ON LINEAR POSITIVE OPERATORS INVOLVING BIORTHOGONAL POLYNOMIAL

GÜRHAN İÇÖZ*, FATMA TAŞDELEN YEŞİLDAL AND SERHAN VARMA

ABSTRACT. In this paper we recall Konhauser polynomials. Approximation properties of these operators are obtained with the help of the Korovkin theorem. The order of convergence of these operators is computed by means of modulus continuity, Peetre's K-functional and the elements of the Lipschitz class. Also, we introduce the r-th order generalization of these operators and we evaluate this generalization by the operators defined in this paper. Finally, we give an application to differential equations.

1. Introduction

In 1965, Konhauser presented the general theory of biorthogonal polynomials [8]. In 1967 [9], he introduced the following pair of biorthogonal polynomials $Y_v^{(n)}(x;k)$ and $Z_v^{(n)}(x;k)$ $(n>-1;\ k\in\mathbb{N}=\{1,2,3,...\})$ which are suggested by the classical Laguerre polynomials $L_v^{(n)}(x)$ given by

$$L_v^{(n)}(x)=Y_v^{(n)}(x;1)=Z_v^{(n)}(x;1).$$

In this work, we are especially interested in $Y_v^{(n)}(x;k)$ $(k\in\mathbb{Z}^+)$ which is defined by

$$Y_v^{(n)}(x;k)=\frac{1}{v!}\sum_{i=0}^{v}\frac{x^i}{i!}\sum_{j=0}^{i}(-1)^j\binom{i}{j}\left(\frac{j+n+1}{k}\right)_v.$$

The classical Meyer-König and Zeller (MKZ) operators are defined in 1960 [11] by

$$M_n(f;x)=(1-x)^{n+1}\sum_{k=0}^{\infty}f\left(\frac{k}{k+n+1}\right)\binom{n+k}{k}x^k,\ x\in[0,1).$$

Key words and phrases. Linear positive operators, Konhauser polynomials, modulus of continuity, Peetre's K-functional, Lipschitz class.

2000 *Mathematics Subject Classification.* 41A25, 41A30,41A36, 33C45, 33C50
Corresponding author.

In order to give the monotonicity properties, Cheney and Sharma introduced the modification of the MKZ operators

$$M_n^*(f;x) = (1-x)^{n+1}\sum_{k=0}^{\infty} f\left(\frac{k}{k+n}\right)\binom{n+k}{k}x^k,\ x\in[0,1). \qquad (1.1)$$

In [4], they also introduced the operators (for $x \in [0,1)$ and $t \in (-\infty, 0]$)

$$P_n(f;x) = \exp\left(\frac{tx}{1-x}\right)\sum_{k=0}^{\infty} f\left(\frac{k}{k+n}\right) L_k^{(n)}(t)\, x^k (1-x)^{n+1} \qquad (1.2)$$

where $L_k^{(n)}(t)$ denotes the Laguerre polynomials. Since $L_k^{(n)}(0) = \binom{n+k}{k}$, then $M_n^*(f;x)$ is the special case of the operators $P_n(f;x)$. Now, We consider the sequence of linear positive operators (similarly in [5]) which is another generalization of the these operators via $Y_v^{(n)}(x;k)$ Konhauser polynomials. For $x \in [0,1)$, $t \in (-\infty, 0]$ and $k < n+1$,

$$(L_n f)(x,t) = \frac{1}{F_n(x,t)}\sum_{v=0}^{\infty} f\left(\frac{vk}{k(v-1)+n+1}\right) Y_v^{(n)}(t;k)\, x^v, \qquad (1.3)$$

where $\{F_n(x,t)\}_{n\in N}$ are the generating functions for the sequence of functions $\left\{Y_v^{(n)}(t;k)\right\}_{v\in N_0}$ given by Carlitz [3] in the form

$$F_n(x,t) = \sum_{v=0}^{\infty} Y_v^{(n)}(t;k)\, x^v, \quad n > 0 \qquad (1.4)$$

and

$$F_n(x,t) = (1-x)^{-\frac{n+1}{k}} \exp\left\{-t\left[(1-x)^{-\frac{1}{k}} - 1\right]\right\} \qquad (1.5)$$

This recurrence relation was given by Srivastava in [12]

$$tY_{v-1}^{(n+1)}(t;k) = (k(v-1)+n+1)\, Y_{v-1}^{(n)}(t;k) - kvY_v^{(n)}(t;k) \qquad (1.6)$$

where $Y_v^{(n)}(t;k) = 0$ for $v \in \mathbb{Z}^-$.

If we choose $k = 1$ in (1.3), then we acquired (1.2). Similarly, if we choose $k = 1$ and $t = 0$ in (1.3), we get (1.1) which are called as Bernstein power series by Cheney and Sharma in [4].

Lemma 1.1. *$L_n f$ is linear and positive operator.*

Proof. For $\forall f,g \in C[0,1)$ and $\forall \alpha, \beta \in \mathbb{R}$, we have

$$\begin{aligned}(L_n(\alpha f+\beta g))(x,t) &= \frac{1}{F_n(x,t)}\sum_{v=0}^{\infty}(\alpha f+\beta g)(\tfrac{vk}{k(v-1)+n+1})Y_v^{(n)}(t;k)\,x^v \\ &= \frac{\alpha}{F_n(x,t)}\sum_{v=0}^{\infty} f\left(\tfrac{vk}{k(v-1)+n+1}\right)Y_v^{(n)}(t;k)\,x^v \\ &\quad +\frac{\beta}{F_n(x,t)}\sum_{v=0}^{\infty} g\left(\tfrac{vk}{k(v-1)+n+1}\right)Y_v^{(n)}(t;k)\,x^v \\ &= \alpha(L_nf)(x,t)+\beta(L_ng)(x,t).\end{aligned}$$

So, the operator L_nf is linear. From the expression of $Y_v^{(n)}(t;k)$ $(k\in\mathbb{Z}^+)$ Konhauser polynomials, we get that

$$\begin{aligned}Y_v^{(n)}(t;k) &= \frac{1}{v!}\sum_{i=0}^{v}\frac{t^i}{i!}\sum_{j=0}^{i}(-1)^j\tbinom{i}{j}\left(\tfrac{j+n+1}{k}\right)_v \\ &= \left(\tfrac{n+1}{k}\right)_0+\left(\tfrac{n+1}{k}\right)_1+\left[\left(\tfrac{n+1}{k}\right)_1-\left(\tfrac{n+2}{k}\right)_1\right]x \\ &\quad +\tfrac{1}{2!}\left\{\left(\tfrac{n+1}{k}\right)_2+\left[\left(\tfrac{n+1}{k}\right)_2-\left(\tfrac{n+2}{k}\right)_2\right]x\right. \\ &\quad \left.+\left[\left(\tfrac{n+1}{k}\right)_2-2\left(\tfrac{n+2}{k}\right)_2+\left(\tfrac{n+3}{k}\right)_2\right]\tfrac{x^2}{2!}\right\}+\tfrac{1}{3!}\cdots \\ &= 1+\left[\tfrac{n+1}{k}-\tfrac{t}{k}\right]+\tfrac{1}{2!}\left[\tfrac{(n+1)(n+k+1)}{k^2}-\tfrac{(2n+k+3)t}{k^2}+\tfrac{t^2}{k^2}\right]+\tfrac{1}{3!}\cdots\end{aligned}$$

If we use $t\in(-\infty,0]$, $n>0$ and $k\in\mathbb{Z}^+$, then we have the positivity of Konhauser polynomials. Also, from (1.5) one can see that $F_n(x,t)$ generating function is positive for $x\in[0,1)$. Hence, we obtain that L_nf operator is positive for $x\in[0,1)$, $t\in(-\infty,0]$, $n>0$ and $k\in\mathbb{Z}^+$. □

The goal of this paper is to define linear positive operators including Konhauser polynomials, to investigate the approximation properties and the rate of convergence of this operators by using modulus of continuity, Peetre's K-functional and Lipschitz class functional, to define the r-th order generalization of the operators and to study the approximation properties and rate of convergence of this r-th order generalization, to give an application to differential equation for the operators.

2. Approximation properties of L_n

Let $x\in[0,1)$, $t\in(-\infty,0]$, b be a real number in the interval $(0,1)$ and $e_i(x)=x^i$, $i=0,1,2$. We have the following theorem for the convergence of the operators L_nf.

To obtain our main results we recall the following theorem.

Theorem 2.1. *(P. P. Korovkin's Theorem) Let $f \in C[a,b]$ and $f(x)$ is bounded function on the real axis. If the three conditions*

$$\begin{array}{lcl} L_n(1;x) & \rightrightarrows & 1 \\ L_n(t;x) & \rightrightarrows & x \\ L_n(t^2;x) & \rightrightarrows & x^2 \end{array}$$

are satisfied for the sequence of linear positive operators $L_n(f;x)$ then the sequence $L_n(f;x)$ converges to $f(x)$.

Now we prove the following theorem to obtain approximation properties of the operators $L_n(f;x)$ with the help of Korovkin's theorem.

Theorem 2.2. *If f is continuous on $[0,b]$ then $(L_nf)(x,t_0)$ converges to $f(x)$ uniformly on $[0,b]$ for each fixed value of the $t_0 \in (-\infty, 0]$.*

Proof. From Lemma 1, it is clear that L_nf is a positive linear operator.

From (1.4), we see from the function $f(s) = 1$, that

$$(L_ne_0)(x,t_0) = 1. \tag{2.1}$$

Consider then the function $f(s) = s$. For it by using (1.6) and (1.4), we have

$$\begin{aligned} (L_ne_1)(x,t_0) &= \frac{1}{F_n(x,t_0)} \sum_{v=0}^{\infty} \tfrac{vk}{k(v-1)+n+1} Y_v^{(n)}(t_0;k)\, x^v \\ &= \frac{1}{F_n(x,t_0)} \sum_{v=1}^{\infty} (Y_{v-1}^{(n)}(t_0;k) - \tfrac{t_0}{k(v-1)+n+1} Y_{v-1}^{(n+1)}(t_0;k)) x^v. \end{aligned}$$

Since

$$\frac{t_0}{F_n(x,t_0)} \sum_{v=1}^{\infty} \tfrac{1}{k(v-1)+n+1} Y_{v-1}^{(n+1)}(t_0;k)\, x^v \leq 0$$

then

$$(L_ne_1)(x,t_0) \geq x. \tag{2.2}$$

On the other hand, we have

$$\begin{aligned} (L_ne_1)(x,t_0) &= \frac{1}{F_n(x,t_0)} \sum_{v=1}^{\infty} (Y_{v-1}^{(n)}(t_0;k) - \tfrac{t_0}{k(v-1)+n+1} Y_{v-1}^{(n+1)}(t_0;k)) x^v \\ &= x - \frac{t_0 x}{F_n(x,t_0)} \sum_{v=0}^{\infty} \tfrac{1}{kv+n+1} Y_v^{(n+1)}(t_0;k)\, x^v. \end{aligned}$$

One can easily see that $\frac{1}{kv+n+1} \leq \frac{1}{n}$ and $F_{n+1}(x,t_0) = (1-x)^{-\frac{1}{k}} F_n(x,t_0)$ from (1.5). Thus, we obtain

$$|(L_ne_1)(x,t_0) - x| \leq \frac{|t_0|\, x}{n} (1-x)^{-\frac{1}{k}}.$$

If we take maximum over $[0, b]$ from two hand-side of last inequality for $x \in [0, b]$, we obtain

$$\|(L_n e_1)(x, t_0) - x\|_{C[0,b]} \le \frac{|t_0|\, b}{n(1-b)}. \tag{2.3}$$

We proceed then to a consideration of the function $f(s) = s^2$. Using

$$(L_n e_2)(x, t_0) = \frac{1}{F_n(x, t_0)} \sum_{v=0}^{\infty} \left(\frac{vk}{k(v-1)+n+1}\right)^2 Y_v^{(n)}(t_0; k)\, x^v$$

and the recurrence formula (1.6) twice, we obtain

$$\begin{aligned}(L_n e_2)(x, t_0) - x^2 \quad = \quad & \left[\frac{1}{F_n(x, t_0)} \sum_{v=2}^{\infty} \frac{k(v-2)+n+1}{k(v-1)+n+1} Y_{v-2}^{(n)}(t_0; k)\, x^v - x^2\right] \\ & - \frac{t_0}{F_n(x, t_0)} \sum_{v=2}^{\infty} \frac{1}{k(v-1)+n+1} Y_{v-2}^{(n+1)}(t_0; k)\, x^v \\ & + \frac{k}{F_n(x, t_0)} \sum_{v=1}^{\infty} \frac{1}{k(v-1)+n+1} Y_{v-1}^{(n)}(t_0; k)\, x^v \\ & - \frac{k t_0}{F_n(x, t_0)} \sum_{v=1}^{\infty} \frac{v}{(k(v-1)+n+1)^2} Y_{v-1}^{(n+1)}(t_0; k)\, x^v.\end{aligned} \tag{2.4}$$

It is obvious that

$$\frac{v+1}{(kv+n+1)^2} \le \frac{1}{n}$$

and using (1.4), we get

$$\left|\frac{k t_0}{F_n(x, t_0)} \sum_{v=1}^{\infty} \frac{v}{(k(v-1)+n+1)^2} Y_{v-1}^{(n+1)}(t_0; k)\, x^v\right| \le \frac{k\,|t_0|\, b}{n(1-b)}. \tag{2.5}$$

Similarly, since $\frac{1}{k(v+1)+n+1} \le \frac{1}{n}$ and using (1.4) we have

$$\left|\frac{t_0}{F_n(x, t_0)} \sum_{v=2}^{\infty} \frac{1}{k(v-1)+n+1} Y_{v-2}^{(n+1)}(t_0; k)\, x^v\right| \le \frac{|t_0|\, b^2}{n(1-b)}, \tag{2.6}$$

in a similar manner, by using the inequality $\frac{1}{kv+n+1} \le \frac{1}{n}$ and (1.4), we have

$$\left|\frac{k}{F_n(x, t_0)} \sum_{v=1}^{\infty} \frac{1}{k(v-1)+n+1} Y_{v-1}^{(n)}(t_0; k)\, x^v\right| \le \frac{kb}{n}. \tag{2.7}$$

Finally, since $\frac{k(v-2)+n+1}{k(v-1)+n+1} \le 1$, we can write

$$\frac{1}{F_n(x,t_0)} \sum_{v=2}^{\infty} \tfrac{k(v-2)+n+1}{k(v-1)+n+1} Y_{v-2}^{(n)}(t_0;k)\, x^v - x^2 \le 0. \tag{2.8}$$

On the other hand, using the expression

$$e_2(s) = s^2 = (s-x)^2 + 2xs - x^2$$

we may write

$$(L_n e_2)(x,t_0) - x^2 = \left(L_n (e_1 - x)^2\right)(x,t_0) + 2x\,(L_n(e_1 - x))(x,t_0).$$

By (2.2) and positivity of L_n, it follows that

$$(L_n e_2)(x,t_0) - x^2 \ge 0.$$

Thus from (2.5), (2.6), (2.7) and (2.8) we can write

$$\left\|(L_n e_2)(x,t_0) - x^2\right\|_{C[0,b]} \le \left[\frac{|t_0|\,b}{n} + \frac{k\,|t_0|}{n} + \frac{k(1-b)}{n}\right]\frac{b}{1-b}. \tag{2.9}$$

So

$$(L_n e_2)(x,t_0) \rightrightarrows x^2$$

on $[0,b]$.

Using the Korovkin's theorem, the proof is completed. □

3. Rate of Convergence

In this section, we compute the rates of convergence by the means of the modulus of continuity, Peetre's K-functional and elements of Lipschitz class.

Let $f \in C[0,b]$. The modulus of continuity of f denoted by $\omega(f;\delta)$, is defined to be

$$\omega(f;\delta) = \sup_{\substack{|s-x|<\delta \\ s,x\in[0,b]}} |f(s) - f(x)|.$$

The modulus of continuity of the function f in $C[0,b]$ gives the maximum oscillation of f in any interval of length not exceeding $\delta > 0$. It is well known that a necessary and sufficient condition for a function f to be in $C[0,b]$ is

$$\lim_{\delta\to 0} \omega(f;\delta) = 0.$$

It is also well known that for any $\delta > 0$ and each $s \in [0,b]$

$$|f(s) - f(x)| \le \omega(f;\delta)\left(1 + \frac{|s-x|}{\delta}\right). \tag{3.1}$$

The next results gives the rate of convergence of the sequence $\{(L_n f)(x,t)\}$ (for all $f \in C[0,b]$) by means of the modulus of continuity.

Theorem 3.1. *For all $f \in C[0,b]$ and fixed $t_0 \in (-\infty, 0]$, we have*

$$\|(L_n f)(x,t_0) - f(x)\|_{C[0,b]} \le \left(1 + (3B)^{1/2}\right) \omega(f;\delta_n)$$

where

$$\delta_n = \frac{1}{\sqrt{n}} \text{ and } B = \max\left\{kb,\ \frac{k|t_0|b}{1-b},\ \frac{3|t_0|b^2}{1-b}\right\}.$$

Proof. Let $f \in C[0,b]$. By linearity and monotonicity of $L_n f$ and using (3.1), we obtain

$$|(L_n f)(x,t_0) - f(x)| \le \omega(f;\delta)\{1 + \frac{1}{\delta}(L_n|e_1 - x|)(x,t_0)\}.$$

By the Cauchy-Schwarz inequality we have

$$|(L_n f)(x,t_0) - f(x)| \le \omega(f;\delta)\left\{1 + \frac{1}{\delta}(A_n(x;t_0))^{\frac{1}{2}}\right\}$$

where

$$A_n(x;t_0) = \frac{1}{F_n(x,t_0)} \sum_{v=0}^{\infty} \left(\frac{vk}{k(v-1)+n+1} - x\right)^2 Y_v^{(n)}(t_0;k)\, x^v. \quad (3.2)$$

This implies that

$$\|(L_n f)(x,t_0) - f(x)\|_{C[0,b]} \le \omega(f;\delta)\left\{1 + \frac{1}{\delta}\left(\sup_{x\in[0,b]} A_n(x;t_0)\right)^{\frac{1}{2}}\right\}. \quad (3.3)$$

For each $x \in [0,b]$, one can write

$$A_n(x;t_0) \le \left|(L_n e_2)(x,t_0) - x^2\right| + 2x\,|(L_n e_1)(x,t_0) - x|.$$

So, by (2.3) and (2.9) we get

$$\begin{aligned} \sup_{x\in[0,b]} A_n(x;t_0) &\le \left\|(L_n e_2)(x,t_0) - x^2\right\| + 2b\,\|(L_n e_1)(x,t_0) - x\| \\ &\le 3B\delta_n^2 \quad (3.4) \end{aligned}$$

where $B = \max\left\{kb,\ \frac{k|t_0|b}{1-b},\ \frac{3|t_0|b^2}{1-b}\right\}$ and $\delta_n = \frac{1}{\sqrt{n}}$. Combining (3.4) with (3.3) we can write

$$\|(L_n f)(x,t_0) - f(x)\|_{C[0,b]} \le \left(1 + (3B)^{1/2}\right) \omega(f;\delta_n).$$

□

$C^2[0,b] :=$ The space of those functions f for which f, f', $f'' \in C[0,b]$. Similarly in [2], we define the following norm in the space $C^2[0,b]$:

$$\|f\|_{C^2[0,b]} := \|f\|_{C[0,b]} + \|f'\|_{C[0,b]} + \|f''\|_{C[0,b]}.$$

We consider the following Peetre's K-functional

$$K(f,\delta_n) = \inf_{g\in C^2[0,b]} \left\{ \|f-g\|_{C[0,b]} + \delta_n \ \|g\|_{C^2[0,b]} \right\}.$$

Theorem 3.2. *If $f \in C[0,b]$ and each fixed value of $t_0 \in (-\infty, 0]$ then we have*

$$\|(L_n f)(x,t_0) - f(x)\|_{C[0,b]} \le 2K(f,\delta_n)$$

where the operators $L_n f$ are defined as (1.3) *and*

$$\delta_n = \frac{2|t_0|b + kb(1-b) + k|t_0|b + 3|t_0|b^2}{4n(1-b)}.$$

Note that $\delta_n \to 0$, when $n \to \infty$.

Proof. Suppose that $g \in C^2[0,b]$, from the Taylor expansion for $g(s)$ function, we have

$$\begin{aligned}|(L_n g)(x,t_0) - g(x)| &\le |(L_n(e_1 - x))(x,t_0)|\,|g'(x)| \\ &+\frac{1}{2}\left|\left(L_n(e_1-x)^2\right)(x,t_0)\right| |g''(x)|. \quad (3.5)\end{aligned}$$

Since

$$\left|\left(L_n(e_1-x)^2\right)(x,t_0)\right| \le \frac{kb(1-b) + k|t_0|b + 3|t_0|b^2}{n(1-b)}, \quad (3.6)$$

taking supremum over $[0,b]$ from two hand-side (3.5) and by (3.6) and (2.3) observe that

$$\|(L_n g)(x,t_0) - g(x)\|_{C[0,b]} \le \tfrac{2|t_0|b+kb(1-b)+k|t_0|b+3|t_0|b^2}{2n(1-b)} \|g\|_{C^2[0,b]}. \quad (3.7)$$

On the other hand, since $L_n f$ is a linear operator, we have

$$\begin{aligned}|(L_n f)(x,t_0) - f(x)| &\le |(L_n(f-g))(x,t_0)| + |f(x) - g(x)| \\ &+ |(L_n g)(x,t_0) - g(x)|. \quad (3.8)\end{aligned}$$

Thus, by using (2.1) and taking supremum over $[0,b]$ from two hand-side of (3.8), we can write

$$\begin{aligned}\|(L_n f)(x,t_0) - f(x)\|_{C[0,b]} &\le 2[\|f-g\|_{C[0,b]} + \tfrac{2|t_0|b+kb(1-b)+k|t_0|b+3|t_0|b^2}{4n(1-b)} \\ &\times \|g\|_{C^2[0,b]}].\end{aligned}$$

If we take infimum over $g \in C^2[0,b]$ from two hand-side of last inequality, by choosing

$$\delta_n = \frac{2|t_0|b + kb(1-b) + k|t_0|b + 3|t_0|b^2}{4n(1-b)}$$

the proof is obvious. □

We will now study the rate of convergence of the positive linear operators L_n means of the Lipschitz class $Lip_M(\alpha)$, for $0 < \alpha \leq 1$. We recall that a function $f \in C[0,b]$ belongs to $Lip_M(\alpha)$ if the inequality

$$|f(t) - f(x)| \leq M|t-x|^{\alpha} \tag{3.9}$$

$(t, x \in [0,b])$ holds.

Theorem 3.3. *For all $f \in Lip_M(\alpha)$, we have*

$$\|(L_n f)(x,t_0) - f(x)\|_{C[0,b]} \leq M(3B)^{\alpha/2}\delta_n^{\alpha}$$

where B and δ_n are the same as in Theorem 3.1.

Proof. Let $f \in Lip_M(\alpha)$ and $0 < \alpha \leq 1$. By (3.9), linearity and monotonicity of $L_n f$, we have

$$|(L_n f)(x,t_0) - f(x)| \leq \frac{M}{F_n(x,t_0)} \sum_{v=0}^{\infty} \left|\tfrac{vk}{k(v-1)+n+1} - x\right|^{\alpha} Y_v^{(n)}(t_0;k)\, x^v.$$

Applying the Hölder inequality with $p = \frac{2}{\alpha}$ and $q = \frac{2}{2-\alpha}$, we get

$$|(L_n f)(x,t_0) - f(x)| \leq M(A_n(x;t_0))^{\alpha/2} \tag{3.10}$$

where $A_n(x;t_0)$ is given by (3.2). Combining (3.4) with (3.10) that

$$\|(L_n f)(x,t_0) - f(x)\|_{C[0,b]} \leq M(3B)^{\alpha/2}\delta_n^{\alpha}$$

whence the result can be obtained. □

4. A Generalization of r-th Order of The Operators L_n

By $C^r[0,b]\,(0 < b < 1,\ r = 0,1,2,\ldots)$, we denote the set of functions f having continuous $r-th$ derivatives $f^{(r)}\left(f^{(0)}(x) = f(x)\right)$ on the segment $[0,b]$.

We consider the following generalization of the positive linear operators L_n defined by (1.3):

$$\left(L_n^{[r]} f\right)(x,t) = \frac{1}{F_n(x;t)} \sum_{v=0}^{\infty} \sum_{i=0}^{r} f^{(i)}\left(\tfrac{vk}{k(v-1)+n+1}\right) \frac{\left(x - \frac{vk}{k(v-1)+n+1}\right)^i}{i!} \times Y_v^{(n)}(t;k)\, x^v \tag{4.1}$$

where $f \in C^r[0,b]$, $r = 0,1,2,\ldots$ and $n \in \mathbb{N}$. We call the operators (4.1) the $r-th$ order of the operators $L_n f$ (for instance [6], [7]). Note that taking $r = 0$ we get the sequence $\{L_n f\}$ defined by (1.3).

Theorem 4.1. *If $f^{(r)} \in Lip_M(\alpha)$ and $f \in C^r[0,b]$ then we have*

$$\left\|\left(L_n^{[r]} f\right)(x,t_0) - f(x)\right\|_{C[0,b]} \leq \frac{M}{(r-1)!}\frac{\alpha}{\alpha+r} B(\alpha,r) \times \left\|\left(L_n |e_1 - x|^{\alpha+r}\right)(x,t_0)\right\|_{C[0,b]} \tag{4.2}$$

where $B(\alpha, r)$ is the beta function and $r, n \in \mathbb{N}$.

Proof. By (4.1) we get

$$f(x) - \left(L_n^{[r]} f\right)(x, t_0) = \frac{1}{F_n(x; t_0)} \sum_{v=0}^{\infty} [f(x) - \sum_{i=0}^{r} f^{(i)}\left(\tfrac{vk}{k(v-1)+n+1}\right) \times \frac{\left(x - \frac{vk}{k(v-1)+n+1}\right)^i}{i!}] Y_v^{(n)}(t_0; k) x^v. \quad (4.3)$$

It is known from Taylor's integral formula in [6],

$$f(x) - \sum_{i=0}^{r} f^{(i)}\left(\tfrac{vk}{k(v-1)+n+1}\right) \frac{\left(x - \frac{vk}{k(v-1)+n+1}\right)^i}{i!} = \frac{\left(x - \frac{vk}{k(v-1)+n+1}\right)^r}{(r-1)!} \times \int_0^1 (1-z)^{r-1} [f^{(r)}\left(\frac{vk}{k(v-1)+n+1} + z\left(x - \tfrac{vk}{k(v-1)+n+1}\right)\right) - f^{(r)}\left(\tfrac{vk}{k(v-1)+n+1}\right)] dz. \quad (4.4)$$

Because of $f^{(r)} \in Lip_M(\alpha)$, one can get

$$\left| f^{(r)}\left(\tfrac{vk}{k(v-1)+n+1} + z\left(x - \tfrac{vk}{k(v-1)+n+1}\right)\right) - f^{(r)}\left(\tfrac{vk}{k(v-1)+n+1}\right) \right| \leq M z^{\alpha} \left| x - \tfrac{vk}{k(v-1)+n+1} \right|^{\alpha}. \quad (4.5)$$

From the well known expansion of the beta function, we can write

$$\int_0^1 z^{\alpha} (1-z)^{r-1} dz = B(1+\alpha, r) = \frac{\alpha}{\alpha + r} B(\alpha, r). \quad (4.6)$$

Now by using (4.6) and (4.5) in (4.4), we conclude that

$$\left| f(x) - \sum_{i=0}^{r} f^{(i)}\left(\tfrac{vk}{k(v-1)+n+1}\right) \frac{\left(x - \frac{vk}{k(v-1)+n+1}\right)^i}{i!} \right| \leq \tfrac{M}{(r-1)!} \tfrac{\alpha}{\alpha+r} B(\alpha, r) \left| x - \tfrac{vk}{k(v-1)+n+1} \right|^{\alpha+r}. \quad (4.7)$$

Taking into consideration (4.3) and (4.7) we have

$$\left| f(x) - \left(L_n^{[r]} f\right)(x, t_0) \right| \leq \tfrac{M}{(r-1)!} \tfrac{\alpha}{\alpha+r} B(\alpha, r) (L_n |e_1 - x|^{\alpha+r})(x, t_0). \quad (4.8)$$

Taking supremum over $[0, b]$ from two hand-side of (4.8), we have (4.2). □

Now, consider the function $g \in C[0, b]$ defined by

$$g(s) = |s - x|^{\alpha+r}. \quad (4.9)$$

Since $g(x) = 0$, Theorem 2.2 yields

$$\lim_n \|(L_n g)(x, t_0) - g(x)\|_{C[0,b]} = 0.$$

So, it follows from Theorem 4.1 that, for all $f \in C^r[0,b]$ such that $f^{(r)} \in Lip_M(\alpha)$, we have

$$\lim_n \left\| \left(L_n^{[r]} f\right)(x,t_0) - f(x) \right\|_{C[0,b]} = 0.$$

Finally, taking into consideration Theorem 3.1 and Theorem 3.3 with $M = b^r$ and observing $g \in Lip_{b^r}(\alpha)$, one can deduce the following results from Theorem 4.1 immediately.

Corollary 4.2. *For all $f \in C^r[0,b]$ such that $f^{(r)} \in Lip_M(\alpha)$, we have*

$$\left\| \left(L_n^{[r]} f\right)(x,t_0) - f(x) \right\|_{C[0,b]} \le \tfrac{M}{(r-1)!} \tfrac{\alpha}{\alpha+r} B(\alpha,r)(1+(3B)^{1/2})\omega(g;\delta_n)$$

where δ_n is the same as in Theorem 3.1 and g is defined by (4.9).

Corollary 4.3. *For all $f \in C^r[0,b]$ such that $f^{(r)} \in Lip_M(\alpha)$, we have*

$$\left\| \left(L_n^{[r]} f\right)(x,t_0) - f(x) \right\|_{C[0,b]} \le \tfrac{Mb^r}{(r-1)!} \tfrac{\alpha}{\alpha+r} B(\alpha,r)(3B)^{\alpha/2}\delta_n^\alpha$$

where δ_n is the same as in Theorem 3.1.

The last two results give us the rates of convergence of the sequence $\left\{\left(L_n^{[r]} f\right)(x,t)\right\}$ by means of the modulus of continuity and the elements of the Lipschitz class $Lip_M(\alpha)$, respectively.

5. An Application to Differential Equations

In this section we give a functional differential equation for $(L_n f)(x,t)$ as defined in (1.3). This equation seems to be fundamental for the investigation of many kinds of linear positive operators. In May [10], Volkov [13] and Alkemade [1] there are equations similar to the equation in Theorem 5.1 below.

Theorem 5.1. *Let $g(s) = \frac{s}{1-s}$. For each $x \in [0,b]$ $(0 < b < 1)$ and $f \in C[0,b]$, $(L_n f)(x,t)$ as defined in* (1.3), *satisfies the functional differential equation*

$$x\frac{d}{dx}(L_n f)(x,t_0) = -x\tfrac{n+1-t_0(1-x)^{-\frac{1}{k}}}{k(1-x)}(L_n f)(x,t_0) + \tfrac{n-k+1}{k}(L_n fg)(x,t_0). \tag{5.1}$$

Proof. Since $f \in C[0,b]$, the power series on the right-hand side of (1.3) converges on $[0,b]$. Hence it is allowed to differentiate this term by term in $[0,b]$. Thus

$$\frac{d}{dx}(L_n f)(x,t_0) = \tfrac{-\frac{\partial}{\partial x}F_n(x;t_0)}{F_n^2(x;t_0)} \sum_{\upsilon=0}^{\infty} f\left(\tfrac{\upsilon k}{k(\upsilon-1)+n+1}\right) Y_\upsilon^{(n)}(t_0;k)\, x^\upsilon$$
$$+\tfrac{1}{F_n(x;t_0)} \sum_{\upsilon=1}^{\infty} f\left(\tfrac{\upsilon k}{k(\upsilon-1)+n+1}\right) Y_\upsilon^{(n)}(t_0;k)\, \upsilon x^{\upsilon-1} \tag{5.2}$$

Using $g\left(\frac{vk}{k(v-1)+n+1}\right) = \frac{vk}{n+1-k}$ and

$$\frac{\partial}{\partial x}F_n(x;t_0) = \frac{n+1-t_0(1-x)^{-\frac{1}{k}}}{k(1-x)}F_n(x;t_0),$$

it follows that

$$x\frac{d}{dx}(L_n f)(x,t_0) = -x\frac{n+1-t_0(1-x)^{-\frac{1}{k}}}{k(1-x)F_n(x;t_0)}\sum_{v=0}^{\infty} f\left(\frac{vk}{k(v-1)+n+1}\right)Y_v^{(n)}(t_0;k)\,x^v + \frac{n-k+1}{kF_n(x;t_0)}\sum_{v=0}^{\infty}(fg)\left(\frac{vk}{k(v-1)+n+1}\right)Y_v^{(n)}(t_0;k)\,x^v.$$

Using (1.3) in this equation, we obtain the proof of the theorem. □

REFERENCES

[1] Alkemade, J. A. H., The Second Moment for the Meyer-König and Zeller Operators, J. Approx. Theory 40 (1984), 261-273.

[2] Bleimann, G., Butzer, P. L. and Hahn, L., A Bernstein-Type operator approximating continuous functions on the semi-axis, Nederl. Akad. Wetensch Indag. Math. 42 (1980), 255-262.

[3] Carlitz, L., A note on certain biorthogonal polynomials, Pacific J. Math. 24 (1968), 425-430.

[4] Cheney, E. W. and Sharma, A., Bernstein Power series, Canad. J. Math. 16 (1964), 241-252.

[5] Doğru, O., Özarslan, M. A. and Taşdelen, F., On positive operators involving a certain class of generating functions, Studia Sci. Math. Hungar. 41 (4) (2004), 415-429.

[6] Kirov, G. H. and Popova, L., A generalization of the linear positive operators, Math. Balkanica 7 (1993), 149-162.

[7] Kirov, G. H., Approximation with Quasi-Splines, Inst. Physics Publ., Bristol (New York, 1992).

[8] Konhauser, J. D. E., Some properties of biorthogonal polynomials, J. Math. Anal. Appl. 11(1-3) (1965), 242-260.

[9] Konhauser, J. D. E., Biorthogonal polynomials suggested by the Laguerre polynomials, Pacific J. Math.(1967), 303-314.

[10] May, C. P., Saturation and inverse theorems for combinations of a class of exponential-type operators, Canad. J. Math. 28 (1976), 1224-1250.

[11] Meyer-König, W. and Zeller, K., Bernsteinsche potenzreihen, Studia Math. 19(1960), 89-94.

[12] Srivastava, H. M., Some biorthogonal polynomials suggested by the Laguerre polynomials, Pasific J. Math.98(1) (1982), 235-250.

[13] Volkov, Yu. I., Certain positive linear operators, Mat. Zametki 23 (1978), 363-368.

Gürhan İÇÖZ
Gazi University, Faculty of Sciences ,
Department of Mathematics,
Teknikokullar TR-06500, Ankara, Turkey.
E-mail address (G. İçöz): gurhanicoz@gazi.edu.tr

Fatma TAŞDELEN YEŞİLDAL
Ankara University, Faculty of Science,
Department of Mathematics,
Tandoğan TR-06100, Ankara, Turkey.
E-mail address (F. Taşdelen Yeşildal): tasdelen@science.ankara.edu.tr

Serhan VARMA
Ankara University, Faculty of Science,
Department of Mathematics,
Tandoğan TR-06100, Ankara, Turkey.
E-Mail address (S. Varma): svarma@science.ankara.edu.tr