contains a path $\langle 0, 1, 1', 2', 2 \rangle$, and hence $d_C(0,2) = 4 < 2m$ for $m \ge 3$. This is a contradiction.

The theorem is proved. □

3.3 Bipanconnected graphs that is bipanpositionable

Theorem 10. *Let n be an even integer with $n \ge 6$. $G(n;1,3)$ is a bipanconnected graph.*

Proof. Let $G = G(n;1,3)$. To show that G is bipanconnected, we prove that there exists a path of length l for $d_G(x,y) \le l \le n-1$ and $l - d_G(x,y)$ is even between any two distinct vertices x, y of G. Without loss of generality, let $x = 0$ and $y \le \lfloor \frac{n}{2} \rfloor$. We define some paths in the following:

$$\begin{aligned} I(i,j) &= \langle i, i+1, i+2, \ldots, j-1, j \rangle; \\ I^{-1}(i,j) &= \langle i, i-1, i-2, \ldots, j+1, j \rangle; \\ S(i) &= \langle i, i+3 \rangle; \\ S^t(i) &= \langle i, i+3, i+6, \ldots, i+3(t-1), i+3t \rangle; \\ S^{-t}(i) &= \langle i, i-3, i-6, \ldots, i-3(t-1), i-3t \rangle. \end{aligned}$$

There are two cases.

Case 1. $l \leq y$. Let $k = \lfloor \frac{y}{3} \rfloor \times 3$. The corresponding path is $\langle 0, I(0, k - \frac{3}{2}(y-l)), S^{\frac{y-l}{2}}(k - \frac{3}{2}(y-l)), I(k, y), y \rangle$.

Case 2. $l > y$. Let $k = \frac{l-y}{2}$.

(2.1) $k < \frac{n+1-y}{3}$. The corresponding path is $\langle 0, I(0, y-l), S^k(y-l), y+3k-1, y+3k-2, S^{-(k-1)}(y+3k-2), y+1, y \rangle$.

(2.2) $k \geq \frac{n+1-y}{3}$. The corresponding path is constructed below.

y (mod 3)	l	the path between 0 and y with length l
0	$n-3$	$\langle 0, 1, 2, n-1, I^{-1}(n-1, y+3), y+3, y \rangle$
	$n-1$	$\langle 0, 1, 2, n-1, I^{-1}(n-1, y+1), y+1, y \rangle$
1	$n-2$	$\langle 0, 1, 2, n-1, I^{-1}(n-1, y+1), y+1, y \rangle$
2	$n-3$	$\langle 0, 1, 2, n-1, I^{-1}(n-1, y+1), y+1, y \rangle$
	$n-1$	$\langle 0, 1, 2, n-1, I^{-1}(n-1, y+1), y+1, y-2, y-1, y \rangle$

The theorem is proved □

It is proved in [4] that $G(n; 1, 3)$ is bipanpositionable if and only if n is an even integer and $n \geq 6$. Therefore, with Theorem 10, we have the following result.

Theorem 11. *$G(n; 1, 3)$ is bipanconnected and bipanpositionable if and only if n is an even integer and $n \geq 6$.*

4 Conclusion

In this paper, we clarify the relationship among the set of pancyclic graphs, the set of panconnected graphs, and the set of panpositionable graphs. It is obvious that panconnected graphs are pancyclic, and panpositionable graphs are pancyclic. We present some examples of pancyclic graphs that are neither panconnected nor panpositionable, some examples of panconnected graphs that are not panpositionable, and some examples of graphs that are panconnected and panpositionable. The existence of panpositionable graphs that are not panconnected is still an open problem. Similarly, for bipartite graphs, we show that bipanconnected graphs are bipancyclic and bipanpositionable graphs are bipancyclic. We present some examples of bipancyclic graphs that are neither bipanconnected nor bipanpositionable, some examples of bipanconnected graphs that are not bipanpositionable, and some examples of graphs that are bipanconnected and bipanpositionable. The existence of bipanpositionable graphs that are not bipanconnected remains an open problem.

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