THE DECK RATIO AND SELF-REPAIRING GRAPHS

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ABSTRACT. In this paper we define, for a graph invariant ψ, the *deck ratio* of ψ by $D_\psi(G) = \frac{\psi(G)}{\sum_{v \in V(G)} \psi(G-v)}$. We give generic upper and lower bounds on D_ψ for monotone increasing and monotone decreasing invariants ψ, respectively.

Then we proceed to consider the Wiener index $W(G)$, showing that $D_W(G) \leq \frac{1}{|V(G)|-2}$. We show that equality is attained for a graph G if and only if every induced P_3 subgraph of G is contained in a C_4 subgraph. Such graphs have been previously studied under the name of *self-repairing graphs.*

We show that a graph on $n \geq 4$ vertices with at least $\frac{n^2-3n+6}{2}$ edges is necessarily a self-repairing graph and that this is the best possible result. We also show that a 2-connected graph is self-repairing iff all factors in its Cartesian product decomposition are.

Finally, some open problems about the deck ratio and about self-repairing graphs are posed at the end of the paper.

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1. Introduction

The *deck* of a graph G is the multiset of vertex-deleted subgraphs $(G - v)_{v \in V(G)}$. The famous unsettled Reconstruction Conjecture of Kelly and Ulam (cf. [7] and [10]) asserts that every graph on $n \geq 3$ vertices can be uniquely determined (up to isomorphism, of course) from its deck. Information on graph reconstruction can be found in [1] and [8], for instance.

In this paper we shall be considering the deck from a somewhat different point of view. Given some graph invariant ψ we shall focus our attention on the ratio $\frac{\psi(G)}{\sum_{v \in V(G)} \psi(G-v)}$. This ratio will be referred to as the *deck ratio* of ψ and denoted by D_ψ. As is easily seen, D_ψ is an invariant as well. A word of caution, however, is necessary to the effect that D_ψ may not be defined for some graphs. An example of this situation will be seen in Section 2 when we consider the Wiener index.

The main purpose of our investigation will be to look for lower and upper bounds on D_ψ (for particular invariants or classes of invariants) and to investigate the structure of extremal graphs which attain the bounds.

A graph invariant ψ will be called *vertex-increasing* (resp. *vertex-decreasing*) if for every graph G and every vertex v of G holds $\psi(G) \geq \psi(G - v)$ (resp. $\psi(G) \leq \psi(G - v)$).

It is easy to establish the following generic bounds on D_ψ for a monotone ψ:

Observation 1.1. *Let ψ be a graph invariant. Let G be a graph on n vertices.*
(a) If ψ is vertex-increasing, then: $D_\psi(G) \geq \frac{1}{n}$.
(b) If ψ is vertex-decreasing, then: $D_\psi(G) \leq \frac{1}{n}$.

Proof. We prove only (a), since the proof of (b) is the same, up to obvious sign reversals. By the monotonicity of ψ, we have for every vertex $v \in V(G)$:

$$\psi(G) \geq \psi(G - v).$$

Summing over all vertices we obtain:

$$n\psi(G) \geq \sum_{v \in V(G)} \psi(G - v).$$

And this is just:

$$D_\psi(G) \geq \frac{1}{n}.$$

□

In the remainder of this paper we shall be considering mostly a particular invariant, namely the *Wiener index* W of a graph. If the graph G is connected, then $W(G)$ is defined as the sum of all distances between pairs of two different vertices. That is:

$$W(G) = \sum \{d(u, v) | u, v \in V(G), u \neq v\}.$$

For disconnected graphs we define the Wiener index to be ∞, following [5]. The Wiener index and its applications to theoretical chemistry have been extensively studied (cf. [4] for a survey). It is also pertinent to remark that for a graph on n vertices the Wiener index may be viewed as a re-scaling by $\binom{n}{2}$ of the *average distance.* (cf. [9] for a survey) of the graph.

In the next section we study the deck ratio of the Wiener index in detail, establishing a tight upper bound and giving a characterization of graphs attaining equality in this bound. These turn out to be exactly the so called *self-repairing graphs*, first introduced by Farley and Proskurowski [6] and studied further by Djelloul and Kouider [3]. We also establish in Section 2 some new properties of self-repairing graphs.

In Section 3 we consider the extremal problem for self-repairing graphs and show that a graph on $n \geq 4$ vertices with at least $\frac{n^2-3n+6}{2}$ edges is necessarily a self-repairing graph and that this is the best possible result. Some open questions about the deck ratio of an invariant and about self-repairing graphs are asked in Section 4.

Our notation and terminology follow mostly that of [2]. in particular, the clique, cycle and path on k vertices are denoted by K_k, C_k, P_k, respectively. The join $G_1 \vee G_2$ of the graphs G_1

and G_2 is formed by adding all possible edges between G_1 and G_2.

2. The deck ratio of the Wiener index

In this section all graphs are assumed to be 2-connected. The reason for this is that the deck ratio of the Wiener index is undefined whenever a graph has a vertex-deleted unconnected subgraph (since the denominator of the deck ratio is infinite in this case).

We introduce another term before proceeding. For any vertex $v \in V(G)$ its *transmission* $\sigma(v)$ is the sum of all distances from v to other vertices. That is:

$$\sigma(v) = \sum\{d(u,v) | u \in V(G)\backslash\{v\}\}.$$

The Wiener index is not a vertex-monotone invariant (in the sense of the previous section). However we can estimate to some extent the change in the Wiener index effected by the deletion of a vertex. The following fact is well-known (cf. [5, Property 2.1]) and easy to prove:

Fact 2.1. *For every graph G and every vertex $v \in V(G)$ holds:*

$$W(G - v) \geq W(G) - \sigma(v).$$

Now we can establish an upper bound on the deck ratio of the Wiener index.

Theorem 2.2. *Let G be a 2-connected graph on n vertices. Then: $D_W(G) \leq \frac{1}{n-2}$.*

Proof. Summing the inequality of Fact 2.1 over all vertices in G we obtain:

$$\sum_{v \in V(G)} W(G - v) \geq nW(G) - \sum_{v \in V(G)} \sigma(v).$$

By the definition of the transmission we see that:

$$\sum_{v \in V(G)} \sigma(v) = 2W(G).$$

Therefore:

$$(n-2)W(G) \le \sum_{v \in V(G)} W(G-v).$$

In other words:

$$D_W(G) \le \frac{1}{n-2}.$$

□

This bound turns out to be tight. Moreover, we can give a characterization of the graphs for which equality is attained. First we note that the triangle K_3 attains equality (as can be easily verified from the definition). For $n \ge 4$ vertices we have the following result:

Theorem 2.3. *Let G be a 2-connected graph on $n \ge 4$ vertices. Then $D_W(G) = \frac{1}{n-2}$ if and only if every induced subgraph of G which is isomorphic to P_3 is contained in some subgraph of G which is isomorphic to C_4.*

Proof. Looking over the proof of Theorem 2.2 we observe that $D_W(G) = \frac{1}{n-2}$ if and only $W(G-v) = W(G) + \sigma(v)$ holds for every vertex v.

Now, this means that for every v the following property must hold: for every two vertices u, w (different from v) the distances between u and w in G and $G-v$ must be the same.

The preceding property is equivalent to the following one for every v: for every two vertices u, w (different from v) there is a shortest path from u to w in G to which v doesn't belong. When the property is formulated in this way, it is clear that we can assume, without loss of generality that both u and w are adjacent to v and that u and w are non-adjacent themselves.

It remains to point out that the subset $\{u, v, w\} \subseteq V(G)$ induces a P_3. On the other hand, the shortest path between u and w which misses v is necessarily of the form (u, z, w) for some $z \in V(G)$ ($z \notin \{u, v, w\}$). Thus the subgraph induced by $\{u, v, w, z\}$ contains a C_4. □

Graphs with this property have been introduced under the name of *self-repairing graphs* by Farley and Proskurowski [6] and studied further by Djelloul and Kouider [3]. In the remainder of this section we establish two new properties of self-repairing graphs.

Corollary 2.4. *Every graph is isomorphic to an induced subgraph of a self-repairing graph.*

Proof. Let G be some graph and let $H = G \vee K_2$. Then it is readily seen that G is an induced subgraph of H, which is a self-repairing graph by Theorem 2.3. □

The second property pertains to the Cartesian product. The Cartesian product of G_1 and G_2 is denoted by $G_1 \square G_2$, being the graph with vertex set $V(G_1) \times V(G_2)$ and with two vertices (v_1, v_2) and (w_1, w_2) being joined by an edge if and only if one of the following holds: (i) $v_1 = w_1$ and $\{v_2, w_2\} \in E(G_2)$ (ii) $v_2 = w_2$ and $\{v_1, w_1\} \in E(G_1)$.

Note that the Cartesian product is commutative, that is: $G_1 \square G_2 = G_2 \square G_1$.

Theorem 2.5. *Let G and $G_1, \ldots, G_m$ be 2-connected graphs so that $G = G_1 \square \ldots \square G_m$. Then G is a self-repairing graph if and only if every G_i, $1 \leq i \leq m$, is a self-repairing graph.*

Proof. Without loss of generality we may assume that $m = 2$.

First let us prove that if G_1 and G_2 are self-repairing then so is $G_1 \square G_2$. Suppose that the vertices $x = (x_1, x_2), y = (y_1, y_2), z = (z_1, z_2)$ induce a subgraph isomorphic to P_3 in $G_1 \square G_2$, with x being adjacent to both y and z. Since both G_1 and G_2 are self-repairing, without loss of generality we can assume that $x_1 = y_1$ and therefore x_2 and y_2 are adjacent in G_2. Now we shall deal with two cases.

Case 1: Assume that $x_1 = z_1$. Then it follows that x_2 and z_2 are adjacent in G_2. Also, $y_1 = z_1$ and thus y_2 and z_2 are non-adjacent in G_2, since otherwise y and z would be adjacent in $G_1 \square G_2$, contrary to assumption.

So we observe that x_2, y_2 and z_2 induce a P_3 in G_2 and by Theorem 2.3 there must be a vertex $w_2 \in V(G_2)$ that is different from x_2 and is adjacent to both y_2 and z_2. Then the vertices

x, y, z and $w = (x_1, w_2)$ belong to a subgraph isomorphic to C_4 in $G_1 \square G_2$.

Case 2: Assume that $x_1 \neq z_1$. Then it follows that x_1 and z_1 are adjacent in G_1 and that $x_2 = z_2$. Now consider the vertex $u = (z_1, y_2) \in V(G_1 \square G_2)$. We see that $u \notin \{x, y, z\}$ and that the vertices x, y, z, u induce a subgraph isomorphic to C_4 in $G_1 \square G_2$.

In both cases we find that $G_1 \square G_2$ is a self-repairing graph by Theorem 2.3.

Conversely, let us prove that if G_1 is *not* a self-repairing graph, then for every graph H, the graph $G_1 \square H$ is *not* a self-repairing graph. Indeed, by Theorem 2.3 we see that there exist distinct vertices $x, y, z \in V(G_1)$ that induce a subgraph isomorphic to P_3 which is not contained in any subgraph isomorphic to C_4. Without loss of generality assume that x is adjacent to both y and z.

Now pick some vertex $w \in V(H)$ and consider the subgraph induced by the vertices $(x, w), (y, w), (z, w) \in V(G_1 \square H)$. It is isomorphic to P_3 and the vertices (y, w) and (z, w) have no common neighbour in $G_1 \square H$, for the existence of such a neighbour would imply that y and z have a common neighbour in G_1. Thus the subgraph induced by $(x, w), (y, w)$ and (z, w) is not contained in any subgraph isomorphic to C_4. By Theorem 2.3 we conclude that $G_1 \square H$ is not a self-repairing graph. □

3. The extremal problem for self-repairing graphs

Let us consider the extremal problem for self-repairing graphs. We shall denote by $g(n)$ the minimum number such that any 2-connected graph on n vertices and with at least $g(n)$ edges is a self-repairing graph. Since the triangle is a self-repairing graph, we have that $g(3) = 3$. In fact we have the following general formula:

Theorem 3.1. *For any $n \geq 3$, $g(n) = \frac{n^2-3n+6}{2}$.*

Proof. First, consider a non-self-repairing 2-connected graph G on n vertices. By Theorem 2.3 there are three vertices $\{x, y, z\}$ that induce a P_3 subgraph which is not contained in any C_4

subgraph. Without loss of generality, suppose that y is adjacent to x and z.

Then x and z have no common neighbours except for y. This means that the sum of their degrees is at most $n-1$. The degree of y may be as high as $n-1$. On the other hand, the degree of any vertex other than x, y or z is at most $n-2$, since it cannot be adjacent to both x and z. Therefore, the graph has at most $\frac{(n-1)+(n-1)+(n-3)(n-2)}{2} = \frac{n^2-3n+4}{2}$ edges. Therefore a 2-connected graph with at least $\frac{n^2-3n+6}{2}$ edges cannot be a non-self-repairing graph.

Now let us construct a 2-connected non-self-repairing graph on n vertices and with $\frac{n^2-3n+4}{2}$ edges. First take a clique on $(n-4)$ vertices labelled $v_1, \ldots, v_{n-4}$. Then take a triangle with vertices labelled w_1, w_2, w_3 and add the following set of edges: $\{(w_i, v_j) | 1 \leq i \leq 2, 1 \leq j \leq n-4\} \cup \{(w_3, v_{n-4})\}$. Finally, take a new vertex y and add the following set of edges: $\{(y, v_j) | 1 \leq j \leq n-5\} \cup \{(y, w_2)\}$.

The graph G thus obtained has, in fact, $\frac{n^2-3n+4}{2}$ edges and is 2-connected. On the other hand, the subgraph induced by $\{w_2, w_3, y\}$ is isomorphic to P_3 and is contained in no subgraph of G which is isomorphic to C_4. Therefore G is not a self-repairing graph. □

We remark that in [6] Farley and Proskurowski have shown that a self-repairing graph on n vertices must have at least $2n-4$ edges and have characterized the self-repairing graphs with exactly $2n-4$ edges. That result provides a natural complement to our Theorem 3.1.

4. Some open problems

Is the converse of Observation 1.1 true? Without loss of generality, we formulate the problem below only for vertex-increasing invariants.

Problem 4.1. *Let ψ be some invariant. Suppose that $D_\psi(G) \geq \frac{1}{n}$ holds for every graph G on n vertices for which D_ψ is defined. Does it then follow that ψ is vertex-increasing?*

Another interesting question about the relationship between ψ and D_ψ is the following:

Problem 4.2. *Let ψ be an NP-hard invariant. Is then D_ψ necessarily an NP-hard invariant?*

In Section 3 we have found the value of the extremal function $g(n)$ and have described a construction of an extremal graph for it. However, this is not the only possible construction. So, we can pose the following problem:

Problem 4.3. *Describe all 2-connected non-self-repairing graphs on n vertices and with $\frac{n^2-3n+4}{2}$ edges.*

Now let H and K be some fixed graphs such that K is isomorphic to a subgraph of H. We make the following definition:

Definition 4.4. *A graph G is said to have property $\mathbb{I}(K, H)$ if every induced subgraph of G which is isomorphic to K is contained in some subgraph of G which is isomorphic to H.*

Obviously, by Theorem 2.3 the 2-connected graphs with property $\mathbb{I}(P_3, C_4)$ are exactly the self-repairing graphs. In fact, it is not hard to show that a connected graph with property $\mathbb{I}(P_3, C_4)$ must be 2-connected and thus a self-repairing graph.

It is also not difficult to see that the graphs with property $\mathbb{I}(2K_1, P_3)$ are exactly the graphs that have diameter at most two.

We now pose the following problem:

Problem 4.5. *Given H and K as in Definition 4.4, describe all graphs with property $\mathbb{I}(K, H)$.*

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