ON THE SYMMETRIC PROPERTIES FOR THE GENERALIZED TWISTED GENOCCHI POLYNOMIALS

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ABSTRACT. In this paper, we study the symmetry for the generalized twisted Genocchi polynomials and numbers. We give some interesting identities of the power sums and the generalized twisted Genocchi polynomials using the symmetric properties for the p-adic invariant q- integral on $\mathbb{Z}_p$.

1. INTRODUCTION

Let p be a fixed odd prime number. Throughout this paper, the symbols $\mathbb{Z}, \mathbb{Z}_p, \mathbb{Q}_p, \mathbb{C}$, and $\mathbb{C}_p$ will denote the ring of rational integers, the ring of $p-$adic integers, the field of $p-$adic rational numbers, the complex number field, and the completion of the algebraic closure of $\mathbb{Q}_p$, respectively. Let $\mathbb{N}$ be the set of natural numbers and $\mathbb{Z}_+ = \mathbb{N}\bigcup\{0\}$. Let v_p be the normalized exponential valuation of $\mathbb{C}_p$ with $|p|_p = p^{-v_p(p)} = 1/p$.

Let $UD(\mathbb{Z}_p)$ be the space of uniformly differentiable function on $\mathbb{Z}_p$. For $f \in UD(\mathbb{Z}_p)$, the p-adic invariant q-integral on $\mathbb{Z}_p$ is defined by Kim [7] :

$$I_{-q}(f) = \int_{\mathbb{Z}_p} f(x)d\mu_{-q}(x) = \lim_{N\to\infty} \frac{1}{[p^N]_{-q}} \sum_{x=0}^{p^N-1} f(x)(-q)^x. \tag{1.1}$$

Note that

$$I_{-1}(f) = \lim_{q\to 1} I_{-q}(f).$$

From the definition of $q-$integral, we have

$$I_{-1}(f_1) + I_{-1}(f) = 2f(0), \quad \text{where } f_1(x) = f(x+1). \tag{1.2}$$

For $n \in \mathbb{N}$, let $f_n(x) = f(x+n)$.

Then we can derive the following equation from (1.2):

$$I_{-1}(f_n) = (-1)^n I_{-1}(f) + 2\sum_{l=0}^{n-1}(-1)^{n-l-1}f(l), \ (see\ [1-8]). \tag{1.3}$$

For a fixed odd positive integer d with $(p,d) = 1$, set

$$X = X_d = \varprojlim_{N} \mathbb{Z}/dp^N\mathbb{Z}, \quad X_1 = \mathbb{Z}_p, \tag{1.4}$$

$$X^* = \bigcup_{\substack{0<a<dp,\\ (a,p)=1}} (a + dp\mathbb{Z}_p),$$

$$a + dp^N\mathbb{Z}_p = \{x \in X \mid x \equiv a \pmod{dp^N}\},$$

where $a \in \mathbb{Z}$ satisfies the condition $0 \le a < dp^N$ (see [1 – 19]).

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It is easy to see that

$$\int_X f(x)d\mu_{-q}(x) = \int_{\mathbb{Z}_p} f(x)d\mu_{-q}(x), \text{ for } f \in UD(\mathbb{Z}_p). \tag{1.5}$$

The ordinary Genocchi polynomials $G_n(x)$ are defined as

$$\frac{t}{e^t - 1}e^{xt} = \sum_{n=0}^{\infty} G_n(x)\frac{t^n}{n!} \tag{1.6}$$

and $G_n = G_n(0)$ are called the Genocchi numbers (see [4, 12 − 13, 16]).

For $n \in \mathbb{N}$,let T_p be the p-adic locally constant space defined by

$$\mathrm{T}_p = \bigcup_{n\geq 1} \mathbb{C}_{p^n} = \lim_{n\to\infty} \mathbb{C}_{p^n} = \mathbb{C}_{p^\infty} \tag{1.7}$$

where $\mathbb{C}_{p^n} = \{\zeta \in \mathbb{C}_p | \zeta^{p^n} = 1 \; for\ some\ n \geq 0\}$ is the cyclic group of order p^n. It is well known that the twisted Genocchi polynomials are defined as

$$\frac{t}{\zeta e^t - 1}e^{xt} = \sum_{n=0}^{\infty} G_{n,\zeta}(x)\frac{t^n}{n!}, \zeta \in T_p \tag{1.8}$$

and the twisted Genocchi numbers $G_{n,\zeta}$ are defined as $G_{n,\zeta} = G_{n,\zeta}(0)$. Let χ be the Dirichlet character with conductor $d \in \mathbb{N}$ with $d \equiv 1(\ mod\ 2)$. Then the generalized twisted Genocchi polynomials $G_{n,\chi,\zeta}(x)$ attached to χ are defined as follows:

$$\frac{2t\sum_{l=0}^{d-1}(-1)^l\chi(l)e^{lt}}{\zeta^d e^{dt} + 1}e^{xt} = \sum_{n=0}^{\infty} G_{n,\chi,\zeta}(x)\frac{t^n}{n!}, \zeta \in T_p. \tag{1.9}$$

In the special case $x = 0$, $G_{n,\chi,\zeta} = G_{n,\chi,\zeta}(0)$ are called the n-th twisted generalized Genocchi numbers attached to χ (see [5 − 11]).

2. Symmetry for the generalized Genocchi polynomials

Let χ be the Dirichlet character with conductor $d \in \mathbb{N}$ with $d \equiv 1(mod2)$. For $\zeta \in T_p$, we have

$$\begin{aligned} t\int_X \zeta^x\chi(x)e^{xt}d\mu_{-1}(x) &= \frac{2t\sum_{l=0}^{d-1}(-1)^l\chi(l)e^{lt}}{\zeta^d e^{dt}+1} \\ &= t\sum_{n=0}^{\infty}\frac{G_{n+1,\chi,\zeta}}{n+1}\frac{t^n}{n!}, \end{aligned} \tag{2.1}$$

where $G_{n,\chi,\zeta}$ are the n-th generalized twisted Genocchi numbers attached to χ. We also see that the generalized twisted Genocchi polynomials attached to χ are given by

$$\begin{aligned} t\int_X \zeta^y\chi(y)e^{(x+y)t}d\mu_{-1}(y) &= \frac{2t\sum_{l=0}^{d-1}(-1)^l\chi(l)e^{lt}}{\zeta^d e^{dt}+1}e^{xt} \\ &= t\sum_{n=0}^{\infty}\frac{G_{n+1,\chi,\zeta}(x)}{n+1}\frac{t^n}{n!}, \end{aligned} \tag{2.2}$$

where $G_{n,\chi,\zeta}(x)$ are the n-th generalized twisted Genocchi polynomials attached to χ.

By (2.1) and (2.2), we can easily see that

$$\int_X \zeta^y \chi(y)(x+y)^n d\mu_{-1}(y) = \frac{G_{n+1,\chi,\zeta}(x)}{n+1} \text{ and } G_{0,\chi,\zeta}(x) = 0. \tag{2.3}$$

In particular,

$$\int_X \zeta^x \chi(x) x^n d\mu_{-1}(x) = \frac{G_{n+1,\chi\zeta}}{n+1} \text{ and } G_{0,\chi,\zeta} = 0.$$

By (1.3) and (1.5), we have that for $n \in \mathbb{N}$,

$$\int_X f(x+n) d\mu_{-1}(x) = (-1)^n \int_X f(x) d\mu_{-1}(x) + 2\sum_{l=0}^{n-1}(-1)^{n-l-1} f(l). \tag{2.4}$$

By taking $f(x) = \zeta^x \chi(x) e^{xt}$ in (2.4), it follows that

$$\begin{aligned} & t\int_X \zeta^{x+nd}\chi(x+nd)e^{(nd+x)t}d\mu_{-1}(x) + t\int_X \zeta^x \chi(x) e^{xt} d\mu_{-1}(x) \\ &= 2t\sum_{l=0}^{nd-1}(-1)^l \zeta^l \chi(l) e^{lt} \\ &= \sum_{k=0}^{\infty}\{2t\sum_{l=0}^{nd-1}(-1)^l\zeta^l\chi(l)l^k\}\frac{t^k}{k!}. \end{aligned} \tag{2.5}$$

For $k \in \mathbb{Z}_+$, let us define the p-adic functional $T_{k,\chi,\zeta}(n)$ as follows:

$$T_{k,\chi,\zeta}(n) = \sum_{l=0}^{n}(-1)^l\zeta^l\chi(l)l^k. \tag{2.6}$$

From (2.5) and (2.6), we observe that for $k, n, d \in \mathbb{N}$,

$$\int_X \zeta^x\chi(x)(nd+x)^k d\mu_{-1}(x) + \int_X \zeta^x \chi(x) x^k d\mu_{-1}(x) = 2T_{k,\chi,\zeta}(nd-1). \tag{2.7}$$

From (2.3) and (2.7), we obtain the following result.

Theorem 2.1 For $k \in \mathbb{Z}_+$ and $n, d \in \mathbb{N}$. We have

$$\frac{G_{k+1,\chi,\zeta}(nd)}{k+1} + \frac{G_{k+1,\chi,\zeta}}{k+1} = 2T_{k,\chi,\zeta}(nd-1), \zeta \in T_p. \tag{2.8}$$

Let $w_1, w_2 \in \mathbb{N}$ with $w_1 \equiv 1 \pmod 2$ and $w_2 \equiv 1 \pmod 2$. Then we set

(2.9)

$$R(\chi,\zeta,w_1,w_2) = \frac{\int_X\int_X \chi(x_1)\chi(x_2)\zeta^{w_1x_1+w_2x_2}e^{(w_1x_1+w_2x_2+w_1w_2x)t}d\mu_{-1}(x_1)d\mu_{-1}(x_2)}{\int_X \zeta^{dw_1w_2x}e^{dw_1w_2xt}d\mu_{-1}(x)}.$$

By the definition of p-adic invariant integral on $\mathbb{Z}_p$, we can derive that

$$\begin{aligned} R(\chi,\zeta,w_1,w_2) &= \sum_{i=0}^{\infty}\frac{G_{i+1,\zeta^{w_1},\chi}(w_2x)}{i+1}\frac{w_1^it^i}{i!}\sum_{k=0}^{\infty}T_{k,\zeta^{w_2},\chi}(dw_1-1)\frac{w_2^kt^k}{k!} \\ &= \sum_{l=0}^{\infty}\{\sum_{i=0}^{l}\binom{l}{i}\frac{G_{i+1,\zeta^{w_1},\chi}(w_2x)}{i+1}T_{l-i,\zeta^{w_2}\chi}(dw_1-1)w_1^iw_2^{l-i}\}\frac{t^l}{l!} \end{aligned} \tag{2.10}$$

From the symmetry of $R(\chi,\zeta,w_1,w_2)$ in w_1 and w_2, we also see that

(2.11)

$$R(\chi,\zeta,w_1,w_2)=\sum_{l=0}^{\infty}\{\sum_{i=0}^{l}\binom{l}{i}\frac{G_{i+1,\zeta^{w_2},\chi}(w_1x)}{i+1}T_{l-i,\zeta^{w_1},\chi}(dw_2-1)w_2^iw_1^{l-i}\}\frac{t^l}{l!}.$$

By the comparing the coefficients on the both sides of (2.10) and (2.11), we have the following result.

Theorem 2.2 Let $\zeta\in T_p$and $d,w_1,w_2\in\mathbb{N}$. Then we have

$$\sum_{i=0}^{l}\binom{l}{i}\frac{G_{i+1,\chi,\zeta^{w_1}}(w_2x)}{i+1}T_{l-i,\chi,\zeta^{w_2}}(dw_1-1)w_1^iw_2^{l-i} \tag{2.12}$$

$$=\sum_{i=0}^{l}\binom{l}{i}\frac{G_{i+1,\chi,\zeta^{w_2}}(w_1x)}{i+1}T_{l-i,\chi,\zeta^{w_1}}(dw_2-1)w_2^iw_1^{l-i},$$

where $w_1,w_2\in\mathbb{N}$ with $w_1\equiv 1(\ mod\ 2)$, $w_2\equiv 1(\ mod\ 2)$.

We also obtain some identities for the generalized twisted Genocchi polynomials. Taking $x=0$ in Theorem 2.2, we have the following corollary.

Corollary 2.3 Let $\zeta\in T_p$ and $d,w_1,w_2\in\mathbb{N}$. Then we have

$$\sum_{i=0}^{l}\binom{l}{i}\frac{G_{i+1,\chi,\zeta^{w_1}}}{i+1}T_{l-i,\chi,\zeta^{w_2}}(dw_1-1)w_1^iw_2^{l-i} \tag{2.13}$$

$$=\sum_{i=0}^{l}\binom{l}{i}\frac{G_{i+1,\chi,\zeta^{w_2}}}{i+1}T_{l-i,\chi,\zeta^{w_1}}(dw_2-1)w_2^iw_1^{l-i},$$

where $w_1,w_2\in\mathbb{N}$ with $w_1\equiv 1(\ mod\ 2)$, $w_2\equiv 1(\ mod\ 2)$.

Now we will derive another interesting identities for the generalized twisted Genocchi polynomials using the symmetric property of $R(\chi,\zeta,w_1,w_2)$.

(2.14)

$$\begin{aligned}
&R(\chi,\zeta,w_1,w_2)\\
&=\frac{1}{2}e^{w_1w_2xt}\{\int_X\chi(x_1)\zeta^{w_1x_1}e^{w_1x_1t}d\mu_{-1}(x_1)\}\times\{\frac{2\int_X\chi(x_2)\zeta^{w_2x_2}e^{w_2x_2t}d\mu_{-1}(x_2)}{\int_X\zeta^{dw_2x_2}e^{dw_1w_2x_1t}d\mu_{-1}(x_1)}\}\\
&=\sum_{l=0}^{dw_1-1}(-1)^l\zeta^l\chi(l)\int_X\chi(x_1)\zeta^{w_1x_1}e^{(x_1+w_2x+\frac{w_2}{w_1}l)w_1t}d\mu_{-1}(x_1)\\
&=\sum_{n=0}^{\infty}\{\sum_{l=0}^{dw_1-1}(-1)^l\zeta^l\chi(l)\frac{G_{n+1,\chi\zeta^{w_1}}(w_2x+\frac{w_2}{w_1}l)}{n+1}w_1^n\}\frac{t^n}{n!}.
\end{aligned}$$

From the symmetric property of $R(\chi,\zeta,w_1,w_2)$, we also see that

$$R(\chi,\zeta,w_1,w_2)=\sum_{n=0}^{\infty}\{\sum_{l=0}^{dw_2-1}(-1)^l\zeta^l\chi(l)\frac{G_{n+1,\chi\zeta^{w_2}}(w_1x+\frac{w_1}{w_2}l)}{n+1}w_2^n\}\frac{t^n}{n!}. \tag{2.15}$$

Comparing the coefficients on the both sides of (2.14) and (2.15), we obtain the following theorem.

Theorem 2.4 Let $\zeta \in T_p$ and $d, w_1, w_2 \in \mathbb{N}$ .Then we have

$$w_1^n \sum_{l=0}^{dw_1-1} (-1)^l \zeta^l \chi(l) \frac{G_{n+1,\chi\zeta^{w_1}}(w_2 x + \frac{w_2}{w_1} l)}{n+1} = w_2^n \sum_{l=0}^{dw_2-1} (-1)^l \zeta^l \chi(l) \frac{G_{n+1,\chi\zeta^{w_2}}(w_1 x + \frac{w_1}{w_2} l)}{n+1}. \tag{2.16}$$

If we take $x = 0$ in Theorem 2.4 , we also derive the interesting identity for the generalized twisted Genocchi numbers as follows:

$$w_1^n \sum_{l=0}^{dw_1-1} (-1)^l \zeta^l \chi(l) \frac{G_{n+1,\chi\zeta^{w_1}}(\frac{w_2}{w_1} l)}{n+1} = w_2^n \sum_{l=0}^{dw_2-1} (-1)^l \zeta^l \chi(l) \frac{G_{n+1,\chi\zeta^{w_2}}(\frac{w_1}{w_2} l)}{n+1}. \tag{2.17}$$

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