Completing the Spectrum of Resolvable $(K_4 - e)$-Designs

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Abstract. In this note, a resolvable $(K_4 - e)$-design of order 296 is constructed. Combined the results of [2, 3, 4], the existence spectrum of resolvable $(K_4 - e)$-design of order v is the set $\{v : v \equiv 16 \pmod{20},\ v \geq 16\}$.

Key words. $K_4 - e$; resolvable; spectrum

1. Introduction

Let K_v be the complete undirected graph with v vertices. Let G be a finite simple undirected graph without isolated vertices. A G-design of order v is a partition of the edges of K_v into subgraphs G_i, each of which is isomorphic to G.

A G-design is *resolvable* if the subgraphs G can be partitioned into parallel classes, such that every vertex of K_v appears exactly once in each class.

Let $K_4 - e$ be the graph obtained from a K_4 on the vertex set $\{a, b, c, d\}$ by removing one edge. We shall use $\{a, b, c; d\}$ to denote the $K_4 - e$ on the vertex set $\{a, b, c, d\}$ missing the edge $\{c, d\}$.

A $(K_4 - e)$-design of order v is a pair $(X, \mathcal{B})$, where $\mathcal{B}$ is an edge-disjoint decomposition of the edge set of K_v with vertex set X, into copies of (*blocks*) $K_4 - e$. It is well known (see Bermond and Schönheim [1] for example) that a $(K_4 - e)$-design of order v exists for all $v \equiv 0, 1 \pmod 5$ and $v \geq 6$.

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A (K_4-e)-design is called *resolvable* if the block set $\mathcal{B}$ can be partitioned into parallel classes, each forming a partition of X. A simple counting argument yields that a resolvable (K_4-e)-design of order v exists only if $v \equiv 16 \pmod{20}$ and $v \geq 16$. A.P. Street posed the existence question concerning resolvable (K_4-e)-designs at the Auburn conference (1994). In [2], Colbourn, Stinson, and Zhu began to investigate the existence problem of resolvable (K_4-e)-designs, and solved it when $v \equiv 16 \pmod{60}$ with two possible exceptions.

Theorem 1. *[2] There exists a resolvable (K_4-e)-design of order v for $v \equiv 16 \pmod{60}$ and $v \geq 16$, except possibly for $v = 496$ or $v = 736$.*

In [3], Ge and Ling continued to study this problem, and almost determined the existence spectrum of resolvable (K_4-e)-designs, leaving two orders 116 and 296 undecided. In addition, Su and the auther [4] constructed a resolvable (K_4-e)-design of order 116. We restate the results as follows.

Theorem 2. *[3, 4] The necessary conditions for the existence of a resolvable (K_4-e)-design of order v, namely, $v \equiv 16 \pmod{20}$, $v \geq 16$, are also sufficient except possibly for $v = 296$.*

In this note, a resolvable (K_4-e)-design of order 296 is constructed. Hence, the existence spectrum of resolvable (K_4-e)-designs is finally determined.

2. Construction

In this section, we will give a direct construction of a resolvable (K_4-e)-design of order 296 using computer search.

Lemma 1. *There exists a resolvable (K_4-e)-design of order 296.*

Proof. Let the vertex set be Z_{296}. The desired design can be obtained by adding 1 (mod 296) to the following base blocks. Here, the last block is a half orbit.

$\{27,213,234;260\}$, $\{268,191,117;146\}$, $\{57,59,72;2\}$, $\{173,35,138;32\}$, $\{254,76,243;281\}$, $\{154,71,100;85\}$, $\{11,76,245;182\}$, $\{177,202,124;183\}$, $\{207,4,86;169\}$, $\{191,94,140;173\}$, $\{188,71,129;34\}$, $\{289,219,229;107\}$, $\{13,273,221;5\}$, $\{268,167,56;183\}$, $\{43,245,205;143\}$, $\{248,102,111;270\}$, $\{95,208,127;137\}$, $\{294,41,228;15\}$, $\{281,182,18;206\}$, $\{84,284,104;11\}$, $\{82,288,190;238\}$, $\{138,218,52;187\}$, $\{179,9,186;271\}$, $\{2,254,78;193\}$, $\{3,98,110;139\}$, $\{12,165,135;269\}$, $\{14,262,101;129\}$, $\{20,177,25;244\}$,

$\{44, 272, 216; 276\}$, $\{0, 148, 1; 149\}$.

Each one of the first 11 blocks can generate 4 parallel classes. Adding $0, 74, 148, 222 \pmod{296}$ to the next 18 blocks gives in total 72 blocks, these 72 blocks with the 2 blocks obtained by adding $0, 74 \pmod{296}$ to the last block can form an initial parallel class. Adding $0, 1, \ldots, 73 \pmod{296}$ to this initial parallel class gives 74 parallel classes. This completes the proof.

3. Conclusion

Now we are in a position to prove our main result.

Theorem 3. *There exists a resolvable $(K_4 - e)$-design of order v if and only if $v \equiv 16 \pmod{20}$, $v \geq 16$.*

Proof. By Theorems 1, 2, and Lemma 1, the conclusion then follows.

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