Edge Colorings of Planar Graphs without adjacent special cycles

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Abstract

Abstract: By applying discharging methods and properties of critical graphs, we proved that every simple planar graph G with $\Delta \geq 5$ is of class 1, if any 4-cycle is not adjacent to a 5-cycle in G.
Keywords: planar graph; edge coloring; maximum degree; cycle

1 Introduction

Only simple graphs are considered in this paper. For a plane graph G, we denote its vertex set, edge set, face set, and maximum degree by $V(G)$, $E(G)$, $F(G)$ and $\Delta(G)$ (or simply Δ), respectively. We use $d(x)$ denote the degree of x, $x \in V(G) \cup F(G)$. A k -, k^+-vertex (or face) is a vertex (or face) of degree k, at least k. A k -vertex, k^+ - vertex adjacent to a vertex x is called a k- neighbor, k^+- neighbor of x. Let $d_k(x)$, $d_{k^+}(x)$ denote the number of k-neighbors, k^+- neighbors of x. A k-cycle is a cycle of length k. Two cycles sharing a common edge are said to be adjacent.

An edge k-coloring of a graph G is a function $\phi : E(G) \rightarrow \{1, 2, \cdots, k\}$ such that any two adjacent edges $e_1, e_2 \in E(G)$ have $\phi(e_1) \neq \phi(e_2)$. The chromatic index $\chi'(G)$ is the smallest integer k such that G admits an edge k -coloring. A graph G is of class 1 if $\chi'(G) = \Delta$ and of class 2 if $\chi'(G) = \Delta + 1$, A critical graph G is a connected graph such that G is of class 2 and $\chi'(G - e) < \chi'(G)$ for each edge $e \in E(G)$. A critical graph of maximum degree Δ is called a Δ-critical graph.

Vizing[1,2] proved that every planar graph with $\Delta \geq 8$ is of class 1 and conjectured that this is true for $6 \leq \Delta \leq 7$, and first presented examples of planar graphs of class 2 for each $\Delta \in \{2, 3, 4, 5\}$. The case $\Delta = 7$ was confirmed by Zhang [3], and Sanders and Zhao [4] independently. Thus, Vizing's conjecture remains open only for the case $\Delta = 6$. References [5,6]

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proved some sufficient conditions for a planar graph of maximum degree six to be of class 1. References [7,8,9,10] proved some sufficient conditions for a planar graph of maximum degree five to be of class 1.

In this paper, we shall prove that every simple planar graph G with $\Delta \geq 5$ is of class 1, if any 4-cycle is not adjacent to a 5-cycle in G.

2 Property of critical graphs

Lemma 1. [2] *Let G be a Δ -critical graph, $\Delta \geq 3$ and $v \in V(G)$, $u \in V(G)$. Then* (1) *v is adjacent to at most one 2-vertex, and at least two Δ -vertices;* (2) *if $d_k(v) \geq 1$, $k \neq \Delta$, then $d_\Delta(v) \geq \Delta - k + 1$;* (3) *if $uv \in E(G)$, then $d(u) + d(v) \geq \Delta + 2$.*

Lemma 2. [3] *Let G be a Δ -critical graph, $xy \in E(G)$, if $d(x) + d(y) = \Delta + 2$, then*

(1) *any $v \in N(\{x,y\}) \backslash \{x,y\}$ is Δ- vertex,*

(2) *any $v \in N(N(\{x,y\})) \backslash \{x,y\}$ satisfies $d(v) \geq \Delta - 1$, and*

(3) *if $d(x) < \Delta$, $d(y) < \Delta$, then any $v \in N(N(\{x,y\})) \backslash \{x,y\}$ is Δ-vertex.*

Lemma 3. [4] *A graph G is not a critical graph if it has distinct vertices x, y, z such that* (1) *$xy \in E(G)$, $xz \in E(G)$, $d(z) < 2\Delta - d(x) - d(y) + 2$;* (2) *$xz$ is in at least $d(x) + d(y) - \Delta - 2$ triangles not containing y.*

Suppose G is a 5-critical graph, then G is 2-connected, implying that the boundary of each face forms a cycle and every edge lies on the boundaries of two faces. Let $s(v)$ denotes the number of all 3-faces incident to vertex $v \in V(G)$. Since any 4-cycle is not adjacent to a 5-cycle in G, hence, we have follow Lemma 4 and Lemma 5.

Lemma 4. *Any 4-vertex v of G is incident to at most two 3-faces, i.e. $s(v) \leq 2$. When $s(v) = 2$, if two 3-faces are adjacent, then v is incident to two 6^+ -faces; if two 3-faces are not adjacent, then v is incident to two 5^+ -faces. When $s(v) = 1$, v is incident to two 6^+ -faces or three 5^+ -faces.*

Lemma 5. *Any 5-vertex v of G is incident to at most three 3-faces, i.e. $s(v) \leq 3$.* (a) *Let v be adjacent to no 2-vertex, when $s(v) = 3$, v is incident to two 6^+ -faces; when $s(v) = 2$, if two 3-faces are adjacent, then v is incident to two 6^+ -faces, if two 3-faces are not adjacent, then v is incident to three 5^+ -faces; when $s(v) = 1$, v is incident to two 6^+ -faces or four 5^+ -faces.* (b) *Let v be adjacent to 2-vertex, when $s(v) = 3$, v is incident to two 6^+ -faces; when $s(v) = 2$, v is incident to at least one 6^+ -face and one 5^+-face or three 5^+-faces; when $s(v) = 1$, v is incident to one 6^+ -face and one 5^+-face or four 5^+-faces.*

3 Main result and its proof

Theorem 6. *Every simple planar graph G with $\Delta \geq 5$ is of class 1, if any 4-cycle is not adjacent to a 5-cycle in G.*

Proof. Suppose on the contrary that G is of class 2. Without loss of generality, we may assume that G is a k-critical ($k \geq 5$). Using Euler's formula, we can derive the following identity:

$$\sum_{x \in V(G)} (d(x) - 4) + \sum_{x \in F(G)} (d(x) - 4) = -8$$

We define an initial charge function $ch(x) = d(x) - 4$, for each $x \in F(G) \cup V(G)$. Next, a modified charge function $ch'(x)$ is defined as a modification of $ch(x)$ by moving some charge locally among vertices and faces according to the following discharging rules. Since the total sum of charges is kept fixed when the discharging is in process, therefore,we have identity (1):

$$\sum_{x \in \{V \cup F\}} ch(x) = \sum_{x \in \{V \cup F\}} ch'(x) = -8$$

For each $x \in F(G) \cup V(G)$, if we can show that $ch'(x) \geq 0$, this contradicts identity (1) to prove the Theorem 1.

Case 1 Every simple planar graph G with $\Delta \geq 7$ is of class 1, if any 4-cycle is not adjacent to a 5-cycle in G.

The case was proved in references [1,2,3,4], we omit here.

Case 2 Every simple planar graph G with $\Delta = 6$ is of class 1, if any 4-cycle is not adjacent to a 5-cycle in G.

We define discharging rules as follows.

R1 Let v be a 6-vertex, v sends 1 to adjacent 2- vertex; v sends $\frac{1}{3}$ to each adjacent 3- vertex, and 4-vertex;

R2 Let v be a 5-vertex, v sends $\frac{1}{3}$ to each adjacent 3- vertex;

R3 Let f be a 3-face, x, y, z are distinct vertices of f, and $d(x) \leq d(y) \leq d(z)$.

R3-1 If f is $(k, 6, 6)$ -face, $k = 2, 3$, then y and z independently sends $\frac{1}{2}$ to f;

R3-2 If f is $(3, 5, 6)$-face, then y sends $\frac{1}{3}$, z sends $\frac{2}{3}$ to f ;

R3-3 If $d(x) \geq 4$, then x, y, z independently sends $\frac{1}{3}$ to f.

R4 Let $d(f) \geq 5$, then f sends $\frac{d(f)-4}{d(f)}$ to each incident vertex.

Suppose $f \in F(G)$, then $d(f) \geq 3$. If $d(f) = 4$, then $ch(f) = 0$, thus $ch'(f) = 0$. If $d(f) \geq 5$, then $ch(f) \geq 1$, thus $ch'(f) = d(f) - 4 - \frac{d(f)-4}{d(f)} \times d(f) = 0$ by R4.

If $d(f) = 3$, then $ch(f) = -1$. Since $\Delta = 6$, f must be one of the following 3-faces: $(2^+, 6, 6)$-face, $(3^+, 5, 6)$-face, $(4, 4, 6)$-face, $(4^+, 5, 5)$- face by Lemma 1, 2, 3. Hence, $ch'(f) = -1 + 1 = 0$ by R3.

Suppose $v \in V(G)$, then $d(v) \geq 2$. Let $d(v) = 2$, then $ch(v) = -2$. v is adjacent to two 6-vertices by Lemma 1, thus, $ch'(v) = -2 + 2 = 0$ by R1.

Let $d(v) = 3$, then $ch(v) = -1$. v is adjacent to three 5^+-vertices by Lemma 1. Hence, $ch'(v) = -1 + \frac{1}{3} \times 3 = 0$ by R1, R2.

Let $d(v) = 4$, then $ch(v) = 0$. v is adjacent to at most one 4-vertex and at least two 6-vertices by Lemma 1, v receives at least $\frac{1}{3} \times 2$ by R1. Since any 4-cycle is not adjacent to a 5-cycle in G, therefore, v is incident to at most two 3-faces. Hence, $ch'(v) \geq \frac{1}{3} \times 2 - \frac{1}{3} \times 2 = 0$ by R3-3.

Let $d(v) = 5$, then $ch(v) = 1$. $min\{d(u) | u \in N(v)\} \geq 3$, $d_3(v) \leq 1$ and $d_6(v) \geq 2$ by Lemma 1. Since any 4-cycle is not adjacent to a 5-cycle in G, therefore, v is incident to at most three 3-faces, and when $s(v) = 3$, v is incident to two 6^+-faces.

If $min\{d(u) | u \in N(v)\} = 3$, then, $d_6(v) = 4$ by Lemma 2. When $s(v) = 3$, v receives at least $\frac{2}{3}$ by R4, v sends at most $\frac{1}{3} \times 4$ to its adjacent 3-vertex and incident 3-faces by R2, R3-2, R3-3, hence, $ch'(v) \geq 1 + \frac{2}{3} - \frac{1}{3} \times 4 > 0$; when $s(v) \leq 2$, v sends at most $\frac{1}{3} \times 3$ to its adjacent 3-vertex and incident 3-faces by R2, R3-2, R3-3, hence, $ch'(v) \geq 1 - \frac{1}{3} \times 3 = 0$.

If $min\{d(u) | u \in N(v)\} \geq 4$, then, $d_4(v) \leq 2$ and $d_6(v) \geq 3$ by Lemma 1. v sends at most $\frac{1}{3} \times 3$ to its incident 3-faces by R3-3, hence, $ch'(v) \geq 1 - \frac{1}{3} \times 3 = 0$.

Let $d(v) = 6$, then $ch(v) = 2$. $min\{d(u) | u \in N(v)\} \geq 2$, v is adjacent to at most one 2-vertex and at least two 6-vertices. Since any 4-cycle is not adjacent to a 5-cycle in G, therefore, v is incident to at most four 3-faces. When v is adjacent to 2-vertex, if $s(v) = 4$, then, v is incident to one 6^+-face and one 7^+-face, if $s(v) = 3$, then, v is incident to two 6^+-faces or two 5^+-faces and one 6^+-face; when v is adjacent to no 2-vertex, and $s(v) = 4$, v is incident to two 6^+-faces.

If $d_2(v) = 1$, then, $d_6(v) = 5$ by Lemma 2. When $s(v) = 4$, v receives at least $\frac{1}{3} + \frac{3}{7}$ by R4, v sends at most $1 + \frac{1}{2} + \frac{1}{3} \times 3$ to its adjacent 2-vertex and incident 3-faces by R1, R3-1, R3-3, hence, $ch'(v) \geq 2 + \frac{1}{3} + \frac{3}{7} - (1 + \frac{1}{2} + \frac{1}{3} \times 3) > 0$; when $s(v) = 3$, v receives at least $\frac{1}{3} \times 2$ by R4, v sends at most $1 + \frac{1}{2} + \frac{1}{3} \times 2$ to its adjacent 2-vertex and incident 3-faces by R1, R3-1, R3-3, hence, $ch'(v) \geq 2 + \frac{1}{3} \times 2 - (1 + \frac{1}{2} + \frac{1}{3} \times 2) > 0$; when $s(v) \leq 2$, we have $ch'(v) \geq 2 - (1 + \frac{1}{2} + \frac{1}{3}) > 0$ by R1, R3-1, R3-3.

If $min\{d(u) | u \in N(v)\} = 3$, then $d_6(v) \geq 4$ by Lemma 1. When $d_3(v) = 2$, or $d_3(v) = 1$ and $d_4(v) = 1$, v is incident to at most two 3-faces by Lemma 3, they are $(6, 6, 6)$-faces, v sends at most $\frac{1}{3} \times 4$ to its adjacent 3-vertices, 4-vertex and incident 3-faces, by R1,R3-3, hence, $ch'(v) \geq 2 - \frac{1}{3} \times 4 > 0$. When $d_3(v) = 1$ and $d_{5^+}(v) = 5$, if $s(v) = 4$, v receives at least $\frac{1}{3} \times 2$ by R4, v sends at most $\frac{1}{2} + \frac{2}{3} + \frac{1}{3} \times 3$ to its adjacent 3-vertex, and incident 3-faces by R1, R3-1, R3-2, R3-3, hence, $ch'(v) \geq 2 + \frac{1}{3} \times 2 - (\frac{1}{2} + \frac{2}{3} + \frac{1}{3} \times 3) > 0$; if $s(v) \leq 3$, v sends at most $\frac{1}{2} + \frac{2}{3} + \frac{1}{3} \times 2$ to its adjacent 3-vertex, and incident 3-faces by R1, R3-1, R3-2, R3-3, hence, $ch'(v) \geq 2 - (\frac{1}{2} + \frac{2}{3} + \frac{1}{3} \times 2) > 0$.

If $min\{d(u)|u \in N(v)\} \geq 4$, then, $d_6(v) \geq 2$, $d_4(v) \leq 3$. When $s(v) = 4$, v receives at least $\frac{1}{3} \times 2$ by R4, v sends at most $\frac{1}{3} \times 7$ to its adjacent 4-vertices, and incident 3-faces by R1,R3-3, hence, $ch'(v) \geq 2 + \frac{1}{3} \times 2 - \frac{1}{3} \times 7 > 0$. When $s(v) \leq 3$, v sends at most $\frac{1}{3} \times 6$ to its adjacent 4-vertices, and incident 3-faces by R1,R3-3, hence, $ch'(v) \geq 2 - \frac{1}{3} \times 6 = 0$. Thus, for each $x \in F(G) \cup V(G)$, we show that $ch'(x) \geq 0$, this contradiction proves Case 2.

Case 3 Every simple planar graph G with $\Delta = 5$ is of class 1, if any 4-cycle is not adjacent to a 5-cycle in G.

We define discharging rules as follows.

R1 Let v be a 5-vertex, we do the following:

R1-1 v sends 1 to each adjacent 2- vertex;

R1-2 If v is adjacent to 3-vertex x, x is adjacent to 4-vertex, then v sends $\frac{1}{2}$ to x; otherwise, v sends $\frac{1}{3}$ to x;

R1-3 If v is adjacent to 5-vertex u, u is adjacent to 2-vertex x, but v is adjacent to no x, then v sends $\frac{1}{7}$ to u.

R1-4 If v is adjacent to 4-vertex y, y is incident to $(3,4,5)$-face f, but v is incident to no f, then v sends $\frac{1}{6}$ by y to 5-vertex of f.

R1-5 Suppose v is adjacent to 4-vertex y, y is incident to two no adjacent 3-faces. If edge yv is incident to 5-face and no $(3,4,5)$-face, then v sends $\frac{2}{15}$ to y.

R1-6 Suppose v is adjacent to 4-vertex y, y is incident to two no adjacent $(4,4,5)$-faces, and one 5-face f_1, one 6^+-face f_2. If two 4-neighbors of y is incident to f_1, then v sends $\frac{1}{15}$ to y.

R2 Let f be a 3-face, x, y, z are distinct vertices of f, and $d(x) \leq d(y) \leq d(z)$.

R2-1 If f is $(k, 5, 5)$ -face, $k = 2, 3$, then y and z independently sends $\frac{1}{2}$ to f;

R2-2 If f is $(3,4,5)$-face, then y sends $\frac{1}{3}$, z sends $\frac{2}{3}$ to f ;

R2-3 If $d(x) \geq 4$, then x, y, z independently sends $\frac{1}{3}$ to f.

R3 Let $d(f) \geq 5$, then f sends $\frac{d(f)-4}{d(f)}$ to each incident vertex. When 2-vertex x is incident to f $(d(f) \geq 5)$, x receives $\frac{d(f)-4}{d(f)}$ from f. Then, x sends $\frac{d(f)-4}{2d(f)}$ to its each adjacent 5-vertex.

Claim Suppose G is a 5-critical graph, and any 4-cycle is not adjacent to a 5-cycle in G. For any 4-vertex $v \in G$, v receives at least $\frac{2}{3}$, if v is incident to two 3-faces according to the previous discharging rules.

Proof For any 4-vertex $v \in G$, v is adjacent to at least two 5-vertices by Lemma 1, $s(v) \leq 2$ by Lemma 4.

When $s(v) = 2$ and v is incident to two adjacent 3-faces, v receives at least $\frac{1}{3} \times 2$ by Lemma 4 and R3.

When $s(v) = 2$ and v is incident to two no adjacent 3-faces, v is incident to two 5^+-faces f_1, f_2 by Lemma 4. (1) If $d(f_1) \geq 6$, $d(f_2) \geq 6$, then v receives at least $\frac{2}{3}$ by R3. (2)If $d(f_1) = 5$, $d(f_2) = 5$, then there is at least two 5-neighbors of v, we denote them by u, so that uv is incident to 5-face, and no $(3,4,5)$-face. Thus, v receives at least $\frac{2}{15} \times 2$ by R1-5, v receives $\frac{1}{5} \times 2$ by Lemma 4 and R3. Hence, $\frac{2}{15} \times 2+\frac{1}{5} \times 2 =\frac{2}{3}$. (3) If $d(f_1) = 5$, $d(f_2) \geq 6$, then v receives $\frac{1}{5} + \frac{1}{3}$ by R3. When v is adjacent to two 4-vertices u_1,u_2, and u_i (i=1,2) is incident to f_1, then two 5-neighbors of v are incident to f_2, now, v is incident to two no adjacent $(4,4,5)$-faces, one 5-face and one 6^+-face, thus, v receives $\frac{1}{15} \times 2$ by R1-6, therefore, v receives $\frac{1}{5} + \frac{1}{3}+\frac{1}{15} \times 2=\frac{2}{3}$; when v is adjacent to at most one 4-vertex, or v is adjacent to two 4-vertices u_i (i=1,2), but u_1 is incident to f_1, u_2 is incident to f_2, then, there is at least one 5-neighbors u of v, so that uv is incident to 5-face, and no $(3,4,5)$-face, thus, v receives at least $\frac{2}{15}$ by R1-5, therefore, v receives $\frac{1}{5} + \frac{1}{3}+\frac{2}{15} = \frac{2}{3}$. This completes the proof of Claim.

Suppose $f \in F(G)$, then $d(f) \geq 3$. If $d(f) = 4$, then $ch(f) = 0$, thus $ch'(f) = 0$. If $d(f) \geq 5$, then $ch(f) \geq 1$, thus $ch'(f) = d(f) - 4-\frac{d(f)-4}{d(f)}\times d(f) = 0$ by R3.

If $d(f) = 3$, then $ch(f) = -1$. Since $\Delta = 5$, f must be one of the following 3-faces: $(2^+,5,5)$-face, (3,4,5)-face, (4,4,4)-face,(4,4,5)- face by Lemma 1, 2, 3. Hence, $ch'(f) = -1 + 1 = 0$ by R2.

Suppose $v \in V(G)$, then $d(v) \geq 2$. Let $d(v) = 2$, then $ch(v) = -2$. v is adjacent to two 5-vertices by Lemma 1, thus, $ch'(v) = -2+2 = 0$ by R1-1.

Let $d(v) = 3$, then $ch(v) = -1$. v is adjacent to at least two 5-vertices and at most one 4- vertex by Lemma 1. By R1-2, if $d_4(v) = 1$, then v receives $\frac{1}{2}\times 2$; if $d_5(v) = 3$, then v receives $\frac{1}{3}\times 3$. Hence, $ch'(v) = -1+1 = 0$

Let $d(v) = 4$, then $ch(v) = 0$. v is adjacent to at most one 3-vertex and at least two 5-vertices by Lemma 1. $s(v) \leq 2$ by Lemma4. When $s(v) = 2$, v receives at least $\frac{2}{3}$ by Claim, v sends at most $\frac{1}{3} \times 2$ to incident 3-faces by R2-2,R2-3, thus, $ch'(v) \geq \frac{2}{3}-\frac{1}{3}\times 2 = 0$. When $s(v) = 1$, v receives at least $\frac{3}{5}$ by Lemma 4 and R3, v sends at most $\frac{1}{3}$ to incident 3-faces by R2-2,R2-3, thus, $ch'(v) \geq \frac{3}{5}-\frac{1}{3}> 0$. When $s(v) = 0$, thus $ch'(v) = 0$.

Let $d(v) = 5$, then $ch(v) = 1$. $min\{d(u)|u \in N(v)\}\geq 2$, $d_2(v)\leq 1$ and $d_5(v)\geq 2$ by Lemma 1. $s(v) \leq 3$ by Lemma 5.

Suppose v sends charges by R1-5, since any 4-cycle is not adjacent to a 5-cycle in G, then v satisfies $1 \leq s(v) \leq 2$, and $min\{d(u)|u \in N(v)\}\geq 3$ by Lemma2. Suppose v sends charges by R1-6, then, v satisfies $1 \leq s(v) \leq 3$, and $min\{d(u)|u \in N(v)\}\geq 4$ by Lemma 1 and 2.

First, let v sends charges to no adjacent 5-vertex of v by R1-4, then

$min\{d(u)|u \in N(v)\}= 4$ and $d_5(v) = 4$ by Lemma 2. When $s(v) = 0$, v sends at most $\frac{1}{7} \times 4$ to four 5-neighbors of v by Lemma2 and R1-3, v sends at most $\frac{1}{6} \times 2$ to its no adjacent 5-vertices by R1-4, thus, $ch'(v) \geq 1 - (\frac{1}{6} \times 2 + \frac{1}{7} \times 4) > 0$. When $1 \leq s(v) \leq 3$, we discuss it as follows.(1) If 4-neighbor y of v is incident to two $(3,4,5)$-faces, then v is incident to at most two 3-faces by Lemma 4 and condition of Theorem 1, this time, edge vy is incident to two 6^+-faces, so, v does not send charge according to R1-5. Therefore, when $1 \leq s(v) \leq 2$, v receives at least $\frac{3}{5}$ by Lemma 5(a) and R3, v sends at most $\frac{1}{7} \times 3$ to three 5-neighbors of v by Lemma2 and R1-3, v sends out at most $\frac{1}{6} \times 2 + \frac{1}{3} \times 2$ by R1-4,R2-3, thus, $ch'(v) \geq 1 + \frac{3}{5} - (\frac{1}{6} \times 2 + \frac{1}{3} \times 2 + \frac{1}{7} \times 3) > 0$. (2) If 4-neighbor y of v is incident to one $(3,4,5)$-face. Let $s(v) = 3$, then, v sends at most $\frac{1}{7} \times 2$ to two 5-neighbors of v by Lemma2 and R1-3, v sends $\frac{1}{6}$ to its no adjacent 5-vertex by R1-4, v sends $\frac{1}{3} \times 3$ to its incident 3-faces by R2-3, v receives at least $\frac{2}{3}$ by Lemma 5(a) and R3, thus, $ch'(v) \geq 1 + \frac{2}{3} - (\frac{1}{7} \times 2 + \frac{1}{6} + \frac{1}{3} \times 3) > 0$. Let $1 \leq s(v) \leq 2$, then, v sends at most $\frac{1}{7} \times 3$ to three 5-neighbors of v by Lemma2 and R1-3, v sends out at most $\frac{1}{6} + \frac{1}{3} \times 2$ by R1-4,R2-3, v sends $\frac{2}{15}$ to its incident 4-vertex by R1-5, v receives at least $\frac{3}{5}$ by Lemma 5(a) and R3, thus, $ch'(v) \geq 1 + \frac{3}{5} - (\frac{1}{6} + \frac{1}{3} \times 2 + \frac{1}{7} \times 3 + \frac{2}{15}) > 0$.

Therefore, we suppose v sends charges only to its adjacent vertices and incident 3-faces, as the following discuss.

Let $d_2(v) = 1$, then $d_5(v) = 4$ by Lemma 2, v receives at least $\frac{1}{7} \times 3$ from three 5-neighbors of v by R1-3. If $s(v) = 3$, then, v receives at least $\frac{2}{3} + \frac{1}{6}$ by Lemma 5(b) and R3, v sends at most $1 + \frac{1}{2} + \frac{1}{3} \times 2$ to its adjacent 2-vertex and incident 3-faces by R1-1, R2-1, R2-3, thus, $ch'(v) \geq 1 + \frac{1}{7} \times 3 + \frac{2}{3} + \frac{1}{6} - (1 + \frac{1}{2} + \frac{1}{3} \times 2) > 0$. If $1 \leq s(v) \leq 2$, then, v receives at least $\frac{1}{5} + \frac{1}{3}$ by Lemma 5(b) and R3, v sends at most $1 + \frac{1}{2} + \frac{1}{3}$ to its adjacent 2-vertex and incident 3-faces by R1-1, R2-1, R2-3, thus, $ch'(v) \geq 1 + \frac{1}{7} \times 3 + \frac{1}{5} + \frac{1}{3} - (1 + \frac{1}{2} + \frac{1}{3}) > 0$. If $s(v) = 0$, then, v sends 1 to its adjacent 2-vertex, thus, $ch'(v) = 1 - 1 = 0$.

Let $min\{d(u)|u \in N(v)\}= 3$, then v is adjacent to at least three 5-vertices and at most two 3- vertices by Lemma 1, and any $x \in N(N(v)) \backslash \{v\}$ satisfies $d(x) \geq 3$ by Lemma 2.

If $s(v) = 0$, v sends out at most $\frac{1}{2} \times 2$ by R1-2, hence, $ch'(v) \geq 1 - \frac{1}{2} \times 2 = 0$. If $1 \leq s(v) \leq 3$, we discuss it as follows:

Suppose $d_3(v) = 2$, then $d_5(v) = 3$. v is incident to at most two 3-faces by Lemma 3, if $s(v) = 2$, then two 3-faces are adjacent. Thus, when $1 \leq s(v) \leq 2$, v receives at least $\frac{1}{3} \times 2$ by Lemma 5(a) and R3, v sends at most $\frac{1}{2} \times 2 + \frac{1}{3} \times 2$ by R1-2, R2-3, hence, $ch'(v) \geq 1 + \frac{1}{3} \times 2 - (\frac{1}{2} \times 2 + \frac{1}{3} \times 2) = 0$.

Suppose $d_3(v) = 1$, $d_4(v) = 1$, then $d_5(v) = 3$ by Lemma 2. (1) When $s(v) = 3$, v receives at least $\frac{1}{3} \times 2$ by Lemma 5(a) and R3. If v is incident to $(3,4,5)$-face and $(3,5,5)$-face, then, v receives $\frac{1}{6} \times 2$ by R1-4, v sends $\frac{1}{2} \times 2 + \frac{1}{3} + \frac{2}{3}$ to its adjacent 3-vertex and incident 3-faces by R1-2, R2-1, R2-2, R2-3, hence, $ch'(v) \geq 1 + \frac{1}{3} \times 2 + \frac{1}{6} \times 2 - (\frac{1}{2} \times 2 + \frac{1}{3} + \frac{2}{3}) = 0$; if v is incident to $(3,4,5)$-face and no $(3,5,5)$-face, then, v receives at least $\frac{1}{6}$ by R1-4, v sends $\frac{1}{2}+\frac{1}{3}\times 2+\frac{2}{3}$ to its adjacent 3-vertex and incident 3-faces by R1-2, R2-2, R2-3, hence, $ch'(v)\geq 1+\frac{1}{3}\times 2+\frac{1}{6}-(\frac{1}{2}+\frac{1}{3}\times 2+\frac{2}{3})=0$; if v is incident to no $(3,4,5)$-face, v sends at most $\frac{1}{2}\times 2+\frac{1}{3}\times 2$ to its adjacent 3-vertex and incident 3-faces by R1-2、R2-1、R2-3, hence, $ch'(v)\geq 1+\frac{1}{3}\times 2-(\frac{1}{2}\times 2+\frac{1}{3}\times 2)=0$. (2) When $1\leq s(v)\leq 2$, v receives at least $\frac{1}{5}\times 3$ by Lemma 5(a) and R3. If v is incident to no $(3,4,5)$-face, v sends at most $\frac{1}{2}\times 2+\frac{1}{3}+\frac{2}{15}$ to its adjacent 3-vertex, 4-vertex and incident 3-faces by R1-2, R2-1, R2-3, R1-5, hence, $ch'(v)\geq 1+\frac{3}{5}-(\frac{1}{2}\times 2+\frac{1}{3}+\frac{2}{15})>0$; if v is incident to $(3,4,5)$-face and $(3,5,5)$-face, then, v receives $\frac{1}{6}\times 2$ by R1-4, v sends at most $\frac{1}{2}\times 2+\frac{2}{3}$ to its adjacent 3-vertex and incident 3-faces by R1-2, R2-1, R2-2, R2-3, hence, $ch'(v)\geq 1+\frac{3}{5}+\frac{1}{6}\times 2-(\frac{1}{2}\times 2+\frac{2}{3})>0$; if v is incident to $(3,4,5)$-face and no $(3,5,5)$-face, then, v sends at most $\frac{1}{2}+\frac{1}{3}+\frac{2}{3}$ by R1-2、R2-2、R2-3, hence, $ch'(v)\geq 1+\frac{3}{5}-(\frac{1}{2}+\frac{1}{3}+\frac{2}{3})>0$.

Suppose $d_3(v)=1$, $d_5(v)=4$. (1)When $s(v)=3$, v receives at least $\frac{2}{3}$ by Lemma 5(a) and R3. If v is incident to two $(3,5,5)$-faces, then, v sends $\frac{1}{3}$ to its adjacent 3-vertex by R1-2, v sends $\frac{1}{2}\times 2+\frac{1}{3}$ to its incident 3-faces by R2-1, R2-3, thus, $ch'(v)\geq 1+\frac{2}{3}-(\frac{1}{2}\times 2+\frac{1}{3}\times 2)=0$; if v is incident to at most one $(3,5,5)$-face, then, v sends at most $\frac{1}{2}$ to its adjacent 3-vertex by R1-2, v sends $\frac{1}{2}+\frac{1}{3}\times 2$ to its incident 3-faces by R2-1, R2-3, thus, $ch'(v)\geq 1+\frac{2}{3}-(\frac{1}{2}\times 2+\frac{1}{3}\times 2)=0$. (2) When $1\leq s(v)\leq 2$, v receives at least $\frac{3}{5}$ by Lemma 5(a) and R3, v sends at most $\frac{1}{2}$ to its adjacent 3-vertex by R1-2, v sends $\frac{1}{2}\times 2$ to its incident 3-faces by R2-1, R2-3, hence, $ch'(v)\geq 1+\frac{3}{5}-(\frac{1}{2}\times 3)>0$.

Let $min\{d(u)|u\in N(v)\}=4$, then, $d_5(v)\geq 2$, $d_4(v)\leq 3$.

If $s(v)=0$, then, v sends at most $\frac{1}{7}\times 4$ to its adjacent 5-vertices by R1-3, hence, $ch'(v)\geq 1-\frac{1}{7}\times 4>0$. If $1\leq s(v)\leq 3$, we discuss it as follows:

Suppose $d_4(v)=3$, then, $d_5(v)=2$. When $s(v)=3$, v receives at least $\frac{1}{3}\times 2$ by Lemma 5(a) and R3, v sends at most $\frac{1}{7}$ to its adjacent 5-vertex by R1-3, v sends at most $\frac{1}{15}\times 3$ to its adjacent 4-vertices by R1-6, v sends at most $\frac{1}{3}\times 3$ to its incident 3-faces by R2-3, hence, $ch'(v)\geq 1+\frac{1}{3}\times 2-(\frac{1}{7}+\frac{1}{15}\times 3+\frac{1}{3}\times 3)>0$; when $1\leq s(v)\leq 2$, v receives at least $\frac{1}{5}\times 3$ by Lemma 5(a) and R3, v sends at most $\frac{1}{7}\times 2$ to its adjacent 5-vertices by R1-3, v sends at most $\frac{2}{15}\times 3$ to its adjacent 4-vertices by R1-5, R1-6, v sends at most $\frac{1}{3}\times 2$ to its incident 3-faces by R2-3, hence, $ch'(v)\geq 1+\frac{1}{5}\times 3-(\frac{1}{7}\times 2+\frac{2}{15}\times 3+\frac{1}{3}\times 2)>0$.

Suppose $d_4(v)=2$, then, $d_5(v)=3$. When $s(v)=3$, v receives at least $\frac{1}{3}\times 2$ by Lemma 5(a) and R3, v sends at most $\frac{1}{7}\times 2+\frac{1}{15}\times 2+\frac{1}{3}\times 3$ to its adjacent 5-vertices, 4-vertices and incident 3-faces by R1-3, R1-6, R2-3, hence, $ch'(v)\geq 1+\frac{1}{3}\times 2-(\frac{1}{7}\times 2+\frac{1}{15}\times 2+\frac{1}{3}\times 3)>0$; when $1\leq s(v)\leq 2$, v receives at least $\frac{1}{5}\times 3$ by Lemma 5(a) and R3, v sends at most $\frac{1}{7}\times 3+\frac{2}{15}\times 2+\frac{1}{3}\times 2$ to its adjacent 5-vertices, 4-vertices and incident 3-faces by R1-3, R1-5, R1-6, R2-3, hence, $ch'(v)\geq 1+\frac{1}{5}\times 3-(\frac{1}{7}\times 3+\frac{2}{15}\times 2+\frac{1}{3}\times 2)>$

0.

Suppose $d_4(v) = 1$, then, $d_5(v) = 4$. When $s(v) = 3$, v receives at least $\frac{1}{3}\times 2$ by Lemma 5(a) and R3, v sends at most $\frac{1}{7}\times 2 + \frac{1}{15} + \frac{1}{3}\times 3$ to its adjacent 5-vertices, 4-vertex and incident 3-faces by R1-3, R1-6, R2-3, hence, $ch'(v) \geq 1 + \frac{1}{3}\times 2 - (\frac{1}{7}\times 2 + \frac{1}{15} + \frac{1}{3}\times 3) > 0$; when $1 \leq s(v) \leq 2$, v receives at least $\frac{1}{5}\times 3$ by Lemma 5(a) and R3, v sends at most $\frac{1}{7}\times 3 + \frac{2}{15} + \frac{1}{3}\times 2$ to its adjacent 5-vertices, 4-vertex and incident 3-faces by R1-3, R1-5, R1-6, R2-3, hence, $ch'(v) \geq 1 + \frac{1}{5}\times 3 - (\frac{1}{7}\times 3 + \frac{2}{15} + \frac{1}{3}\times 2) > 0$.

Let $min\{d(u)|u \in N(v)\} = 5$, then, $d_5(v) = 5$. When $s(v) = 0$, v sends at most $\frac{1}{7}\times 5$ to its adjacent 5-vertices by R1-3, hence, $ch'(v) \geq 1 - \frac{1}{7}\times 5 > 0$. When $1 \leq s(v) \leq 2$, v receives at least $\frac{1}{5}\times 3$ by Lemma 5(a) and R3, v sends at most $\frac{1}{7}\times 4 + \frac{1}{3}\times 2$ to its adjacent 5-vertices, and incident 3-faces by R1-3, R2-3, hence, $ch'(v) \geq 1 + \frac{1}{5}\times 3 - (\frac{1}{7}\times 4 + \frac{1}{3}\times 2) > 0$. When $s(v) = 3$, v receives at least $\frac{1}{3}\times 2$ by Lemma 5(a) and R3, v sends at most $\frac{1}{7}\times 3 + \frac{1}{3}\times 3$ to its adjacent 5-vertices, and incident 3-faces by R1-3, R2-3, hence, $ch'(v) \geq 1 + \frac{1}{3}\times 2 - (\frac{1}{7}\times 3 + \frac{1}{3}\times 3) > 0$. Thus, for each $x \in F(G) \cup V(G)$, we show that $ch'(x) \geq 0$, this contradiction proves Case 3.

Therefore, we complete the proof of the Theorem according to Case 1, 2, 3. □

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