# SMALL COMPLETE CAPS IN GALOIS SPACES

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ABSTRACT. Some new families of complete caps in Galois affine spaces $AG(N,q)$ of dimension $N \equiv 0 \pmod 4$ and odd order $q \leq 127$ are constructed. No smaller complete caps appear to be known.

## 1. INTRODUCTION

A $k$-cap in an (affine or projective) Galois space over the finite field with $q$ elements $\mathbb{F}_q$, is a set of $k$ points no three of which are collinear. A $k$-cap is said to be complete if it is not contained in a $(k+1)$-cap. A plane $k$-cap is also called a $k$-arc.

The central problem on caps is determining the maximal and minimal sizes of complete caps in a given space, see the survey papers [14],[1] and the references therein. For the size of the smallest complete cap in the affine space $AG(N,q)$ of dimension $N$ over $\mathbb{F}_q$, the trivial lower bound is $\sqrt{2}q^{\frac{N-1}{2}}$. Complete caps of size about $q^{N/2}$ are known to exist for $q$ even, see [16, 10, 8, 12]; the same holds for $q$ odd, provided that $N$ is even [7]. In this paper, the case $q$ odd, $N \equiv 0 \pmod 4$ will be dealt with. Under these assumptions, complete caps of size $k \leq \frac{1}{2}q^{\frac{N}{2}}$ were obtained by Giulietti, provided that $q \geq 76^2$ [7]. For $25 < q < 76^2$, the smallest known complete caps appear to have size $q^{N/2}$. The aim of this paper is to construct smaller complete caps for $q$ in this range.

Our main result is the following.

**Theorem 1.1.** *Assume that $N \equiv 0 \pmod 4$. Then there exists a complete cap in $AG(N,q)$ of size $n_q q^{\frac{N-2}{2}}$, with $(q, n_q)$ as follows:*

| $q$ | 27 | 29 | 53 | 67 | 73 | 81 | 83 | 89 |
|---|---|---|---|---|---|---|---|---|
| $n_q$ | 23 | 25 | 35 | 42 | 45 | 49 | 50 | 54 |

| $q$ | 97 | 101 | 103 | 107 | 109 | 113 | 121 | 127 |
|---|---|---|---|---|---|---|---|---|
| $n_q$ | 55 | 61 | 60 | 63 | 65 | 66 | 71 | 74 |

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2000 *Math. Subj. Class.*: 51E22.
*Keywords*: Affine space, Complete cap, Complete arc.

## 2. Complete caps from bicovering arcs

Throughout this section, $q$ is assumed to be odd and $N$ divisible by 4. Let $q' = q^{\frac{N-2}{2}}$. Fix a basis of $\mathbb{F}_{q'}$ as a linear space over $\mathbb{F}_q$, and identify points in $AG(N,q)$ with vectors of $\mathbb{F}_{q'} \times \mathbb{F}_{q'} \times \mathbb{F}_q \times \mathbb{F}_q$. Also, let $c$ be a fixed non-square in $\mathbb{F}_q$. Note that as $\frac{N-2}{2}$ is odd, $c$ is a non-square in $\mathbb{F}_{q'}$ as well.

For an arc $A$ in $AG(2,q)$, let

$$K_A = \{(\alpha, \alpha^2, u, v) \in AG(N,q) \mid \alpha \in \mathbb{F}_{q'}, \ (u,v) \in A\}.$$

As noticed in [7], the set $K_A$ is a cap whose completeness in $AG(N,q)$ depends on the bicovering properties of $A$ in $AG(2,q)$.

According to Segre [17], given three pairwise distinct points $P, P_1, P_2$ on a line $\ell$ in $AG(2,q)$, $P$ is external or internal to the segment $P_1P_2$ depending on whether

$$(2.1) \qquad (x - x_1)(x - x_2) \quad \text{is a non-zero square in } \mathbb{F}_q \text{ or not,}$$

where $x$, $x_1$ and $x_2$ are the coordinates of $P$, $P_1$ and $P_2$ with respect to any affine frame of $\ell$.

**Definition 2.1.** *Let $A$ be a complete cap in $AG(2,q)$. A point $P \in AG(2,q) \setminus A$ is said to be bicovered by $A$ if there exist $P_1, P_2, P_3, P_4 \in A$ such that $P$ is both external to the segment $P_1P_2$ and internal to the segment $P_3P_4$. If every $P \in AG(2,q) \setminus A$ is bicovered by $A$, then $A$ is said to be a bicovering arc. If there exists precisely one point $Q \in AG(2,q) \setminus A$ which is not bicovered by $A$, then $A$ is said to be almost bicovering, and $Q$ is called the center of $A$.*

Our main tool is the following result from [7].

**Proposition 2.2.** *If $A$ is a bicovering $k$-arc, then $K_A$ is a complete cap in $AG(N,q)$ of size $kq^{(N-2)/2}$. If $A$ is almost bicovering with center $Q = (x_0, y_0)$, then either*

$$K = K_A \cup \{(\alpha, \alpha^2 - c, x_0, y_0) \mid \alpha \in \mathbb{F}_{q'}\}$$

*or*

$$K = K_A \cup \{(\alpha, \alpha^2 - c^2, x_0, y_0) \mid \alpha \in \mathbb{F}_{q'}\}$$

*is a complete cap in $AG(N,q)$ of size $(k+1)q^{(N-2)/2}$.*

## 3. New complete caps

The starting point of our investigation are some known families of complete arcs in the projective plane $PG(2,q)$ over $\mathbb{F}_q$. Given a complete $k$-arc $A$ in $PG(2,q)$, a natural question in this context is whether for a line $\ell$ disjoint from $A$, the arc $A$ is bicovering (or almost bicovering) in the affine plane $PG(2,q) \setminus \ell \cong AG(2,q)$.

We fix the following identification of $AG(2,q)$ with $PG(2,q)\setminus\ell$. Assume that $\ell$ as equation $a_1X_1 + a_2X_2 + a_3X_3 = 0$. Then for a point $P$ in $PG(2,q)\setminus\ell$ with homogeneous coordinates $(x_1,x_2,x_3)$, let $\pi_\ell(P)$ be the point in $AG(2,q)$ defined as follows:

$$\pi_\ell(P) = \begin{cases} \left(\frac{x_2}{a_1x_1+a_2x_2+a_3x_3}, \frac{x_3}{a_1x_1+a_2x_2+a_3x_3}\right), & \text{if } a_1 \neq 0, \\ \left(\frac{x_1}{a_1x_1+a_2x_2+a_3x_3}, \frac{x_3}{a_1x_1+a_2x_2+a_3x_3}\right), & \text{if } a_1 = 0, a_2 \neq 0, \\ \left(\frac{x_2}{a_1x_1+a_2x_2+a_3x_3}, \frac{x_1}{a_1x_1+a_2x_2+a_3x_3}\right), & \text{if } a_1 = a_2 = 0. \end{cases}$$

For a $k$-arc $A$ in $PG(2,q)$ and for a line $\ell$ disjoint from $A$, let

$$A_\ell := \{\pi_\ell(P) \mid P \in A\} \subseteq AG(2,q).$$

We have investigated the bicovering properties of $A_\ell$ for the following arcs $A$ in $PG(2,q)$:

- the Chao-Kaneta complete 22-arc $CK_{22}$ in $PG(2,27)$ [3];
- the Chao-Kaneta complete 24-arc $CK_{24}$ in $PG(2,29)$ [2];
- Zirilli arcs in $PG(2,q)$ [19, 20], for $53 \le q \le 127$.

We recall the definition of a Zirilli arc. Let $\mathcal{C}$ be a non-singular plane projective curve of degree 3 defined over $\mathbb{F}_q$, and let $\mathcal{C}(\mathbb{F}_q) \subseteq PG(2,q)$ be the set of points of $\mathcal{C}$ with coordinates in $\mathbb{F}_q$. It is well-known that $\mathcal{C}(\mathbb{F}_q)$ can be given the structure of an abelian group. If the size of $\mathcal{C}(\mathbb{F}_q)$ is even, then the coset of the subgroup of index 2 in $\mathcal{C}(\mathbb{F}_q)$ is an arc. This was first noticed by Zirilli. The completeness of Zirilli arcs has been investigated in several papers [18, 15, 11, 6, 5]. The Zirilli arc arising from a curve with affine equation $Y^2 = f(X)$, with $f$ a polynomial of degree 3, will be denoted by $Z(f(X))$.

A computer assisted search has produced the results of Table 1. In Table 1, the letter $B$ stands for bicovering, whereas $AB$ indicates that $A_\ell$ is almost bicovering. Also, $\omega$ stands for an element in $\mathbb{F}_{27}$ satisfying $\omega^3 + 2\omega + 1 = 0$, $\alpha \in \mathbb{F}_{81}$ is such that $\alpha^4 + 2\alpha^3 + 2 = 0$ and $\gamma \in \mathbb{F}_{121}$ satisfies $\gamma^2 + 7\gamma + 2 = 0$. Taking into account Proposition 2.2 these results provide a proof of Theorem 1.1.

**Remark 3.1.** Other complete arcs have been checked, for instance those described in [13, 9, 4]. No further example of a bicovering or almost bicovering arc has been obtained. On the other hand, other almost bicovering Zirilli arcs with size larger than those described in Table 1 have been found out.

| $q$ | $A$ | $\ell$ | $\lvert A_\ell \rvert$ | B or AB | Center |
|---|---|---|---|---|---|
| 27 | $CK_{22}$ | $X_1+\omega^8X_2+\omega^{12}X_3=0$ | 22 | AB | $(\omega^2,\omega^{14})$ |
| 29 | $CK_{24}$ | $X_1+27X_2+27X_3=0$ | 24 | AB | $(5,22)$ |
| 53 | $Z(X^3+X)$ | $X_1+21X_2+24X_3=0$ | 34 | AB | $(57,1)$ |
| 67 | $Z(X^3+X^2+39X)$ | $X_2+4X_3=0$ | 42 | B | |
| 73 | $Z(X^3+5)$ | $X_1+41X_2+72X_3=0$ | 45 | B | |
| 81 | $Z(X^3+\alpha X^2+\alpha^{19})$ | $X_1+\alpha^{65}X_2+\alpha^{52}X_3=0$ | 49 | B | |
| 83 | $Z(X^3+X^2+67)$ | $X_1+78X_2+35X_3=0$ | 50 | B | |
| 89 | $Z(X^3+X^2+33)$ | $X_1+87X_2+X_3=0$ | 54 | B | |
| 97 | $Z(X^3+X^2+13)$ | $X_1+37X_2+43X_3=0$ | 55 | B | |
| 101 | $Z(X^3+8)$ | $X_1+86X_2+70X_3=0$ | 61 | B | |
| 103 | $Z(X^3+X^2+20)$ | $X_1=0$ | 60 | B | |
| 107 | $Z(X^3+2X^2+18)$ | $X_1+40X_2+43X_3=0$ | 63 | B | |
| 109 | $Z(X^3+8)$ | $X_1+4X_2+64X_3=0$ | 65 | B | |
| 113 | $Z(X^3+X^2+110)$ | $X_1+69X_2+11X_3=0$ | 66 | B | |
| 121 | $Z(X^3+\gamma X^2+\gamma^{21})$ | $X_1+\gamma^{65}X_2+\gamma^{17}X_3=0$ | 71 | B | |
| 127 | $Z(X^3+X^2+24)$ | $X_1+34X_2+7X_3=0$ | 74 | B | |

TABLE 1. Bicovering or almost bicovering arcs $A_\ell$ in $AG(2,q)$

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