Permutations with interval cycles

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Abstract

We study permutations of the set $[n] = \{1, 2, \ldots, n\}$ written in cycle notation, for which each cycle forms an increasing or decreasing interval of positive integers. More generally, permutations whose cycle elements form arithmetic progressions are considered. We also investigate the class of generalised interval permutations, where each cycle can be rearranged in increasing order to form an interval of consecutive positive integers..

1 Interval Permutations

The partitions of the set $[n] = \{1, 2, \ldots, n\}$ into k nonempty subsets of consecutive integers are enumerated by $\binom{n-1}{k-1}$ since this is the number of ways of inserting $k-1$ separators between the sequence of numbers $1, 2, \ldots, n$, $1 \leq k \leq n$. We will obtain analogous results for permutations of $[n]$, written in the cycle notation. Taking different orderings of the elements of the cycles into account gives several different analogues of the set partition case.

Firstly, a permutation p of $[n]$ will be called an *interval permutation* if every cycle of p consists of one increasing or decreasing sequence of consecutive integers.

Even though the standard notation places the least member of a cycle in the first position, we adopt the convention to reckon only with the permutations in which the members of a cycle have been shifted so as to exhibit the maximal number of pairs of consecutive integers. An interval permutation is the unique member of its cycle class in which every v-cycle consists of v increasing or decreasing consecutive integers. Later in Section 5 we will also study the class of generalised interval permutations, where each cycle can be rearranged in increasing order to form an interval of consecutive integers.

Denote the set of interval permutations of $[n]$ with k cycles (also k-permutations below) by $R(n,k)$, and let $r(n,k) = |R(n,k)|$. Also let $r(n) = r(n,1) + r(n,2) + \cdots + r(n,n)$.

Example 1 : *We have* $r(4,2) = 5$, *the enumerated permutations being*

$$(12)(34),\ (1)(234),\ (1)(432),\ (321)(4),\ (123)(4).$$

Observe that $(1)(432) = (1)(243)$ *in standard cycle notation.*

Theorem 2 *The numbers* $r(n,k)$ *satisfy the recurrence*

$$r(n,k) = r(n-1,k-1) + r(n-1,k) + r(n-3,k-1) \quad (1)$$

$$r(0,0) = 1,\ r(1,1) = r(2,1) = 1;\ r(n,0) = r(0,k) = 0,\ n,k > 0;$$
$$r(n,1) = 2,\ n \geq 3.$$

Proof. There are two ways to find an element $p \in R(n,k)$:
(i) insert the singleton (n) into any $q \in R(n-1,k-1)$, giving $r(n-1,k-1)$ permutations.
(ii) put the integer n into a $q \in R(n-1,k)$ as follows:
(a) if $(n-2,n-1) \in q$, then put n either before or after $n-1$ to make p in 2 possible ways;
(b) if $(n-2,n-1) \notin q$, then put n immediately before or after $n-1$ (depending on whether $n-1$ lies in a v−cycle, $v > 2$, with decreasing or increasing sequence), or into the cycle $(n-1)$, to make p in exactly one way. In (a) the number of permutations is $2r(n-3,k-1)$ since there are $r(n-3,k-1)$ ways to form the required part of $R(n-1,k)$ by inserting the cycle $(n-2,n-1)$ into each permutation in $R(n-3,k-1)$. Thus in (b) there are $r(n-1,k) - r(n-3,k-1)$ possibilities. Therefore the total number of permutations from (ii) is
$2.r(n-3,k-1) + r(n-1,k) - r(n-3,k-1) = r(n-1,k) + r(n-3,k-1)$.
Hence altogether the main result follows. The starting values follow from the definition, except possibly the last. Note that for $n > 2$ the only members of $R(n,1)$ are $(1,2,\ldots,n)$ and its reversal $(n,n-1,\ldots,1)$. Hence $r(n,1) = 2$.

Let $G_k(x) = \sum_{n\geq 0} r(n,k)x^n$. Then (1) translates to $G_k(x) = xG_{k-1}(x) + xG_k(x) + x^3G_{k-1}(x)$, that is,

$$G_k(x) = \frac{x+x^3}{1-x}G_{k-1}(x),\ G_0(x) = 1,$$

which may be iterated to give

$$\sum_{n\geq 0} r(n,k)x^n = \left(\frac{x+x^3}{1-x}\right)^k. \tag{2}$$

By summing (2) over k we also obtain

$$\sum_{n\geq 0} r(n)x^n = \frac{x-1}{x^3+2x-1}. \tag{3}$$

The explicit forms below follow from (2) by a routine procedure.

Corollary 1

$$r(n,k) = \sum_{j=0}^{\lfloor (n-k)/2 \rfloor} \binom{k}{j}\binom{n-2j-1}{k-1},$$

where $\lfloor N \rfloor$ *is the greatest integer* $\leq N$. *Hence*

$$r(n) = \sum_k \sum_{j=0}^{\lfloor (n-k)/2 \rfloor} \binom{k}{j}\binom{n-2j-1}{k-1},$$

Asymptotic estimates for $r(n)$ as well as a different symbolic approach to obtaining the generating functions (2) and (3) will be given in Section 3.

2 Interval Permutations without Unit Cycles

Denote the set of interval k−permutations of $[n]$ without unit cycles by $U(n,k)$, and let $u(n,k) = |U(n,k)|$. Also let $u(n) = u(n,1)+u(n,2)+\cdots+u(n,\lfloor n/2 \rfloor)$.

The following recurrence is obtained by modifying the derivation of (1). Elements of $U(n,k)$ in which n belongs to a 2-cycle are enumerated by $u(n-2,k-1)$, and those in which n belongs to a t−cycle, $t > 2$, are enumerated by $u(n-1,k) + u(n-3,k-1)$.

Theorem 3 *The numbers* $u(n,k)$ *satisfy the recurrence*

$$u(n,k) = u(n-1,k) + u(n-2,k-1) + u(n-3,k-1), \quad n \geq 3,\ k \geq 1, \tag{4}$$

$$u(0,0) = 1,\ u(1,1) = 0;\ u(2,1) = 1,\ u(n,1) = 2 \text{ if } n \geq 3.$$

Corollary 2 *The generating functions for $u(n,k)$ and $u(n)$ are*

$$\sum_{n\geq 0} u(n,k)x^n = \left(\frac{x^2+x^3}{1-x}\right)^k,$$

$$\sum_{n\geq 0} u(n)x^n = \frac{1-x}{1-x-x^2-x^3}.$$

The explicit formulas below follow from routine coefficient extraction, and the generating function for the $m-$generalized Fibonacci numbers [6] namely

$$\sum_{n\geq 0} F_n^{(m)}x^n = \frac{x}{1-x-x^2-\cdots-x^m}.$$

Corollary 3

$$u(n,k) = \sum_{j=0}^{n-2k} \binom{k}{j}\binom{n-j-k-1}{k-1};$$

$$u(n) = \sum_{k\geq 0}\sum_{j=0}^{n-2k} \binom{k}{j}\binom{n-j-k-1}{k-1} = F_{n+1}^{(3)} - F_n^{(3)}$$

where $F_n^{(3)}$ is the nth Tribonacci number.

Remark 4 *The standard sequence of Tribonacci numbers begins with* $0,1,1,2,4,7,13,24,44,\ldots$, *that is,*

$$F_0^{(3)} = 0,\ F_1^{(3)} = F_2^{(3)} = 1,\ F_n^{(3)} = F_{n-1}^{(3)} + F_{n-2}^{(3)} + F_{n-3}^{(3)}.$$

On the other hand the sequence $u(n)$, $n \geq 1$, begins as $0,1,2,3,6,11,20,37,68,\ldots$, *which is just $T(n+1)$, where $T(n)$, $n \geq 0$, is given by* $0,1,0,1,2,3,6,11,20,37,68,\ldots$.
Notice that $T(n)$ satisfies the same recurrence as the numbers $F_n^{(3)}$ but with the initial values
$T(0) = 0$, $T(1) = 1$, $T(2) = 0$ *(see [2] and [5]).*

3 Bijections and Asymptotic Estimates

The sequence of the number of interval permutations $r(n)$, $n \geq 0$: $1,1,2,5,11,24,53,\ldots$, is given in [8, A052980] as a non-combinatorial representation of the generating function in (3), while in the Encyclpedia of Combinatorial Structures ([7, ECS 1053]) it appears under the title, 'a simple regular expression'. In terms of regular expression syntax, these simple regular expressions are of the type $((a|bbb)Z^*)^*$, where for any set of letters x, the notation x^* denotes a sequence of zero or more phrases x.

We show a bijection between interval permutations, these simple regular expressions, and also, with a certain class of bi-coloured compositions. We will call a composition of n *bi-coloured* if every part $j \geq 1$ can occur in a standard colour, but all parts $j \geq 3$ are also available in a second colour. We will use j to denote a part of the standard colour and j' to denote a part of the second colour.

For the bijection, begin with a interval permutation with cycles arranged from left to right in terms of the size of the smallest element of the cycle. We form a bi-coloured composition of n by summing the lengths of the cycles of the interval permutation, where an increasing cycle of length j correspond to a part j and a decreasing cycle of length j corresponds to a part j'. Then for the regular expressions, each part j is mapped to $aZZ\ldots Z$ where there are $j-1$ letters Z and each part $j' \geq 3$ is mapped to $bbbZZ\ldots Z$, with $j'-3$ letters Z.

For example, if we start with the interval permutation $\pi = (4321)(567)(8)(13,12,11,10,9)$ whose cycles have been ordered in terms of the least elements of each cycle, then our bijection produces the bi-coloured composition $4'+3+1+5'$. Then applying our translation rules, we obtain the regular expression $(bbbZ)(aZZ)(a)(bbbZZ)$, or removing the superflous brackets, we can write this as $bbbZaZZabbbZZ$.

In the other direction, let us start with the regular expression $l = aaZZbbbbbbZZaZ$. We introduce brackets to help determine the sizes of the components.

So $l = (a)(aZZ)(bbb)(bbbZZ)(aZ)$. This maps to the bi-coloured composition $1+3+3'+5'+2$. Assigning, in order, the numbers 1 up to n into cycles with lengths corresponding to the composition parts, gives the interval permutation $(1)(234)(765)(12,11,10,9,8)(13,14)$.

The bijection with bi-coloured compositions allows us to give a simple derivation of the generating functions for interval permutations via the symbolic method. Firstly the generating function for parts of a bi-coloured composition, is

$$x + x^2 + 2\sum_{j\geq 3} x^j = \frac{x+x^3}{1-x},$$

A bi-coloured composition with exactly k parts is a sequence of k such parts with generating function

$$\sum_{n\geq 0} r(n,k)x^n = \left(\frac{x+x^3}{1-x}\right)^k,$$

and any bi-coloured composition is just a sequence of parts, with generating function

$$\sum_{n\geq 0} r(n)x^n = \frac{1}{1-\left(\frac{x+x^3}{1-x}\right)} = \frac{1-x}{1-2x-x^3}.$$

3.1 Asymptotic estimates

The dominant singularity of this rational generating function is a simple pole at the smallest positive root ρ of $1-2x-x^3=0$. We find $\rho \approx 0.4533976515$ and via singularity analysis, [3],

$$r(n) \sim A\rho^{-n}, \tag{5}$$

where $A = \frac{1-\rho}{\rho(3\rho^2+2)} \approx 0.4607198419$ and $\rho^{-1} \approx 2.2055694304$.

For example $r(25) = 178424817$ whereas the asymptotic estimate gives 178424816.99998. A consideration of the two complex zeroes of $1-2x-x^3 = 0$ shows that in general the error in (5) is exponentially small, namely $O(0.68^n)$.

Interval permutations without unit cycles correspond to bi-coloured compositions with no parts of size 1. The parts generating function is now

$$x^2 + 2\sum_{j\geq 3} x^j = \frac{x^2+x^3}{1-x}$$

and hence we recover Corollary 2.2, via the symbolic method. Asymptotically

$$u(n) \sim B\tau^{-n}, \tag{6}$$

where $B \approx 0.2821918053$ and $\tau^{-1} \approx 1.8392867552$, where τ is the positive zero of $1-x-x^2-x^3=0$.

3.2 The number of cycles in interval permutations

To count cycles in interval permutations we use an auxilliary variable u. The bivariate generating function is

$$R1(x,u) = \frac{1}{1-u\left(x-x^2-\frac{2x^3}{1-x}\right)}.$$

The generating function for the cumulated number of cycles over all interval permutations of $[n]$ is then

$$\left.\frac{\partial R1(x,u)}{\partial u}\right|_{u=1} = \frac{(1-x)\left(x^3+x\right)}{\left(1-2x-x^3\right)^2}.$$

Via singularity analysis, the coefficient $a(n)$ of x^n in this satisfies

$$a(n) \sim \frac{(1-\rho)\left(\rho^2+1\right)n}{\rho\left(3\rho^2+2\right)^2}\rho^{-n}.$$

Dividing by $r(n)$ shows that the mean number of cycles in a random permutation of n elements is asymptotically Bn where $B = \frac{\rho^2+1}{3\rho^2+2} \approx 0.4607198419$.

Similarly, the bivariate generating function for the number of unit cycles in interval permutations is

$$R2(x,u) = \frac{1}{1-ux+x^2+\frac{2x^3}{1-x}}.$$

The generating function for the cumulated number of unit cycles over all interval permutations of $[n]$ is then

$$\left.\frac{\partial R2(x,u)}{\partial u}\right|_{u=1} = \frac{(1-x)\left(x-x^2\right)}{\left(1-2x-x^3\right)^2}.$$

Extracting the coefficient of x^n via singularity analysis and dividing by $r(n)$ gives the mean number of unit cycles in a random permutation of n elements is asymptotically Dn where $D = \frac{1-\rho}{3\rho^2+2} \approx 0.20888929435$.

As a consequence of the asymptotic results for cycles and unit cycles above, we see that as $n \to \infty$ the proportion of cycles in a interval permutation that are unit cycles tends to the constant $\frac{1-\rho}{\rho^2+1} = \rho$, where $\rho \approx 0.4533976515$ as shown previously.

4 AP Permutations

In this section we extend interval permutations to *arithmetic progression* (AP) permutations after the set partitions scenario (see [1]). Let $CP(n,d)$ denote the set of permutations of $[n]$ in which the members of each cycle form an increasing (or decreasing) AP with common difference $\pm d$ (some fixed $d > 0$), unit cycles are ignored, but the identity permutation is assumed to have $d = n$.

Let $cp(n,d) = |CP(n,d)|$ and $cp(n) = |Cp(n)|$, where $CP(n) = \bigcup_d CP(n,d)$.

Proposition 1

$$cp(n,d) = r(q+1)^{\nu} r(q)^{d-\nu}, \tag{7}$$

where $n = qd+\nu$, $0 \le \nu < d$ and $r(N)$ is given by (3).

Proof. We apply the result of Section 1 namely, the permutations of $[m]$ into cycles of increasing/decreasing consecutive integers are enumerated by $r(m)$. Note that any such permutation, when viewed as an AP, has common difference(s) $d = \pm 1$.

To enumerate the elements of $CP(n,d)$ we first partition $[n]$ into a complete set of residue classes modulo d. Asumme that $R(\lambda_j)$ is the class with least element j and size λ_j, $1 \le j \le d$. Then there are $r(\lambda_j)$ permutations of $R(\lambda_j)$ into arithmetic cycles with common difference(s) $\pm d$. Consequently, each element of $CP(n,d)$ is obtained by combining the subpartitions across the $R(\lambda_j)$, one at a time from each $R(\lambda_j)$, in cartesian product fashion. Thus $cp(n,d) = r(\lambda_1)r(\lambda_2)\cdots r(\lambda_d)$. Note that when the $R(\lambda_j)$ are arranged in the increasing order of least elements the λ_j form a unique nonincreasing partition $\lambda_1 + \cdots + \lambda_d$ of n given by $\lambda_j = q+1$ if $j \le \nu$, and $\lambda_j = q$ otherwise, where $n = qd + \nu$, $1 \le \nu < d$. Hence the result.

If we sum (7) over d, $1 \le d \le n$, and delete the $n-1$ excess copies of the identity permutation, we obtain the next result.

Corollary 4

$$cp(n) = \sum_{\substack{d=1 \\ n=qd+\nu,\, 1\le\nu<d}}^{n} r(q+1)^{\nu} r(q)^{d-\nu} - n + 1. \tag{8}$$

Table 1 shows the numbers $cp(n,d)$ for $n = 1, 2, \ldots, 10$. Note that the last column is given by $cp(n) = \sum_{d=1}^{n} cp(n,d) - n + 1$, where $cp(n,n)$ counts the identity permutation and $cp(n,d) = 0$ when $d > n$.

Let $CP(n,d,\bar{1})$ denote the subset of $CP(n,d)$ without unit cycles, so that $1 \le d \le \lfloor n/2 \rfloor$. Since the interval permutations of v without unit cycles are enumerated by $F^{(3)}_{v+1} - F^{(3)}_{v}$, we obtain

Corollary 5 *The number $cp(n,d,\bar{1})$ of elements of $CP(n,d)$ without unit cycles is given by*

$$cp(n,d,\bar{1}) = \left(F^{(3)}_{q+2} - F^{(3)}_{q+1}\right)^{\nu} \left(F^{(3)}_{q+1} - F^{(3)}_{q}\right)^{d-\nu},$$

$n = qd + \nu$, $1 \le \nu < d$.

Hence we obtain

Corollary 6 *The number $cp(n,\bar{1})$ of AP permutations of $[n]$ without unit cycles, in which the cycles of each permutation have the same absolute common difference, is given by*

$$cp(n,\bar{1}) = \sum_{\substack{d=1 \\ n=qd+\nu,\, 1\le\nu<d}}^{\lfloor n/2 \rfloor} \left(F_{q+2}^{(3)} - F_{q+1}^{(3)}\right)^{\nu} \left(F_{q+1}^{(3)} - F_{q}^{(3)}\right)^{d-\nu}. \tag{9}$$

Table 1: The numbers $cp(n,d)$ of AP permutations, $d>0$, $n=1,\ldots,10$

$n\backslash d$	1	2	3	4	5	6	7	8	9	10	$cp(n)$
1	1										1
2	2	1									2
3	5	2	1								6
4	11	4	2	1							15
5	24	10	4	2	1						37
6	53	25	8	4	2	1					88
7	117	55	20	8	4	2	1				201
8	258	121	50	16	8	4	2	1			453
9	569	264	125	40	16	8	4	2	1		1021
10	1255	576	275	100	32	16	8	4	2	1	2260

Remark: We note some Observations from Table 1. If n is even, then

$$cp(n,n/2) = 2^{n/2},\ n>0,$$

but since $cp(n,n)=1$ for each n it is convenient to include the case $n=0$. If n is odd, we have

$$cp(n,(n+1)/2) = 2^{(n-1)/2}.$$

Hence

$$cp(n,\lfloor\frac{n+1}{2}\rfloor) = 2^{\lfloor n/2\rfloor}. \tag{10}$$

More generally, there are $\lfloor(n+2)/2\rfloor$ occurrences of powers of 2 on each row:

$$cp(n,\lfloor\frac{n+1}{2}\rfloor+j) = 2^{\lfloor n/2\rfloor-j},\quad j=0,1,\ldots,\lfloor\frac{n}{2}\rfloor. \tag{11}$$

To see this, if $d=\lfloor\frac{n+1}{2}\rfloor+j$, where $j=0,1,\ldots,\lfloor\frac{n}{2}\rfloor$ is fixed, then each of the numbers i for $i=1,2,\ldots\lfloor\frac{n}{2}\rfloor-j$ can either appear as a unit cycle or in a 2-cycle $(i,i+d)$. No other pairs of i values are possible as cycles, so $cp(n,d)=2^{\lfloor n/2\rfloor-j}$.

Note that (10) implies that $cp(n,n-i)$ becomes stationary from $i=N/2$ along a right-downward diagonal for each even number N, $0\le N\le n$. That is:

If $d = \frac{n}{2}$ *is an integer, then* $cp(n+j, n+j-d) = 2^d$, $j = 0, 1, \ldots$. (12)

Similarly, for column $d = \lfloor (n+1)/2 \rfloor - 1$ we find that for odd $n \geq 3$, $cp(n, \lfloor n/2 \rfloor) = 5 \times 2^{\lfloor n/2 \rfloor - 1}$ and for even $n \geq 6$, $cp(n, n/2-1) = 25 \times 2^{n/2-3}$. In the odd case, the factor 5 arises from the possible arrangements of $\{1, 1+d, 1+2d\}$ as one-, two- or three-cycles. In the even case, the factor 5^2 arises from the possible arrangements of $\{1, 1+d, 1+2d\}$ and of $\{2, 2+d, 2+2d\}$ as one-, two- or three-cycles.

The following general result includes all the cases of $cp(n, d)$ computed above, and in fact, represents a compact rule for obtaining all the entries in Table 1.

Theorem 5 *Let* $m > 0$, $d_0 \geq 0$, *be fixed integers, and let* $c_m = cp(m, 1) = r(m)$ $(c_0 = 1)$. *Then*

$$cp(md + d_0, d) = c_{m+1}^{d_0} c_m^{d-d_0}, \quad d = d_0, d_0+1, d_0+2, \ldots, \tag{13}$$

Alternatively,

$$cp(n, \frac{n-d_0}{m}) = c_{m+1}^{d_0} c_m^{(n-d_0(m+1))/m}, \quad n = mj + d_0,\ j \geq d_0.$$

Proof. This is essentially a restatement of (7) with the notation $c_m = cp(m, 1) = r(m)$.

Theorem 5 implies the following class of congruences.

Corollary 7

$$cp(md + d_0, d) \equiv 0 \ (mod\ \ r(m)r(m+1)),\ m \geq 2,\ 0 < d_0 \leq d.$$

In particular, for each integer $d > 0$, *we have*

$$cp(md, d) \equiv 0 \ (mod\ \ r(m)),\ \ m \geq 2.$$

We can also get elementary bounds for the magnitude of $cp(n, d)$ by using the earlier asymptotic results for $r(n)$. Since $r(q)^d < cp(n, d) < r(q+1)^d$, we have as $n \to \infty$,

$$A^d \rho^{-d\lfloor n/d \rfloor} < cp(n, d) < \left(\frac{A}{\rho}\right)^d \rho^{-d\lfloor n/d \rfloor}$$

which implies

$$(A\rho)^d \rho^{-n} < cp(n, d) < \left(\frac{A}{\rho}\right)^d \rho^{-n}.$$

5 Generalised Interval Permutations

Define a generalized interval permutation (G-interval permutation) as one in which the increasing rearrangement of each cycle forms an interval $[a, b] = \{a, a+1, \ldots, b\}$, $1 \leq a \leq b$.
Example (1243)(5)(6978) is a G-interval permutation since the respective sets of elements of cycles are $[1,4], [5,5], [6,9]$, but (1283)(45)(697) is not a G-interval permutation since the first cycle gives $(1238) \neq [1,8]$. We introduce the notation
$GP(n) = \{G\text{-interval permutations of } n\}$,
$GP(n,k) = \{G\text{-interval } k\text{-permutations of } n\}$.
$|GP(n)| = gp(n)$, $|GP(n,k)| = gp(n,k)$.

A recurrence formula can be obtained for $gp(n,k)$ by enlarging the scope of derivation of (1). Things simplify considerably.

Besides the $gp(n-1,k-1)$ permutations with the 1-cycle (n), an element of $GP(n,k)$ is obtained from a member of $GP(n-1,k)$ in which $n-1$ lies in a j-cycle, $j > 0$, by inserting n into j possible positions. Since there are $(j-1)!$ distinct j-cycles, it follows that the total number of G-interval k-permutations of $[n-1]$ in which $n-1$ belongs to a j-cycle is exactly $(j-1)!jgp(n-1-j,k-1) = j!gp(n-1-j,k-1)$. Hence $gp(n,k) = gp(n-1,k-1) + \sum_{j\geq 1} j!gp(n-j-1,k-1)$.

Theorem 6

$$gp(0,0) = 1, \quad gp(n,k) = \sum_{j=0}^{n-k} j!gp(n-j-1,k-1). \tag{14}$$

Corollary 8

$$gp(0) = 1, \quad gp(n) = \sum_{j=0}^{n-1} j!gp(n-j-1). \tag{15}$$

5.1 G-interval Permutations without Unit Cycles

Denote the set of G-interval k-permutations of $[n]$ without unit cycles by $GU(n,k)$, and let $gu(n,k) = |GU(n,k)|$. Also let $gu(n) = gu(n,1) + gu(n,2) + \cdots + gu(n, \lfloor n/2 \rfloor)$.

We can use a similar procedure to those of Section 2 to obtain: $gu(n,k) = gu(n-2,k-1) + \sum_{j\geq 2} j!gu(n-j-1,k-1)$.

Theorem 7 *For* $n \geq 2, k \geq 1$,

$$gu(n,1) = (n-1)!, \quad gu(n,k) = \sum_{j=1}^{n-2k+1} j!gu(n-j-1,k-1), \quad k \geq 2. \tag{16}$$

Corollary 9 *The sum* $iu(n) = \sum_{k\geq 1} iu(n,k)$ *is given by*

$$gu(n) = (n-1)! + \sum_{j=1}^{n-3} j!gu(n-j-1), \quad n \geq 2. \tag{17}$$

Proof. We have

$$\begin{aligned} gu(n) &= (n-1)! + \sum_{k\geq 2} \sum_{j=1}^{n-2k+1} j!gu(n-j-1,k-1) \\ &= (n-1)! + \sum_{j=1}^{n-3} j! \sum_{k\geq 2} gu(n-j-1,k-1) \end{aligned}$$

The sequence $gu(n)$, $n \geq 1$, begins as

$$0, 1, 2, 7, 28, 137, 798, 5443, 42688, 378733, 3749250, \ldots$$

(This sequence is not yet in Sloane [8]).

5.2 G-interval AP Permutations

Let $GCP(n,d)$ = denote the set of permutations of $[n]$ in which the increasing rearrangement of each cycle forms an AP with common difference d (some fixed $d > 0$), unit cycles are ignored, but the identity permutation is assumed to have $d = n$. Let $gcp(n,d) = |GCP(n,d)|$ and $gcp(n) = |GCp(n)|$, where $GCP(n) = \bigcup_d GCP(n,d)$.

The following results follow from straightforward adaptations of similar ones established in Section 4.

Proposition 2

$$gcp(n,d) = gp(q+1)^{\nu} gp(q)^{d-\nu}, \tag{18}$$

where $n = qd + \nu$, $0 \leq \nu < d$ *and* $gp(N)$ *is given by* (8).

Corollary 10

$$gcp(n) = \sum_{\substack{d=1 \\ n=qd+\nu,\, 1\leq \nu < d}}^{n} gp(q+1)^{\nu} gp(q)^{d-\nu} - n + 1. \tag{19}$$

Table 2: The numbers $gcp(n,d)$ of G-interval AP permutations, $d > 0$, $n = 1, \ldots, 10$

$n\backslash d$	1	2	3	4	5	6	7	8	9	10	$gcp(n)$
1	1										1
2	2	1									2
3	5	2	1								6
4	15	4	2	1							19
5	54	10	4	2	1						67
6	235	25	8	4	2	1					270
7	1237	75	20	8	4	2	1				1341
8	7790	225	50	16	8	4	2	1			8089
9	57581	810	125	40	16	8	4	2	1		58579
10	489231	2916	375	100	32	16	8	4	2	1	492676

Table 2 shows the numbers $gcp(n,d)$ for $n = 1, 2, \ldots, 10$; the last column is given by $gcp(n) = \sum_{d=1}^{n} gcp(n,d) - n + 1$, where $gcp(n,n)$ counts the identity permutation and $gcp(n,d) = 0$ when $d > n$.

Remark: The observations on Table 1 remain valid for Table 2.

Let $GCP(n,d,\bar{1})$ denote the subset of $GCP(n,d)$ without unit cycles, so that $1 \leq d \leq \lfloor n/2 \rfloor$. Since there is no special formula for $gu(n)$ the formula for $gcp(n,d,\bar{1}) = |GCP(n,d,\bar{1})|$ takes the form of (18).

6 A symbolic Approach to generalised interval permutations

A cycle of k consecutive numbers can be arranged in $(k-1)!$ different ways. Therefore a generalised interval permutation corresponds to a multi-coloured composition of n in which a part of size k can appear in $(k-1)!$ different colours. It follows that the generating function for G-interval permutations is

$$\sum_{n \geq 0} gp(n) x^n = \frac{1}{1 - \sum_{k \geq 1} (k-1)!\, x^k}.$$

Note that this a purely formal power series which diverges for all $x \neq 0$. Similarly the generating function for generalised interval permutations without unit cycles is

$$\sum_{n \geq 0} gu(n) x^n = \frac{1}{1 - \sum_{k \geq 2} (k-1)!\, x^k}.$$

In view of the divergent nature of these formal power series, we cannot apply singularity analysis to find asymptotic estimates. Instead, there is a Theorem of Bender which applies in such cases (see e.g. [4]):

Theorem 8 *Let* $A(z) = \sum_{n\geq 1} a_n z^n$ *and* $B(z) = \sum_{n\geq 0} b_n z^n$ *be two power series with* $B(z) = F(z, A(z))$ *where* $F(z,y)$ *is analytic near* $(0,0)$. *Assume that* $a_{n-1} = o(a_n)$ *and for some* $r > 0$

$$\sum_{r\leq k\leq n-r} |a_k a_{n-k}| = O(a_{n-r}).$$

Then

$$b_n = \sum_{0\leq k\leq r-1} d_k a_{n-k} + O(a_{n-r})$$

where

$$d_k = [z^k]\frac{\partial}{\partial y}F(z,y)|_{y=A(z)}.$$

For $gp(n)$ we apply this Theorem with $A(z) = \sum_{k\geq 1}(k-1)!z^k$ and $F(z,y) = \frac{1}{1-y}$. The conditions of the theorem are easily verified for any $r > 0$. Taking $r = 4$ we find

$$\frac{\partial}{\partial y}F(z,y)|_{y=A(z)} = \frac{1}{(1-A(z))^2} = 1 + 2z + 5z^2 + 14z^3 + O(z^4).$$

Hence

$$\begin{aligned} gp(n) &= (n-1)! + 2(n-2)! + 5(n-3)! + 14(n-4)! + O(n-5)! \\ &= (n-1)!\left(1 + \frac{2}{n} + \frac{7}{n^2} + \frac{31}{n^3} + O(n^{-4})\right). \end{aligned}$$

For $gu(n)$ we take $A(z) = \sum_{k\geq 2}(k-1)!z^k$ and find in the same way that

$$\begin{aligned} gu(n) &= (n-1)! + 2(n-3)! + 4(n-4)! + O(n-5)! \\ &= (n-1)!\left(1 + \frac{2}{n^2} + \frac{10}{n^3} + O(n^{-4})\right). \end{aligned}$$

The bivariate generating function for generalised interval permutations with u marking cycles is

$$G1(x,u) := \sum_{n,k\geq 0} gp(n,k)x^n u^k = \frac{1}{1 - u\sum_{k\geq 1}(k-1)!\,x^k}.$$

The generating function for the cumulated number of cycles over all G-interval permutations of $[n]$ is then

$$\left.\frac{\partial G1(x,u)}{\partial u}\right|_{u=1} = \frac{\sum_{k\geq 1}(k-1)!\,x^k}{\left(1-\sum_{k\geq 1}(k-1)!\,x^k\right)^2}.$$

Via Bender's theorem, together with the previous asymptotic estimate for $gp(n)$, we deduce that the average number of cycles in a G-interval permutation is

$$1+\frac{2}{n}+\frac{6}{n^2}+\frac{28}{n^3}+O(n^{-4}).$$

Similarly, the bivariate generating function with u marking unit cycles in G-interval permutations is

$$G2(x,u) := \frac{1}{1-ux-\sum_{k\geq 2}(k-1)!\,x^k}.$$

From this, the generating function for the cumulated number of unit cycles over all G-interval permutations of $[n]$ is

$$\left.\frac{\partial G2(x,u)}{\partial u}\right|_{u=1} = \frac{x}{\left(1-\sum_{k\geq 1}(k-1)!\,x^k\right)^2}.$$

A further application of Bender's theorem, leads to the result that the average number of unit cycles in a G-interval permutation of $[n]$ is

$$\frac{2}{n}+\frac{4}{n^2}+\frac{16}{n^3}+O(n^{-4}).$$

Remark 9 *Since there are $(n-1)!$ n-cycles, all of which belong to the set of G-interval permutations on $[n]$, we conclude from the more precise estimates above, that almost all G-interval permutations on $[n]$ consist of a single n-cycle. The same conclusion applies to G-interval permutations without unit cycles.*

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