# End-regular and End-orthodox joins of split graphs *

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**Abstract**

A graph $X$ is said to be End-regular (resp., End-orthodox, End-inverse) if its endomorphism monoid $End(X)$ is a regular (resp., orthodox, inverse) semigroup. In this paper, End-regular (resp., End-orthodox, End-inverse) graphs which are the join of split graphs $X$ and $Y$ are characterized. It is also proved that $X+Y$ is never End-inverse for any split graphs $X$ and $Y$.

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## 1 Introduction and preliminaries

Endomorphism monoids of graphs are generalizations of automorphism groups of graphs. In recent years much attention has been paid to endomorphism monoids of graphs and many interesting results concerning graphs and their endomorphism monoids have been obtained. The aim of this research is try to establish the relationship between graph theory and algebraic theory of semigroups and to apply the theory of semigroups to graph theory. Just as Petrich and Reilly pointed out in [11], in the great

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range of special classes of semigroups, regular semigroups take a central position from the point of view of richness of their structural "regularity". So it is natural to ask for which graph $G$ the endomorphism monoid of $G$ is regular (such an open question raised in [10]). However, it seems difficult to obtain a general answer to this question. So the strategy for solving this question is finding various kinds of conditions of regularity for various kinds of graphs. In [12], The connected bipartite graphs whose endomorphism monoids are regular were explicitly found. An infinite family of graphs with regular endomorphism monoids were provided in [7], and the joins of two trees with regular endomorphism monoids were also characterized. Hou, Luo and Cheng [5] explored the endomorphism monoid of $\overline{P_n}$, the complement of a path $P_n$ with $n$ vertices. It was shown that $End(\overline{P_n})$ is an orthodox monoid. The split graphs with regular endomorphism monoids were studied in [4] and [6], respectively. The split graphs with orthodox endomorphism monoids were characterized in [3]. Moreover, Split graphs whose half-strong (resp, locally-srong, quasi-strong) endomorphisms form a monoid were characterized in [9]. In this paper, we continue to explore the endomorphisms monoids of the joins of split graphs and characterize such graphs whose endomorphism monoids are regular (resp., orthodox, inverse).

The graphs considered in this paper are finite undirected graphs without loops and multiple edges. Let $X$ be a graph. The vertex set of $X$ is denoted by $V(X)$ and the edge set of $X$ is denoted by $E(X)$. The cardinality of the set $V(X)$ is called the *order* of $X$. If two vertices $x_1$ and $x_2$ are adjacent in graph $X$, the edge connecting $x_1$ and $x_2$ is denoted by $\{x_1, x_2\}$ and write $\{x_1, x_2\} \in E(X)$. For a vertex $v$ of $X$, denote by $N_X(v)$ (or briefly by $N(v)$) the set $\{x \in V(X) | \{x, v\} \in E(X)\}$ and called it the *neighborhood* of $v$ in $X$, the cardinality of $N_X(v)$ is called the *degree* or *valency* of $v$ in $X$ and is denoted by $d_X(v)$. A subgraph $H$ is called an *induced subgraph* of $X$ if for any $a, b \in H$, $\{a, b\} \in H$ if and only if $\{a, b\} \in V(X)$. We denote by $K_n$ a complete graph with $n$ vertices. A *clique* of a graph $X$ is the maximal complete subgraph of $X$. The *clique number* of $X$, denoted by $\omega(X)$, is the maximal order among the cliques of $X$.

Let $X$ and $Y$ be two graphs. The *join* of $X$ and $Y$, denoted by $X + Y$, is a graph with $V(X+Y) = V(X) \cup V(Y)$ and $E(X+Y) = E(X) \cup E(Y) \cup \{\{a, b\} | a \in V(X), b \in V(Y)\}$.

Let $G$ be a graph. A subset $K \subseteq V(G)$ is said to be *complete* if $\{a, b\} \in E(G)$ for any two vertices $a, b \in K$. A subset $S \subseteq V(G)$ is said to be *independent* if $\{a, b\} \notin E(G)$ for any two vertices $a, b \in S$. A graph $X$ is called a *split graph* if its vertex set $V(X)$ can be partitioned into disjoint (non-empty) sets $S$ and $K$, such that $S$ is an independent set and $K$ is a complete set. We can always assume that any split graph $X$ has an unique partition $V(X) = K \cup S$, where $K$ is a maximal complete set and $S$ is an independent set. Since $K$ is a maximal complete set of $X$, it is easy to see that for any $y \in S$, $0 \le d_X(y) \le n-1$ where $n = |K|$.

Let $X$ and $Y$ be two graphs. A mapping from $V(X)$ to $V(Y)$ is called a *homomorphism* if $\{a, b\} \in E(X)$ implies that $\{f(a), f(b)\} \in E(Y)$. A homomorphism from $X$ to itself is called an *endomorphism* of $X$. An endomorphism $f$ of $X$ is said to be *half-strong* if $\{f(a), f(b)\} \in E(X)$ implies that there exist $c \in f^{-1}(a)$ and $d \in f^{-1}(b)$ such that $\{c, d\} \in E(X)$. Denote by $End(X)$ and $hEnd(X)$ the set of all endomorphisms and half-strong endomorphisms of $X$. It is known that $End(X)$ is a monoid with respect to the composition of mappings and is called the endomorphism monoid (or briefly monoid) of $X$. Denote by $Idpt(X)$ the set of all idempotents of $End(X)$. It is known that every idempotent endomorphism is half-strong.

A *retraction* is a homomorphism $f$ from a graph $X$ to a subgraph $Y$ of $X$ such that the restriction $f|_Y$ of $f$ to $V(Y)$ is the identity map on $V(Y)$. It is easy to see that the idempotents of $End(X)$ are retractions. Let $f$ be an endomorphism of a graph $X$. A subgraph of $G$ is called the *endomorphic image* of $G$ under $f$, denoted by $I_f$, if $V(I_f) = f(V(G))$ and $\{f(a), f(b)\} \in E(I_f)$ if and only if there exist $c \in f^{-1}(f(a))$ and $d \in f^{-1}(f(b))$ such that $\{c, d\} \in E(G)$. By $\rho_f$ we denote the equivalence relation on $V(X)$ induced by $f$, i.e., for $a, b \in V(X)$, $(a, b) \in \rho_f$ if and only if $f(a) = f(b)$. Denote by $[a]_{\rho_f}$ the equivalence class containing $a \in V(X)$ with respect to $\rho_f$.

An element $a$ of a semigroup $S$ is called *regular* if there exists $x \in S$ such that $axa = a$. A semigroup $S$ is called regular if its elements are regular. A semigroup $S$ is called *orthodox* if $S$ is regular and the set of all idempotents forms a subsemigroup, that is, a regular semigroup is orthodox if the product of its any two idempotents is still an idempotent. An inverse semigroup is a regular semigroup in which the idempotents commute and inverse semigroups are orthodox semigroups. A graph $X$ is said to be End-regular (resp., End-orthodox, End-inverse) if its endomorphism monoid $End(X)$ is regular (resp., orthodox, inverse). Clearly, End-inverse graphs are End-orthodox and End-orthodox graphs are End-regular.

The reader is referred to [1] and [2] for all the notation and terminology not defined here. We list some known results which will be used frequently in the sequel to end this section.

**Lemma 1.1**([7]) Let $X$ be a graph and $f \in End(X)$. Then
(1) $f \in hEnd(X)$ if and only if $I_f$ is an induced subgraph of $X$.
(2) If $f$ is regular, then $f \in hEnd(X)$.

**Lemma 1.2** ([8]) Let $X$ be a graph and $f \in End(X)$. Then $f$ is regular if and only if there exists $g, h \in Idpt(X)$ such that $\rho_g = \rho_f$ and $I_h = I_f$.

**Lemma 1.3** ([7]) Let $X$ and $Y$ be two graphs. If $X+Y$ is End-regular, then both $X$ and $Y$ are End-regular.

**Lemma 1.4** ([7]) Let $X$ be a graph. Then $X$ is End-regular if and only if $X + K_n$ is End-regular for any $n \geq 1$.

The following are some known results about split graphs which are essential for our consideration.

**Lemma 1.5**([4]) Let $X$ be a split graph with $V(X) = K \cup S$, where $S = \{y_1, y_2, ..., y_m\}$ is an independent set and $K = \{k_1, k_2..., k_n\}$ is a maximal complete set. Then a mapping $f$ on the set $V(X)$ is a retraction of $X$ if and only if the following conditions holds:

(1) For $x \in K$, $f(x) = x$; for $y \in S$, either $f(y) \in K \backslash N(y)$, or $f(y) \in S$, in this case, $N(y) \subseteq N(f(y))$ and $f^2(y) = f(y)$.

(2) For some $x_i \in K$ such that $f(x_i) = y_j \in S$, where $y_j$ is some vertex in $X$ which is adjacent to every vertices of $K$ except $x_i$; for $x \in K \setminus \{x_i\}$, $f(x) = x$; $f(y_j) = y_j$; for $y \in N(x_i) \cap S$, $f(y) \in N(y_j) \setminus N(y)$; for $y \in S \setminus (N(x_i) \cup \{y_j\})$, either $f(y) \in K \setminus N(y)$ or $f(y) \in S$, in this case, $N(y) \subseteq N(f(y))$ and $f^2(y) = f(y)$.

**Lemma 1.6**([6]) Let $X$ be a connected split graph with $V(X) = K \cup S$, where $S$ is an independent set and $K$ is a maximal complete set, $|K| = n$. Then $X$ is End-regular if and only if there exists $r \in \{1, 2, ..., n-1\}$ such that $d(x) = r$ for any $x \in S$.

**Lemma 1.7**([6]) A non-connected split graph $X$ is End-regular if and only if $X$ exactly consists of a complete graph and several isolated vertices.

**Lemma 1.8**([4]) Let $X$ be a split graph with $V(X) = K \cup S$, where $S$ is an independent set and $K$ is a maximal complete set. If for some $y_i \neq y_j$, $N(y_i) \subseteq N(y_j)$, then $X$ is not End-orthodox.

# 2 End-orthodox split graphs

We will characterize the End-orthodox split graphs in this section. Since End-orthodox split graphs are End-regular, we always assume our graphs are End-regular in this section. To our aim, we first describe the idempotent endomorphisms (retractions) of End-regular split graphs.

**Lemma 2.1** Let $X$ be a connected End-regular split graph with $V(X) = K \cup S$, where $S = \{y_1, y_2, ..., y_m\}$ is an independent set and $K = \{k_1, k_2..., k_n\}$ is a maximal complete set. Then $f \in End(X)$ is an idempotent if and only if one of the following conditions holds:

(1) For $x \in K$, $f(x) = x$; for $y \in S$, either $f(y) \in K \setminus N(y)$, or $f(y) \in S$ with $N(y) = N(f(y))$ and $f^2(y) = f(y)$.

(2) If $d(y) = n - 1$ for all $y \in S$, there exists $x_i \in K$ such that $f(x_i) = y_j \in S$, where $y_j$ is some vertex in $X$ which is adjacent to every vertices of $K$ except $x_i$; for $x \in K \setminus \{x_i\}$, $f(x) = x$; $f(y_j) = y_j$; for $y \in N(x_i) \cap S$, $f(y) \in K$; for $y \in S \setminus (N(x_i) \cup \{y_j\})$, $f(y) \in S$ with $N(y) = N(f(y))$ and $f^2(y) = f(y)$.

**Proof** It follows from Lemmas 1.5 and 1.6.

Note that a non-connected End-regular split graph consists of a complete subgraph of it and several isolated vertices.

**Lemma 2.2** Let $X$ be a non-connected End-regular split graph with $V(X) = K \cup S$, where $S = \{y_1, y_2 ..., y_m\}$ is an independent set and $K = \{k_1, k_2 ..., k_n\}$ is a complete set. Then $f \in End(X)$ is an idempotent if and only if for $k_i \in K$, $f(k_i) = k_i$; for $y \in S$, either $f(y) \in K$, or $f(y) \in S$ with $f^2(y) = f(y)$.

**Proof** Note that an endomorphism of graph maps a clique to a clique with the same order.

**Theorem 2.3** Let $X$ be a connected split graph with $V(X) = K \cup S$, where $S = \{y_1, y_2, ..., y_m\}$ is an independent set and $K = \{k_1, k_2 ..., k_n\}$ is a maximal complete set. Then $X$ is End-orthodox if and only if

(i) $d(y) = r$ for any $y \in S$, where $r \in \{1, 2, ...n - 1\}$;

(ii) $N_X(y_i) \neq N_X(y_j)$ whenever $i \neq j$ for $i, j \in \{1, 2, ..., m\}$.

**Proof** The necessity follows from Lemmas 1.6 and 1.8.

Conversely, if the condition (1) holds, by Lemma 1.6, $X$ is End-regular. In the following, we only need to show that the composition of any idempotent endomorphisms of $X$ is also an idempotent in each cases.

Assume $r < n - 1$. Let $f$ be an arbitrary retractions of $X$. Then by Lemma 2.1, for $x \in K$, $f(x) = x$; for $y \in S$, either $f(y) \in K \setminus N(y)$, or $f(y) = y$. It is a routine to show that the composition of any such two retractions is also a retraction of $X$.

Assume $r = n - 1$. Let $f$ be an arbitrary retractions of $X$. Without loss of generality, suppose $y_j$ is not adjacent to $k_j$ for $i = 1, 2, \cdots, m$. Then by Lemma 2.1, $f$ acts in one of the following ways:

(1) for any $x \in K$, $f(x) = x$; for $y_j \in S$, either $f(y_j) = x_j$, or $f(y_j) = y_j$.

(2) $f(x_i) = y_i \in S$ for some $x_i \in K$; $f(x) = x$ for any $x \in K \setminus \{x_i\}$; $f(y_i) = y_i$; $f(y_j) = k_j$ for any $j \neq i$.

It is straightforward to see that the composition of any such two retractions is still a retraction of $X$. The proof is complete.

**Remark 2.4** Fan characterized the connected End-orthodox split graphs in [4], but the main result Theorem 2.5 [4] has a mistake.

Theorem 2.5 [4]: Let $G$ be a split graph with $V(G) = K \cup S$, where $K = \{k_1, k_2..., k_n\}$ is a maximal complete set and $S = \{y_1, y_2, ..., y_m\}$ is an independent set. Then $X$ is End-orthodox if and only if exactly one of the following conditions holds:

(1) $d(y) < n - 1$ for all $y \in S$, Moreover, $N_X(y_i) \not\subseteq N_X(y_j)$ for any $i \neq j$ $(i, j \in \{1, 2, ..., m\})$.

(2) $d(y) = n - 1$ for all $y \in S$, Moreover, $m \leq n$ and after reindexing $\{x_j, y_j\} \notin E(G)$ for every $j \in \{1, 2, ..., m\}$.

In fact, the condition (1) of Theorem 2.5 [4] is not enough to ensure a split graphs being regular. For example, let $X$ be a connected split graph with vertex set $V(X) = K \cup S$, where $K = \{1, 2, 3, 4\}$, $S = \{5, 6\}$, $N_X(5) = \{1, 2\}$, $N_X(6) = \{3\}$. Obviously $X$ satisfies condition (1) of Theorem 2.5 in [4]. Lemma 1.5 implies that $X$ is not End-regular. So that the vertices in the independent set have the same valency is necessary for End-regular and also for End-orthodox.

The next theorem characterizes the non-connected End-orthodox split graphs.

**Theorem 2.5** Let $X$ be a non-connected End-regular split graph with $V(X) = K \cup S$, where $S = \{y_1, y_2, ..., y_m\}$ is an independent set and $K = \{k_1, k_2..., k_n\}$ is a complete set. Then $X$ is End-orthodox if and only if $m = 1$.

**Proof** Suppose $m \neq 1$, then $S$ contains at least two vertices $y_1$ and $y_2$. Let $f$ be an idempotent that maps $y_1$ to some $k_i \in K$ and fixes the others, and let $g$ be an idempotent that maps $y_2$ to $y_1$ and fixes the others. Now $gf(y_2) = y_1$, but $gf(y_1) = k_i \neq y_1$. Hence $gf$ is not an idempotent and $X$ is not orthodox.

Conversely, suppose $m = 1$ and $f$ is an idempotent endomorphism of $X$, then $f(k_i) = k_i$ and either $f(y_1) = k_j$ $(k_j \in K)$ or $f(y_1) = y_1$. It is a routine matter to show that the composition of any two such idempotents is still an idempotent. Hence $X$ is End-orthodox.

# 3 End-regular joins of split graphs

The End-regular split graphs have been characterized in Lemma 1.6 and Lemma 1.7. In this section, we will characterize the End-regular graphs which are the join of split graphs.

Let $X$ be a split graph with $V(X) = V(K_n) \cup S_1$, where $S_1 = \{x_1, \cdots, x_p\}$ is an independent set and $V(K_n) = \{k_1, k_2, \cdots, k_n\}$ is a maximal complete set. Let $Y$ be another split graph with $V(Y) = V(K_m) \cup S_2$, where

$S_2 = \{y_1, y_2, \cdots, y_q\}$ is an independent set and $V(K_m) = \{r_1, r_2, \cdots, r_m\}$ is a maximal complete set. Then the vertex set $V(X+Y)$ of $X+Y$ can be partitioned into three parts $V(K_{n+m})$, $S_1$ and $S_2$, i.e., $V(X+Y) = V(K_{n+m}) \cup S_1 \cup S_2$, where $V(K_{n+m}) = V(K_n) \cup V(K_m)$ is a complete set, $S_1$ and $S_2$ are independent sets. Obviously the subgraph of $X+Y$ induced by $V(K_{n+m})$ is a complete graph and the subgraph of $X+Y$ induced by $S_1 \cup S_2$ is a complete bipartite graph. Hence in graph $X+Y$, $N_{X+Y}(x_i) = N_X(x_i) \cup V(Y)$ for $x_i \in S_1$, $i \in \{1, 2, \cdots, p\}$ and $N_{X+Y}(y_i) = N_Y(y_i) \cup V(X)$ for $y_i \in S_2$, $i \in \{1, 2, \cdots, q\}$. It is easy to see that $X+Y$ is a split graph adding to the edge set $\{\{x_i, y_j\} \mid x_i \in S_1, y_j \in S_2\}$. By Lemma 1.3, we know if $X+Y$ is End-regular, then both of $X$ and $Y$ are End-regular, so we always assume that $X$ and $Y$ are End-regular split graphs in the sequel unless otherwise stated. Moreover, let $d_1$ be the valency of the vertices of $S_1$ in $X$ and $d_2$ be the valency of the vertices of $S_2$ in $Y$. Clearly, if $X$ (resp., $Y$) is connected, then $0 < d_1 \leq n-1$ (resp., $0 < d_2 \leq n-1$); if $X$ (resp., $Y$) is non-connected, then $d_1 = 0$ (resp., $d_2 = 0$).

**Lemma 3.1** Let $X$ and $Y$ be two End-regular split graphs. If $X+Y$ is End-regular, then $d_1 + m = d_2 + n$.

**Proof** Suppose $d_1 + m \neq d_2 + n$, without loss of generality, let $d_1 + m < d_2 + n$. As $d_1 < n$, for any $x \in S_1$, $x$ is not adjacent to exactly $n - d_1$ vertices of $V(K_n)$ in $X$, so $x$ is not adjacent to exactly $n - d_1$ vertices of $V(K_{n+m})$ in $X+Y$, take such a vertex and write $k_x$. Similarly, for any $y \in S_2$, $y$ is not adjacent to exactly $m - d_2$ vertices of $V(K_{n+m})$ in $X+Y$, take such a vertex and write $r_y$. Let $x_1$ be a vertex of $S_1$ and $y_1$ be a vertex of $S_2$. Since $|V(K_{n+m}) \cap N_{X+Y}(x_1)| = d_1 + m < d_2 + n = |V(K_{n+m}) \cap N_{X+Y}(y_1)|$, there exists a permutation $\tau$ on $V(K_{n+m})$ such that $\tau(V(K_{n+m}) \cap N_{X+Y}(x_1)) \subset V(K_{n+m}) \cap N_{X+Y}(y_1)$.

Let $f$ be a mapping from $V(X+Y)$ to itself defined by

$$f(x) = \begin{cases} y_1, & if\ x = x_1, \\ \tau(x), & if\ x \in V(K_{n+m}), \\ \tau(k_x), & if\ x \in S_1 \setminus \{x_1\}, \\ \tau(r_x), & if\ x \in S_2. \end{cases}$$

Then $f \in End(X+Y)$. Since $|V(K_{n+m}) \cap N_{X+Y}(x_1)| < |V(K_{n+m}) \cap N_{X+Y}(y_1)|$, $I_f$ is not an induced subgraph of $X+Y$. Hence $f \notin hEnd(X+Y)$. It follows from Lemma 1.1 that $X+Y$ is not End-regular. A contradiction. Therefore $d_1 + m = d_2 + n$.

**Lemma 3.2** Let $X$ and $Y$ be two End-regular split graphs with $d_1 + m = d_2 + n$. If $X+Y$ is End-regular, then

(1) There are no two vertices $x_1, x_2 \in S_1$, such that $N_X(x_1) \cup N_X(x_2) = V(K_n)$.

(2) There are no two vertices $y_1, y_2 \in S_2$, such that $N_Y(y_1) \cup N_Y(y_2) = V(K_m)$.

**Proof** (1) Suppose there exist two vertices $x_1$ and $x_2$ of $S_1$ such that $N_X(x_1) \cup N_X(x_2) = V(K_n)$. Then $(V(K_n) \backslash N_X(x_1)) \cap (V(K_n) \backslash N_X(x_2)) = \phi$. Let $y_1$ be a vertex of $S_2$. Then there exists a permutation $\tau$ of $V(K_{n+m})$ such that $\tau(V(K_n) \backslash N_X(x_1)) = V(K_m) \backslash N_Y(y_1)$ and $\tau(V(K_n) \backslash N_X(x_2)) = V(K_m) \setminus N_X(x_2)$.

For $x \in S_1$ and $y \in S_2$, $k_x$ and $r_x$ have the same meaning as in the proof of Lemma 3.1. Let $f$ be a mapping from $V(X+Y)$ to itself defined by

$$f(x) = \begin{cases} y_1, & if\ x = x_1, \\ x_2, & if\ x = x_2, \\ \tau(x), & if\ x \in V(K_{n+m}), \\ \tau(k_x), & if\ x \in S_1 \setminus \{x_1, x_2\}, \\ \tau(r_x), & if\ x \in S_2. \end{cases}$$

Then $f \in End(X+Y)$. It is easy to see $\{y_1, x_2\} \in E(X+Y)$. But $f^{-1}(x_2) = x_2$, $f^{-1}(y_1) = x_1$ and $\{x_1, x_2\} \notin E(X+Y)$. Therefore $f \notin hEnd(X+Y)$ and so $X+Y$ is not End-regular.

A similar argument may show that if there exist two vertex $y_1, y_2 \in S_2$ such that $N_Y(y_1) \cup N_Y(y_2) = V(K_m)$, then $X+Y$ is not End-regular.

We next prove the conditions in Lemma 3.1 and 3.2 are the sufficient conditions such that $X+Y$ being End-regular. Note that in case of $m+d_1 = n+d_2$, $X+Y$ has a unique clique (of order $n+m$) if and only if $d_1 \le n-2$ and $d_2 \le m-2$. So we can go process into two cases: $d_1 \le n-2$, $d_2 \le m-2$ and $d_1 = n-1$, $d_2 = m-1$. First we have

**Lemma 3.3** Let $X$ and $Y$ be two End-regular split graphs with $d_1 \le n-2$, $d_2 \le m-2$ and $m+d_1 = n+d_2$. Then for any endomorphism $f$ of $X+Y$, $I_f$ is an induced subgraph of $X+Y$ (i.e., $End(X+Y) = hEnd(X+Y)$) if and only if

(1) There are no two vertices $x_1, x_2 \in S_1$ such that $N_X(x_1) \cup N_X(x_2) = V(K_n)$.

(2) There are no two vertices $y_1, y_2 \in S_2$ such that $N_Y(y_1) \cup N_Y(y_2) = V(K_m)$.

**Proof** Necessity follows from the proof of Lemma 3.2.

Conversely, if there are no two vertices $x_1, x_2 \in S_1$ such that $N_X(x_1) \cup N_X(x_2) = V(K_n)$, then for any two vertices $s_1, s_2 \in S_1$, there is no endomorphism $f$ such that $f(s_1) \in S_1$ and $f(s_2) \in S_2$. Otherwise, since $f(V(K_{n+m})) = V(K_{n+m})$ and the numbers of vertices in $V(K_{n+m})$ which is adjacent to $f(s_i)$ and $s_i$ (i=1,2) are equal, we have $f(V(K_n) \backslash N_X(s_1)) \subset V(K_n)$ and $f(V(K_n) \setminus N_X(s_2)) \subset V(K_m)$. Note that $N_X(s_1) \cup N_X(s_2) \neq$

$V(K_n)$, $(V(K_n) \setminus N_X(s_1)) \cap (V(K_n) \setminus N_X(s_2)) \neq \phi$. Hence $V(K_n) \cap V(K_m) \neq \phi$. A contradiction. Similarly, if there are no two vertices $y_1, y_2 \in S_2$ such that $N_Y(y_1) \cup N_Y(y_2) = V(K_m)$, then for any two vertices $s_1, s_2 \in S_2$, there is no endomorphism $f$ such that $f(s_1) \in S_1$ and $f(s_2) \in S_2$.

Let $f \in End(X+Y)$ and let $a, b \in I_f$ with $\{a, b\} \in E(X+Y)$. We need to prove that there exist $c \in f^{-1}(a)$, $d \in f^{-1}(b)$ such that $\{c, d\} \in E(X+Y)$. If both of $a$ and $b$ are in $f(V(K_{n+m}))$, then there exist $c \in f^{-1}(a)$, $d \in f^{-1}(b)$ such that $\{c, d\} \in E(X+Y)$ since $f(V(K_{n+m})) = V(K_{n+m})$. If exactly one of $a$ and $b$ is in $f(V(K_{n+m}))$, without loss of generality, assume that $a \in f(V(K_{n+m}))$, $b \notin f(V(K_{n+m}))$, then there exists a vertex $c \in V(K_{n+m})$ such that $f(c) = a$. Suppose that $\{c, v\} \notin E(X+Y)$ for any vertex $v \in f^{-1}(b)$, let $u \in f^{-1}(b)$. Then $u$ is adjacent to exactly $m + d_1$ vertices in $V(K_{n+m}) \setminus \{c\}$, say, $x_1, x_2, \cdots, x_{m+d_1}$. So $b$ is adjacent to $f(x_1)$, $f(x_2), \cdots, f(x_{m+d_1})$. Clearly $f(x_1)$, $f(x_2), \cdots, f(x_{m+d_1})$, $a$ are distinct. We get that $b$ is adjacent to $m+d_1+1$ vertices in $V(K_{n+m})$, a contradiction. If both $a$ and $b$ are not in $f(V(K_{n+m}))$ and $\{c, d\} \notin E(X+Y)$ for any $c \in f^{-1}(a)$, $d \in f^{-1}(b)$, then $f^{-1}(a)$ and $f^{-1}(b)$ are contained in the same $S_i (i = 1, 2)$. From the discussion in the last paragraph, we have $a = f(f^{-1}(a))$ and $b = f(f^{-1}(b))$ are in the same $S_i$ $(i = 1, 2)$ and so $\{a, b\} \notin E(X+Y)$, a contradiction, as required.

**Lemma 3.4** Let $X$ and $Y$ be two End-regular split graphs with $d_1 \leq n-2$, $d_2 \leq m-2$. Then $X+Y$ is End-regular if and only if

(1) $m + d_1 = n + d_2$,

(2) There are no two vertices $x_1, x_2 \in S_1$ such that $N_X(x_1) \cup N_X(x_2) = V(K_n)$,

(3) There are no two vertices $y_1, y_2 \in S_2$ such that $N_Y(y_1) \cup N_Y(y_2) = V(K_m)$.

**Proof** Necessity follows immediately from Lemmas 3.1 and 3.2.

Conversely, let $f \in End(X+Y)$. To show that $f$ is regular, we need to prove that there exist two idempotents $g$ and $h$ in $End(X)$ such that $\rho_g = \rho_f$ and $I_h = I_f$.

Since $d_1 \leq n-2$ and $d_2 \leq m-2$, $f(V(K_{n+m})) = V(K_{n+m})$ and for any $x \in S_1 \cup S_2$, there exists a vertex $k_x \in V(K_{n+m})$ such that $x$ is not adjacent to $k_x$. Let $h$ be the mapping from $V(X+Y)$ to itself defined by

$$h(x) = \begin{cases} x, & if\ x \in f(X+Y), \\ k_x, & if\ x \in V(X+Y) \setminus f(X+Y). \end{cases}$$

Then $h \in End(X+Y)$ and $h(V(K_{n+m})) = V(K_{n+m})$. If $x \in f(X+Y)$, then $h^2(x) = h(x) = x$; If $x \in V(X+Y) \setminus f(X+Y)$, then $h^2(x) = h(k_x) = k_x = h(x)$ since $k_x \in V(K_{n+m}) \subseteq f(X+Y)$. Hence $f \in Idpt(X+Y)$.

Clearly, $I_f$ and $I_h$ have the same set of vertices. Note that an idempotent endomorphism is half-strong. It follows from Lemmas 1.1 and 3.3 that both $I_h$ and $I_f$ are induced subgraph of $X+Y$. Therefore $I_h = I_f$.

Since $f(V(K_{n+m})) = V(K_{n+m})$, $[x]_{\rho_f}$ contains at most one vertex of $V(K_{n+m})$ for any $x \in V(X+Y)$. Without loss of generality, suppose that $V(X+Y)/\rho_f = \{[k_1]_{\rho_f}, [k_2]_{\rho_f} \cdots [k_n]_{\rho_f}, [r_1]_{\rho_f} \cdots [r_m]_{\rho_f}, [s_1]_{\rho_f} \cdots [s_t]_{\rho_f}\}$, where $s_i \in S_1 \cup S_2$. Let $g$ be a mapping from $V(X+Y)$ to itself defined by

$$g(x) = \begin{cases} k_i, & if\ x \in [k_i]_{\rho_f}, \\ r_i, & if\ x \in [r_i]_{\rho_f}, \\ s_i, & if\ x \in [s_i]_{\rho_f}. \end{cases}$$

Then $g \in End(X+Y)$. If any $x \in [k_i]_{\rho_f}$, then $g^2(x) = g(k_i) = k_i = g(x)$; If $x \in [r_i]_{\rho_f}$, then $g^2(x) = g(r_i) = r_i = g(x)$; And if $x \in [s_i]_{\rho_f}$, then $g^2(x) = g(s_i) = s_i = g(x)$. Hence $g^2 = g$. Clearly, $\rho_g = \rho_f$, as required.

**Lemma 3.5** Let $X$ and $Y$ be two End-regular split graphs with $d_1 = n-1$ and $d_2 = m-1$. Then $X+Y$ is End-regular if and only if
(1) $N_X(x_1) = N_X(x_2)$ for any $x_1, x_2 \in S_1$,
(2) $N_Y(y_1) = N_Y(y_2)$ for any $y_1, y_2 \in S_2$,.

**Proof** Necessity follows immediately from Lemma 3.2.

Conversely, since $N_X(x_1) = N_X(x_2)$ for any $x_1, x_2 \in S_1$ and $d_1 = n-1$, there is an unique vertex $k$ in $K_1$ such that $\{x, k\} \notin E(X+Y)$ for any $x \in S_1$. Similarly, there is an unique vertex $r$ in $K_2$ such that $\{y, r\} \notin E(X+Y)$ for any $y \in S_2$. Now the subgraph of $X+Y$ induced by $S_1 \cup S_2 \cup \{k, r\}$ is a complete bipartite graph, we denote it by $K_{m_1,n_1}$. Hence $X+Y$ is isomorphic to $K_{n+m-2} + K_{m_1,n_1}$. Since $K_{m_1,n_1}$ is End-regular (see[12]), by Lemma 1.4, $X+Y$ is End-regular.

Now we are ready for our main result in this section.

**Theorem 3.6** Let $X$ and $Y$ be two split graphs with $V(X) = V(K_n) \cup S_1$, $V(Y) = V(K_m) \cup S_2$, respectively. Then $X+Y$ is End-regular if and only if
(1) $X$ is End-regular, that is there is a positive integer $d_1$ such that $d_X(x) = d_1$ for any $x \in S_1$,
(2) $Y$ is End-regular, that is there is a positive integer $d_2$ such that $d_Y(y) = d_2$ for any $y \in S_2$,
(3) $m + d_1 = n + d_2$,
(4) There are no two vertices $x_1, x_2 \in S_1$ such that $N_X(x_1) \cup N_X(x_2) = V(K_n)$,
(5) There are no two vertices $y_1, y_2 \in S_2$ such that $N_Y(y_1) \cup N_Y(y_2) = V(K_m)$.

**Proof** It follows directly from Lemmas 1.3, 1.6, 3.4 and 3.5.

# 4 End-orthodox joins of split graphs

The End-regular graphs which are the join of split graphs are characterized in section 3 and the End-orthodox split graphs are characterized in section 2. Furthermore, we will characterize the End-orthodox graphs which are the join of split graphs in this section. We also prove that the endomorphism monoids of such graphs are not inverse. To these aims, we first give the following lemmas which is essential for our consideration.

**Lemma 4.1** Let $G_1$ and $G_2$ be two graphs. If $G_1+G_2$ is End-orthodox, then both of $G_1$ and $G_2$ are End-orthodox.

**Proof** Since $G_1 + G_2$ is End-orthodox, $G_1 + G_2$ is End-regular. By Lemma 1.3, both of $G_1$ and $G_2$ are End-regular. To show $G_1$ is End-orthodox, we only need to prove that the composition of any two idempotent endomorphisms of $G_1$ is also an idempotent.

Let $f_1$ and $f_2$ be two idempotents in $End(G_1)$. Define two mappings $g_1$ and $g_2$ from $V(X+Y)$ to itself by

$$g_1(x) = \begin{cases} f_1(x), & if\ x \in G_1, \\ x, & if\ x \in G_2, \end{cases} \quad g_2(x) = \begin{cases} f_2(x), & if\ x \in G_1, \\ x, & if\ x \in G_2. \end{cases}$$

Then $g_1$ and $g_2$ are two idempotents of $End(G_1 + G_2)$ and so $g_1g_2$ is also an idempotent of $End(G_1 + G_2)$ since $G_1 + G_2$ is End-orthodox. Clearly, $f_1f_2 = (g_1g_2)|_{G_1}$, the restriction of $g_1g_2$ to $G_1$. Hence $f_1f_2$ is an idempotent of $End(G_1)$, as required.

A similar argument will show that $G_2$ is also End-orthodox.

**Lemma 4.2** Let $G$ be a graph. Then $G$ is End-orthodox if and only if $G + K_n$ is End-orthodox for any positive integer $n$.

**Proof** If $G + K_n$ is End-orthodox, then by Lemma 4.1, $G$ is End-orthodox.

Conversely, for any positive integer $n$, by Lemma 1.4, if $X$ is End-regular, then $X + K_n$ is End-regular. Let $f$ be an idempotent of $End(G + K_n)$. Note that $\omega(G+K_n) = \omega(G)+n$, $V(K_n) \subset I_f$ and $f|_{K_n} = 1|_{K_n}$, the identity mapping on $K_n$. Hence $f(V(G)) \subseteq V(G)$ and $f|_G \in Idpt(G)$.

If $f_1$ and $f_2$ are two idempotents of $End(G + K_n)$, let $g_1 = f_1|_G$ and $g_2 = f_2|_G$. Then $g_1, g_2 \in Idpt(G)$ and so $g_1g_2 \in Idpt(G)$. Now $(f_1f_2)|_{K_n} = 1|_{K_n}$ and $(f_1f_2)|_G = g_1g_2$ imply that $f_1f_2$ is an idempotent of $End(G+K_n)$. Consequently $G + K_n$ is End-orthodox.

Let $X$ and $Y$ be two split graphs. If $X+Y$ is End-orthodox, then $X+Y$ is End-regular and both of $X$ and $Y$ are End-orthodox and so are regular. So, in this case, we can represent graphs $X$ and $Y$ as in section 3 (see the first paragraph in section 3). Furthermore, we have $N_X(x_1) \neq N_X(x_2)$ for any two vertices $x_1, x_2 \in S_1$ and $N_Y(y_1) \neq N_Y(y_2)$ for any two vertices $y_1, y_2 \in S_2$. The following lemma describes the idempotent endomorphisms of certain End-regular graphs $X + Y$.

**Lemma 4.3** Let $X$ and $Y$ be two split graphs and $X + Y$ be End-regular with $d_1 \leq n-2, d_2 \leq m-2$. If $N_X(x_1) \neq N_X(x_2)$ for any two vertices $x_1, x_2 \in S_1$ and $N_Y(y_1) \neq N_Y(y_2)$ for any two vertices $y_1, y_2 \in S_2$, then $f \in End(X+Y)$ is a retraction (idempotents) if and only if

(1) $f(x) = x$ for any $x \in V(K_{n+m})$.

(2) For any $y \in S_1 \cup S_2$, either $f(y) \in V(K_{n+m}) \backslash N_{X+Y}(y)$, or $f(y) = y$.

**Proof** Note that under the hypothesis of lemma, $X + Y$ has an unique maximum clique $K_{n+m}$.

As in section 3, we go precess into two cases.

**Lemma 4.4** Let $X$ and $Y$ be two End-regular split graphs with $d_1 \leq n-2, d_2 \leq m-2$. Then $X + Y$ is End-orthodox if and only if

(1) $X + Y$ is End-regular,

(2) $N_X(x_1) \neq N_X(x_2)$ for any two vertices $x_1, x_2 \in S_1$,

(3) $N_Y(y_1) \neq N_Y(y_2)$ for any two vertices $y_1, y_2 \in S_2$.

**Proof** Necessity is obvious.

Conversely, since $X + Y$ is End-regular, we only need to prove that the composition of two idempotent endomorphisms is also an idempotent. Let $f$ be an arbitrary idempotent of $End(X+Y)$. Then $f|_{V(K_{n+m})} = 1|_{V(K_{n+m})}$ and either $f(x) = x$ or $f(x) = k_x$, for any $x \in S_1 \cup S_2$, where $k_x$ is a vertex in $V(K_{n+m})$ such that $\{x, k_x\} \notin E(X+Y)$. Now the assertion follows immediately.

**Lemma 4.5** Let $X$ and $Y$ be two End-regular split graphs with $d_1 = n-1$ and $d_2 = m-1$. Then $X + Y$ is End-orthodox if and only if $|S_1| = |S_2| = 1$.

**Proof** Necessity is obvious.

Conversely, $X + Y$ is isomorphic to $K_{n+m-2} + C_4$. Since $C_4$ is End-orthodox, it follows from Lemma 4.2 that $X + Y$ is End-orthodox.

Now we are ready for our main result in this section.

**Theorem 4.6** Let $X$ and $Y$ be two split graphs with vertex set $V(X) = V(K_n) \cup S_1$, $V(Y) = V(K_m) \cup S_2$, respectively. Then $X+Y$ is End-orthodox if and only if

(1) $X$ is End-regular, that is there exists a positive integer $d_1$ such that $d_X(x) = d_1$ for any $x \in S_1$,

(2) $Y$ is End-regular, that is there exists a positive integer $d_1$ such that $d_Y(y) = d_2$ for any $y \in S_2$,

(3) $m + d_1 = n + d_2$,

(4) $N_X(x_1) \cup N_X(x_2) \neq V(K_n)$ for any two vertices $x_1, x_2 \in S_1$,

(5) $N_Y(y_1) \cup N_Y(y_2) \neq V(K_m)$ for any two vertices $y_1, y_2 \in S_2$,

(6) $N_{X+Y}(s_1) \neq N_{X+Y}(s_2)$ for any two vertices $s_1, s_2 \in S_1 \cup S_2$.

**Proof** If $X + Y$ is orthodox, then $X + Y$ is regular and so both of $X$ and $Y$ are regular. Now it follows immediately from Theorem 3.6, Lemma 4.4 and 4.5.

In conjunction with Theorem 3.6, we obtain another version of the previous theorem as follows:

**Theorem 4.6*** Let $X$ and $Y$ be two split graphs with vertex set $V(X) = V(K_n) \cup S_1$, $V(Y) = V(K_m) \cup S_2$, respectively. Then $X + Y$ is End-orthodox if and only if

(1) $X + Y$ is End-regular;

(2) Both of $X$ and $Y$ are End-orthodox.

As an application of Theorem 3.6, we prove that the endomorphism monoid of the join of two split graphs can not be an inverse semigroup.

**Theorem 4.7** Let $X$ and $Y$ be split graphs with vertex set $V(X) = V(K_n) \cup S_1$, $V(Y) = V(K_m) \cup S_2$, respectively. Then $End(X + Y)$ can not be an inverse semigroup, that is, $X + Y$ is always not End-inverse.

**Proof** Since inverse semigroups are orthodox semigroups, we may suppose that $X + Y$ is End-orthodox. There two cases:

**Case 1** $X$ and $Y$ are two split graphs satisfied conditions in Lemma 4.4. Let $x_1$ be a vertex in $S_1$. Then there exist two vertices $k_1$ and $k_2$ in $V(K_n)$ such that $\{x_1, k_1\}, \{x_1, k_2\} \notin E(X+Y)$ since $d_1 \leq n-2$. Let $f$ and $g$ be two mappings from $V(X+Y)$ to itself such that $f(x_1) = k_1$, $f(x) = x$ for $x \in V(X+Y) \setminus \{x_1\}$ and $g(x_1) = k_2$, $g(x) = x$ for $x \in V(X+Y) \setminus \{x_1\}$, respectively. Then $f$ and $g$ are two idempotents of $End(X + Y)$ such that $fg = g$ and $gf = f$. Clearly, $f \neq g$. Hence $gf \neq fg$ and $End(X + Y)$ is not an inverse semigroup.

**Case 2** $X$ and $Y$ are two split graphs satisfied conditions in Lemma 4.5. Let $x_1$ be a vertex in $S_1$. Then there exists exactly one vertex in $V(K_n)$ which is not adjacent to $x_1$, say, $k_1$. Let $f$ and $g$ be two mappings from $V(X+Y)$ to itself such that $f(x_1) = k_1$, $f(x) = x$ for $x \in V(X+Y) \setminus \{x_1\}$ and $g(k_1) = x_1$, $g(x) = x$ for $x \in V(X + Y) \setminus \{k_1\}$, respectively. Then $f$ and $g$ are two idempotents of $End(X + Y)$ such that $fg = f$ and $gf = g$. Clearly $f \neq g$. Hence $gf \neq fg$ and $End(X+Y)$ is not an inverse semigroup.

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## References


[1] C.Godsil and G.Royle, *Algebraic Graph Theory*, Springer-verlag, New York, (2000).

[2] J.M.Howie, *Fundamentals of Semigroup Theory*, Clarendon Press, Uxford, (1995).

[3] S. Fan, *Retractions of Split Graphs and End-orthodox Split Graphs*, Discrete Mathematics, **257** (2002), 161-164.

[4] S. Fan, *The Regularities of the Endomorphism Monoid of Split Graphs*, Acta Mathrmatica Sinica, **40** (1997), 419-421.

[5] H. Hou, Y. Luo, Z. Cheng, *The Endomorphism Monoid of* $\overline{P_n}$, European Journal of Combinatorics, **29** (2008), 1173-1185.

[6] W. Li and J. Chen, *Endomorphism-regularity of Split Graphs*, European Journal of Combinatorics , **22** (2001), 207-216.

[7] W. Li, *Graphs with Regular Monoid*, Discrete Mathematics, **265** (2003) 105-118.

[8] W. Li, *A Regular Endomorphism of a Graph and Its Inverse*, Mathematik, **41** (1994) 189-198.

[9] Y. Luo, Y. Qin, H. Hou, *Split Graphs Whose Half-strong (resp, Locally-srong, Quasi-strong) Endomorphisms Form a Monoid*, submitted to European Journal of Combinatorics.

[10] L. Marki, *Problems Raised at the Problem Session of the Colloquium on Semigroups in Szeged*, August 1987, Semigroup Forum 37 (1988), 367-373.

[11] P. Petrich, Norman R. Reilly, *Completely Regular Semigroups*, John Wiley & Sons, Inc., New York, 1999.

[12] E.Wilkeit, *Graphs with Regular Endomorphism Monoid*, Arch.Math, **66** (1996), 344-352.