On The Recursive Sequence Order-k

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November 22, 2011

Abstract

In this paper, we use a simple method to derive different recurrence relations on the recursive sequence order-k and their sums, which are more general than that given in literature [J.Feng, More Identities on the Tribonacci Numbers, Ars Combinatoria, 100(2011), 73-78]. By using the generating matrices, we get more identities on the recursive sequence order-k and their sums, which are more general than that given in literature [E.Kılıç, Tribonacci Sequences with Certain Indices and Their Sums, Ars Combinatoria, 86(2008), 13-22]

1 Introduction

The recursive sequence order-k is like the Fibonacci, Tribonacci, Tetranacci,... sequences. The sequence starts with k predetermined terms and each term afterwards is sum of the preceding k terms, that is,

$$L_n = \sum_{i=1}^{k} L_{n-i} \tag{1}$$

where $L_0 = 0$, $L_1 = 1$, $L_2 = 2^0 = 1$, $L_3 = 2^1 = 2$,...,$L_{k-1} = 2^{k-3}$.

If we take $k = 2$ in equation (1), $\{L_n\}$ sequence is be Fibonacci sequence.If we take $k = 3$ in equation (1), $\{L_n\}$ sequence is be Tribonacci sequence.

Terms of negative subscript $\{L_n\}$ sequence are calculated by

$$L_n = L_{n+k} - \sum_{i=1}^{k-1} L_{n+i}$$

2 The Recursive Sequence Order-k With Certain Indices And Their Sums

Define generating matrix U and V_n as shown, respectively

$$U = \begin{bmatrix} 1 & 1 & \dots & 1 & 1 \\ 1 & 0 & \dots & 0 & 0 \\ 0 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \dots & 1 & 0 \end{bmatrix}_{k\times k}$$

$$V_n = \begin{bmatrix} L_{n+1} & \sum_{i=0}^{k-2} L_{n-i} & \dots & \sum_{i=0}^{1} L_{n-i} & L_n \\ L_n & \sum_{i=1}^{k-1} L_{n-i} & \dots & \sum_{i=1}^{2} L_{n-i} & L_{n-1} \\ L_{n-1} & \sum_{i=2}^{k} L_{n-i} & \dots & \sum_{i=2}^{3} L_{n-i} & L_{n-2} \\ \vdots & \vdots & & \vdots & \vdots \\ L_{n-k+2} & \sum_{i=k-1}^{2k-3} L_{n-i} & \dots & \sum_{i=k-1}^{k} L_{n-i} & L_{n-k+1} \end{bmatrix}_{k\times k}$$

Theorem 1 *If $n > 0$, then $V_n = U^n$*

Proof. By direct computation, we have $V_n = UV_{n-1}$ from which it follows that $V_n = U^{n-1}V_1$. By the definations of matrix U and V_n, one can see that $V_1 = U$ and thus proof is seen. ■

Let for $n > 0$

$$S_n = \sum_{i=0}^{n} L_i$$

for $n < 0$

$$S_n = \sum_{i=-1}^{n} L_i$$

where L_n, nth term of the recursive sequence order-k.

Define Y and Z_n as following;

$$Y = \begin{bmatrix} 1 & 0 & 0 & \dots & 0 & 0 \\ 1 & 1 & 1 & \dots & 1 & 1 \\ 0 & 1 & 0 & \dots & 0 & 0 \\ 0 & 0 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 1 & 0 \end{bmatrix}_{(k+1)\times(k+1)}$$

$$Z_n = \begin{bmatrix} 1 & 0 \; 0 \; 0 \; \dots \; 0 \\ S_n & \\ S_{n-1} & \\ S_{n-2} & V_n \\ \vdots & \\ S_{n-k+1} & \end{bmatrix}_{(k+1)\times(k+1)}$$

Lemma 2 *If $n \geq k$, then $S_n = 1 + \sum_{i=1}^{k} S_{n-i}$*

Proof. Induction on n ■

Theorem 3 *If $n > 0$, then $Y^n = Z_n$*

Proof. Using Lemma 2 and direct computation, we have $Z_n = YZ_{n-1}$ from which it follows that $Z_n = Y^{n-1}Z_1$. By direct computations, $Z_1 = Y$ from which the conclusion follows. ■

By the defination of matrix Z_n, we write $Z_{n+m} = Z_n Z_m = Z_m Z_n$ for all $n, m > 0$. From a matrix multplication we have the following Corollary without proof.

Corollary 4 *For $n, m > 0$,*

$$S_{n+m} = S_n + S_m L_{n+1} + S_{m-1} \sum_{i=0}^{k-2} L_{n-i} + \dots + S_{m-k+2} \sum_{i=0}^{1} L_{n-i} + S_{m-k+1} L_n$$

The roots of characteristic equation of the recursive sequence order-k, $x^k - \sum_{i=0}^{k-1} x^k = 0$, are $\lambda_1, \lambda_2, \lambda_3, \dots, \lambda_k$.

Spikerman and Joyner mentioned binet's formula and roots of the recursive sequence of order k. Acoordingly, we know that the roots of characteristic equation of the recursive sequence order-k is different.

Define the diagonal matrix K and W as shown;

$$K = \begin{bmatrix} 1 & 0 & 0 & \dots & 0 \\ 0 & \lambda_1 & 0 & \dots & 0 \\ 0 & 0 & \lambda_2 & \dots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \dots & \lambda_k \end{bmatrix}_{(k+1)\times(k+1)}$$

$$W = \begin{bmatrix} 1 & 0 & 0 & \dots & 0 \\ -\frac{1}{k-1} & \lambda_1^{k-1} & \lambda_2^{k-1} & \dots & \lambda_k^{k-1} \\ \vdots & \vdots & \vdots & & \vdots \\ -\frac{1}{k-1} & \lambda_1 & \lambda_2 & \dots & \lambda_k \\ -\frac{1}{k-1} & 1 & 1 & \dots & 1 \end{bmatrix}_{(k+1)\times(k+1)}$$

One can check that $YW = WK$. Since the roots $\lambda_1, \lambda_2, \lambda_3, \dots, \lambda_k$ are distinct, it follows that $\det W \neq 0$.

Theorem 5 *If $n > 0$, then* $S_n = \frac{1}{k-1}(L_{n+1} + \sum_{i=0}^{k-2} L_{n-i} + \dots + \sum_{i=0}^{1} L_{n-i} + L_n - 1)$

Proof. Since $YW = WK$ and $\det W \neq 0$, we write $W^{-1}YW = K$. Thus the matrix Y is similar to the matrix K. Then $Y^n W = WK^n$. By Theorem 3, we write $Z_n W = WK^n$. Equating (2.1)th elements of the equation theorem is proven. ∎

Define N and P_n as shown;

$$N = \begin{bmatrix} 2 & 0 & \dots & 0 & -1 \\ 1 & 0 & \dots & 0 & 0 \\ 0 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \dots & 1 & 0 \end{bmatrix}_{(k+1)\times(k+1)}$$

$$P_n = \begin{bmatrix} S_{n+1} & -S_{n-k+1} & \dots & -S_{n-1} & -S_n \\ S_n & -S_{n-k} & \dots & -S_{n-2} & -S_{n-1} \\ S_{n-1} & -S_{n-k-1} & \dots & -S_{n-3} & -S_{n-2} \\ \vdots & \vdots & & \vdots & \vdots \\ S_{n-k+1} & -S_{n-2k+1} & \dots & -S_{n-k-1} & -S_{n-k} \end{bmatrix}_{(k+1)\times(k+1)}$$

Lemma 6 *The sequence $\{S_n\}$ satisfies the following recursion for $n > k$*

$$S_n = 2S_{n-1} - S_{n-k-1}$$

where $S_0 = 0,\ S_1 = 1,\ S_2 = 2,\ S_3 = 4, \dots, S_{k-1} = 2^{k-2}$.

Theorem 7 *If $n > k+1$, then $N^n = P_n$.*

Proof. From Lemma 6, we write $P_n = NP_{n-1}$. By a simple inductive argument, we write $P_n = N^n P_1$. By the definations of matrix N and P_n, one can see that $P_1 = N$ and thus proof is seen. ∎

Define the Vandermonde matrix W_1 and diagonal matrix K_1 as follows:

$$K_1 = \begin{bmatrix} \lambda_k & 0 & \dots & 0 & 0 \\ 0 & \lambda_{k-1} & \dots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \dots & \lambda_1 & 0 \\ 0 & 0 & \dots & 0 & 1 \end{bmatrix}_{(k+1)\times(k+1)}$$

$$W_1 = \begin{bmatrix} \lambda_1^k & \lambda_2^k & \dots & \lambda_k^k & 1 \\ \lambda_1^{k-1} & \lambda_2^{k-1} & \dots & \lambda_k^{k-1} & 1 \\ \vdots & \vdots & & \vdots & \vdots \\ \lambda_1 & \lambda_2 & \dots & \lambda_k & 1 \\ 1 & 1 & \dots & 1 & 1 \end{bmatrix}_{(k+1)\times(k+1)}$$

Let m_i be a $k \times 1$ matrix such that

$$m_i = \begin{bmatrix} \lambda_1^{n-i+k+1} & \lambda_2^{n-i+k+1} & \lambda_3^{n-i+k+1} & \dots & \lambda_k^{n-i+k+1} & 1 \end{bmatrix}^T$$

and $W_j^{(i)}$ be a $(k+1) \times (k+1)$ matrix obtained from W_1 by replacing the jth column of W_1^T by m_i.

Theorem 8 *For $n > k+1$, $b_{ij} = \frac{\det W_j^{(i)}}{\det W_1}$ where $B_n = [b_{ij}]$.*

Proof. One can see that $NW_1 = W_1K_1$. Since $\lambda_1, \lambda_2, \lambda_3, \dots, \lambda_k$ are different and W_1 is a Vandermonde matrix, W_1 is invertible. Thus we write $W_1^{-1}NW_1 = K_1$ and so $N^nW_1 = W_1K_1^n$. By the Theorem 7, $P_nW_1 = W_1K_1^n$. Thus we have the following equations system:

$$\lambda_1^k b_{i1} + \lambda_1^{k-1} b_{i2} + \dots + \lambda_1 b_{i(k-1)} + b_{ik} = \lambda_1^{n-i+k+1}$$

$$\lambda_2^k b_{i1} + \lambda_2^{k-1} b_{i2} + \dots + \lambda_2 b_{i(k-1)} + b_{ik} = \lambda_2^{n-i+k+1}$$

$$\vdots$$

$$\lambda_k^k b_{i1} + \lambda_k^{k-1} b_{i2} + \dots + \lambda_k b_{i(k-1)} + b_{ik} = \lambda_k^{n-i+k+1}$$

$$b_{i1} + b_{i2} + \dots + b_{i(k-1)} + b_{ik} = 1$$

where $B_n = [b_{ij}]$. By Cramer solution of the above system. the proof is seen.

■

3 More Recurrence Identities On The Recursive Sequence Order-k

Suppose we want an identitiy of the form (w, h ,n are positive integers and $s \geq 0$)

$$L_{w(n+h)+s} = \sum_{i=1}^{k} x_i L_{w(n+1-i)} \tag{2}$$

we write an augmented matrix A_n^*;

$$A_n^* = \begin{bmatrix} L_{wn} & L_{w(n-1)} & \cdots & L_{w(n+1-k)} & L_{w(n+h)+s} \\ L_{w(n+1)} & L_{wn} & \cdots & L_{w(n+2-k)} & L_{w(n+h+1)+s} \\ L_{w(n+2)} & L_{w(n+1)} & \cdots & L_{w(n+3-k)} & L_{w(n+h+2)+s} \\ \vdots & \vdots & & \vdots & \vdots \\ L_{w(n+k-1)} & L_{w(n+k-2)} & \cdots & L_{wn} & L_{w(n+h+k-1)+s} \end{bmatrix}$$

and we calculate coefficients x_i $(1 \leq i \leq k)$.

For example; in (2), we take $k = 3$, that is for Tribonacci numbers. If $w = 3$, $h = 2$, $s = 1$, (2) becomes

$$L_{3(n+2)+1} = 81L_{3n} - 63L_{3(n-1)} + 13L_{3(n-2)}$$

and in (2), we take $k = 5$, that is for Pentanacci numbers. If $w = 6$, $h = 1$, $s = 0$, (2) becomes

$$L_{6(n+1)} = 57L_{6n} + 42L_{6(n-1)} + 22L_{6(n-2)} + 7L_{6(n-3)} + L_{6(n-4)}$$

4 Applications

In this section, we mention from applications of second and third section. In second section, for $k = 3$, it done by E. Kılıç. In third section, for $k = 3$, it done by J.Feng except for determinantal representations. In this section, we touch on determinantal representations for $k = 3$.

Generalized Tribonacci sequence is defined by

$$T_{w(n+1)+s} = x_1 T_{wn+s} + x_2 T_{w(n-1)+s} + x_3 T_{w(n-2)+s} \tag{3}$$

where w,n are positive integers and $s \geq 0$.

Theorem 9 *Constructed the $n \times n$ matrix:*

$$M_T^{(1)}(n) = \begin{bmatrix} T_{tw+s} & m & T_{(t+2)w+s} & & & & \\ 1 & T_{tw+s} & T_{(t+1)w+s} & x_3 & & & \\ & 1 & T_{tw+s} & -x_2 & & & \\ & & 1 & x_1 & \ddots & & \\ & & & 1 & \ddots & \ddots & x_3 \\ & & & & \ddots & \ddots & -x_2 \\ & & & & & 1 & x_1 \end{bmatrix} \quad (4)$$

where $m = T_{tw+s}^2 - T_{(t+1)w+s}$. Accordingly we have $\left|M_T^{(1)}(n)\right| = T_{w(n+t-1)+s}$.

Proof. In defination of $M_T^{(1)}$ we take $n=1$. Then $M_T^{(1)} = [T_{tw+s}]$ thus $\left|M_T^{(1)}\right| = T_{tw+s}$. In defination of $M_T^{(1)}$ we take $n=2$. Then $M_T^{(1)} = \begin{bmatrix} T_{tw+s} & m \\ 1 & T_{tw+s} \end{bmatrix}$ thus $\left|M_T^{(1)}\right| = T_{(t+1)w+s}$. Suppose that for $n=i$, $\left|M_T^{(1)}(i)\right| = T_{(t+i-1)w+s}$ be correct. We show this equality is correct for $n = i+1$.

$$M_T^{(1)}(i+1) = \begin{bmatrix} T_{tw+s} & m & T_{(t+2)w+s} & & & & \\ 1 & T_{tw+s} & T_{(t+1)w+s} & x_3 & & & \\ & 1 & T_{tw+s} & -x_2 & & & \\ & & 1 & x_1 & \ddots & & \\ & & & 1 & \ddots & \ddots & x_3 \\ & & & & \ddots & \ddots & -x_2 \\ & & & & & 1 & x_1 \end{bmatrix}$$

By using expansion of a determinant, $\left|M_T^{(1)}(i+1)\right|$ is calculated. ∎

Theorem 10 *Constructed the $n \times n$ matrix:*

$$M_T^{(2)}(n) = \begin{bmatrix} T_{tw+s} & -m & T_{(t+2)w+s} & & & & \\ 1 & T_{tw+s} & T_{(t+1)w+s} & x_3 & & & \\ & 1 & -T_{tw+s} & x_2 & & & \\ & & 1 & x_1 & \ddots & & \\ & & & 1 & \ddots & \ddots & x_3 \\ & & & & \ddots & \ddots & x_2 \\ & & & & & 1 & x_1 \end{bmatrix} \quad (5)$$

where $m = T_{tw+s}^2 - T_{(t+1)w+s}$. Accordingly we have $per(M_T^{(2)}(n)) = T_{w(n+t-1)+s}$.

Proof. Proof can done according to proof of Theorem 11 ∎

Example 11 *If $w = 3$, $h = 1$, $s = 0$, $k = 3$, (2) becomes*

$$T_{3(n+1)} = x_1 T_{3n} + x_2 T_{3(n-1)} + x_3 T_{3(n-2)}$$

the augmented matrix A_1^* can be transformed to

$$A_1^* = \begin{bmatrix} T_3 & T_0 & T_{-3} & T_6 \\ T_6 & T_3 & T_0 & T_9 \\ T_9 & T_6 & T_3 & T_{12} \end{bmatrix} = \begin{bmatrix} 2 & 0 & -1 & 13 \\ 13 & 2 & 0 & 81 \\ 81 & 13 & 2 & 504 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 7 \\ 0 & 1 & 0 & -5 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

Thus we have identity

$$T_{3(n+1)} = 7T_{3n} - 5T_{3(n-1)} + T_{3(n-2)}$$

If we set $t = 1$ in (4) and (5) for $n = 3$, the determinant of

$$\begin{bmatrix} 2 & -9 & 81 \\ 1 & 2 & 13 \\ 0 & 1 & 2 \end{bmatrix}$$

are the Tribonacci number T_9 and permanent of

$$\begin{bmatrix} 2 & 9 & 81 \\ 1 & 2 & 13 \\ 0 & 1 & -2 \end{bmatrix}$$

are the Tribonacci number T_9.
If we set $t = 0$ in (4) and (5) for $n = 4$, the determinant of

$$\begin{bmatrix} 0 & -2 & 13 & 0 \\ 1 & 0 & 2 & 1 \\ 0 & 1 & 0 & 5 \\ 0 & 0 & 1 & 7 \end{bmatrix}$$

are the Tribonacci number T_9 and permanent of

$$\begin{bmatrix} 0 & 2 & 13 & 0 \\ 1 & 0 & 2 & 1 \\ 0 & 1 & 0 & -5 \\ 0 & 0 & 1 & 7 \end{bmatrix}$$

are the Tribonacci number T_9.

References

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