Diatonic Graphs

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Abstract

It has been known for at least 2500 years that mathematics and music are directly related. This article explains and extends ideas originating with Euler involving labeling parts of graphs with notes in such a way that other parts of the graphs correspond in a natural way to chords. The principal focus of this research is the notion of diatonic labelings of cubic graphs, that is, labeling the edges with pitch classes in such a way that vertices are incident with edges labeled with the pitch classes of a triad in a given diatonic scale. The pitch classes are represented in a natural way with elements of $\mathbb{Z}_{12}$, the integers modulo twelve.

Several classes of cubic graphs are investigated and shown to be diatonic. Among the graphs considered are Platonic Solids, cylinders, and Generalized Petersen Graphs. It is shown that there are diatonic cubic graphs on n vertices for even $n \geq 14$. Also it is shown that there are cubic graphs on n vertices that do not have diatonic labellings for all even $n \geq 4$. The question of forbidden subgraphs is investigated, and a forbidden subgraph for diatonic graphs, or "clash", is demonstrated.

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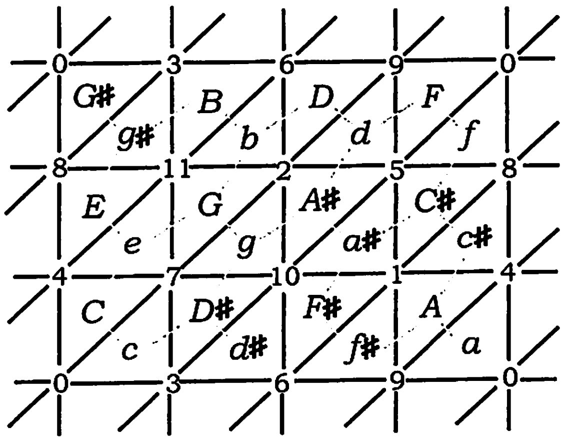

FIGURE 1. the tonnetz

It has been known for at least 2500 years that mathematics and music are directly related, and in fact some of the giants of mathematics have given attention to the connection. Music served as the gateway for Pythagoras and his followers to discover mathematics' value in natural physics. It is surely by Pythagoras' knowledge that we have our current harmonic intervals [2]. In particular, Pythagoras and his followers are credited with the diatonic scales commonly used in Western music [8]. The focus of this article will be a line of investigation pioneered by Euler [10], who introduced what is now called a "tonnetz". This is a tiling of a surface with triangles in which vertices are labeled with notes in the chromatic scale in such a way that the vertices of each triangular face constitute a chord. This is illustrated in Figure 1.

One might generalize this idea in a number of ways. Any triangulation of the plane might be vertex-labeled. A cubic planar graph might be face-labeled or edge-labeled subject to the condition that the faces or edges incident with any vertex should constitute a chord. Dropping the restriction of planar, the idea may be explored for cubic graphs with the edges labeled. Then, there is the possibility of allowing or requiring different sets of notes and chords. Some preliminaries are necessary.

0	1	2	3	4	5	6	7	8	9	10	11
C	C♯ / D♭	D	D♯ / E♭	E	F	F♯ / G♭	G	G♯ / A♭	A	A♯ / B♭	B
P1	A1 / m2	M2	A2 / m3	M3	P4	A4 / d5	P5	A5 / m6	M6	A6 / m7	M7

TABLE 1. The pitch classes and intervals

1. Preliminaries

Two kinds of scales are prominent in Western Music, the *chromatic* scale and the various *diatonic* scales. The chromatic scales consists of 12 equally distributed notes often represented as *pitch classes*, denoted pcs, in music theory. It is interpreted that the pc 0 represents the note C, 1 represents C♯ or D♭, 2 represents D, and so forth. We define the least non-negative integer congruent to $y - x$ mod 12, where x and y are pcs, as the *pitch class interval*, denoted pci. We use $\mathbb{Z}_{12}$ to denote both the set of pcs and the group of pcis.

Shown in Table 1 are the musical notes and intervals within an octave. The second row lists the letter name for the notes, and the third row lists the names of the intervals where P stands for perfect, M for major, m for minor, A for augmented, and d for diminished. The integers in the first row may be considered as either pc or pcis, depending on whether they represent the notes in the second row or the intervals in the third row. Thus while C♯ and D♭ have different names, they are essentially the same note. In music, these notes are said to be *enharmonically equivalent*, since both are represented by pc 1 in the first row of Table 1. In a similar fashion, G♯ and A♭ are enharmonically equivalent since both are represented by pc 8. In addition, the interval from C to C♯ is an augmented unison (A1) while the interval from C to D♭ is a minor second (m2). Once again, these intervals are both represented by pci 1 as they are enharmonically equivalent. In general, raising (lowering) a perfect interval results in an augmented (diminished) interval while a raised minor interval becomes major and a lowered major interval becomes minor.

With these conventions the addition $x + i = y$ will be understood to mean that the pc x raised by the pci i yields pc y. So we now have a numerical interpretation of the twelve pcs of the chromatic scale, and a musical interpretation of the addition.

In Western music theory, a *chord* is defined to be the basic element of harmony and consists of three or more pcs. A chord with three pcs is called a *triad*, and these chords are the ones that hold our particular interest. A triad is consists of a root pc and two other pcs that are a third and a fifth above the root. A *major* triad contains a major third and a perfect fifth, while a *minor* triad contains a minor third and a perfect fifth [3]. For example, the F triad is composed of the pcs F (the unison), A (the major third), and C (the perfect fifth), and the d triad is composed of the pcs D (the unison), F (the minor third), and A (the perfect fifth).

The second prominent kind of scale is the diatonic scale, which is constructed by a model based upon ratio intervals used by the Greeks [2, 8]. At first, it would seem logical to divide the keyboard evenly to construct a scale. This would be synonymous with finding a subgroup of $\mathbb{Z}_{12}$. However, this is not the case. In fact, the diatonic scale can be described as a maximally even set. As this concept does not pertain to the emphasis of this article, we refer the reader to [6] for definitions.

The other diatonic scales are translations in $\mathbb{Z}_{12}$ of the C-scale. For example, the key of E is obtained by adding 4 to each of the numerical values in the key of C. For our purposes, it suffices to consider only the chromatic scale and the diatonic scale in the key of C. We now define the sets of triads for the chromatic and diatonic scales.

Definition 1.1.

(i) $\mathcal{P}$ -The set of all major and minor triads of the chromatic scale (24 in total) which will be called the *consonance triad set.* These triads are: $\{\{0,4,7\},\{1,5,8\},\{2,6,9\},\{3,7,10\},\{4,8,11\},\{5,9,0\},$ $\{6,10,1\},\{7,11,2\},\{8,0,3\},\{9,1,4\},\{10,2,5\},\{11,3,6\},\{0,3,7\},$ $\{1,4,8\},\{2,5,9\},\{3,6,10\},\{4,7,11\},\{5,8,0\},\{6,9,1\},\{7,10,2\},$ $\{8,11,3\},\{9,0,4\},\{10,1,5\},\{11,2,6\}\}$.

(ii) $\mathcal{D}$ -The set of triads of the key of C: {C, d, e, F, G, a, b° } = $\{\{0,4,7\},\{2,5,9\},\{4,7,11\},\{5,9,0\},\{7,11,2\},\{9,0,4\},\{11,2,5\}\}$ (7 in total) which will be called *diatonic set.*

Each triad is represented by a triple in which the first pc represents the root, the second represents the third, and the third represents the fifth. However, at times we may represent a given triad as a permutation of the triple listed here depending on which pc is of interest.

Note that the C scale is the set of pcs $\{0,2,4,5,7,9,11\}$. We denote this set, $\mathbb{D}_{12}$. The B triad is composed of the three pcs B, D♯, F♯. However, we observe that D♯ and F♯ are not in the key of C. To compensate in traditional Western music, we construct the triad B with a minor third, D, and a diminished fifth, F. We call this the B diminished, denoted b°. Also note that $b^\circ \notin \mathcal{P}$, so $\mathcal{D} \not\subseteq \mathcal{P}$.

Definition 1.2. $\mathcal{D}_x = \{y \mid y \in \mathbb{D}_{12}, x \neq y, \{x,y\} \subset X \in \mathcal{D}\}$. Such a y is said to be compatible with x.

Remark 1.3.

(D1) $\forall x \in \mathbb{D}_{12}, x \notin \mathcal{D}_x$. That is, no pc appears twice in one triad.
(D2) Let $\mathcal{D}^1_x = \{X \mid X \in \mathcal{D}, \{x\} \subset X\}\ \forall x \in \mathbb{D}_{12}$. Then $|\mathcal{D}^1_x| = 3$. That is, a pc appears in exactly three distinct triads. Elements of $\mathcal{D}^1_x$ are called x-triads.
(D3) $\forall x \in \mathbb{D}_{12}, |\mathcal{D}_x| = 4$ and $|\mathbb{D}_{12} - (\mathcal{D}_x \bigcup x)| = 2$. That is, there are exactly four pcs compatible with x, and there are exactly two pcs that are neither x nor compatible with x in the diatonic set.
(D4) $\forall x, y \in \mathbb{D}_{12}$, if $\{x, y\} \subset X$ for some $X \in \mathcal{D}$, then there is at most one $Y \in \mathcal{D}$ such that $Y \neq X$ and $\{x, y\} \subset Y$. A pair of pcs appear in at most two triads.
(D5) $\forall x, y \in \mathbb{D}_{12}, x \in \mathcal{D}_y \Leftrightarrow y \in \mathcal{D}_x$. Compatibility satisfies the symmetric property.

Lemma 1.4. *If $\{x, y, z\}$ is a triad of the diatonic set, there is no pc $w \in \mathbb{D}_{12}$ such that $\{x, w\}$, $\{y, w\}$, and $\{z, w\}$ are subsets of triads in the diatonic set.*

Proof. Assume the contrary. That is, assume there exists a pc $w \in \mathbb{D}_{12}$ such that $\{x, w\}$, $\{y, w\}$, and $\{z, w\}$ are all subsets of triads in the diatonic set. Now, by (D3) $|\mathcal{D}_w| = 4$, so there is another pc v such that $v \in \mathcal{P}_w$. Thus, $\{x, w, v\}$, $\{y, w, v\}$, and $\{z, w, v\}$ must be triads of the diatonic set. However, this contradicts (D4) as $\{w, v\}$ appears in three triads. □

2. Edge Harmonies

We now have the tools and knowledge to begin our discussion. Label the edges of a graph with pcs in $\mathbb{Z}_{12}$ in such a manner that the vertices correspond to a common triad. As a simple first example, consider K_4 labeled as in Figure 2. Here each vertex is incident with edges labeled 0, 4, and 7. Hence each vertex corresponds to the C major triad. Such labels are easily characterized. First, some definitions:

Definition 2.1. A cubic graph is said to have a *harmony* if the edges can be labeled with elements of $\mathbb{Z}_{12}$ so that the labels of the edges incident with any vertex correspond to a triad. Such graphs are said to be *harmonic.*

Definition 2.2. A harmonic graph is:

(i) *tonal* if only one triad is induced by the edge labeling.
(ii) *diatonic* if every triad of $\mathcal{D}$ is induced by the graph's labeling.
(iii) *consonant* if every triad of $\mathcal{P}$ is induced by the graph's labeling.

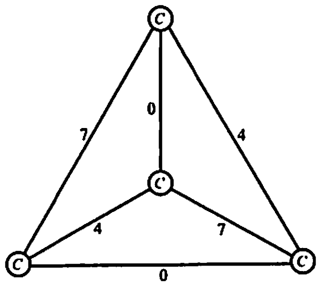

FIGURE 2. K_4 - the tetrahedron tonal

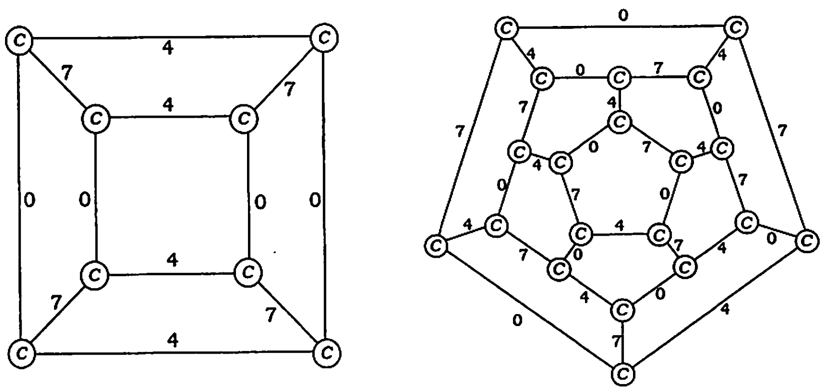

FIGURE 3. the cube and dodecahedron tonal

Nice examples of tonal graphs are the three cubic Platonic Solids: the tetrahedron, the cube, and the dodecahedron, shown in Figures 2 and 3 with labelings.

Characterizing tonal graphs is rather easy. The chromatic index of a graph G is the minimum integer k such that the edges of G can be labeled with k colors, no two incident edges having the same color. Recall Vizing's Theorem [5]:

Theorem 2.3 (Vizing's Theorem). *The chromatic index of any graph G is either $\Delta(G)$ or $1 + \Delta(G)$.*

Hence cubic graphs have chromatic index either three or four. Graphs with chromatic index Δ and $\Delta + 1$ are called respectively class 1 and class 2.

Theorem 2.4. *A cubic graph is tonal if and only if it is of class 1.*

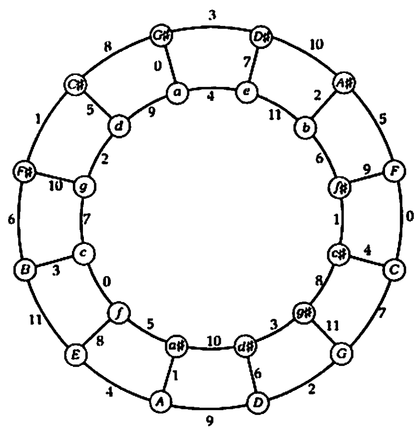

FIGURE 4. $P(12,1)$ consonant

Proof. Color the edges in three colors, calling the colors 0, 4, and 7. Every vertex is incident with the edges of one triad. The argument is reversible. □

Now the Generalized Petersen Graph $P(n,2)$ is hamiltonian and therefore class 1 and therefore tonal if $n \not\equiv 5 \bmod 6$ [1]. The Petersen Graph $P(5,2)$ is class 2 and thus not tonal.

The emphasis of our discussion is on diatonic graphs. However, as an easy example of a consonant graph, consider $P(12,1)$ labeled as in Figure 4.

Now we turn to the more interesting property of diatonic graphs. First, some examples: Figure 5 show the Generalized Petersen Graphs $P(7,1)$, $P(7,2)$, $P(7,3)$, and $P(14,5)$ with appropriate labelings. Additionally, we can further characterize such diatonic harmonies.

Theorem 2.5. *Let G be a diatonic graph. Then G has at least 14 vertices.*

Proof. Let G be a diatonic graph. The sum of the degrees of the vertices is equal to twice the number of edges. Since G is cubic, $3n = 2e$, and so G has an even number of vertices.

Now, G is diatonic so each of the seven triads of $\mathbb{D}_{12}$ appears on at least one vertex. Thus $n > 7$. We can assume that the seven triads appear on seven of the vertices. Create a listing of the n triads appearing on the n vertices of G. In the listing, all seven triads appear at least once, so each pc is written at least three times. Each edge is incident with two vertices,

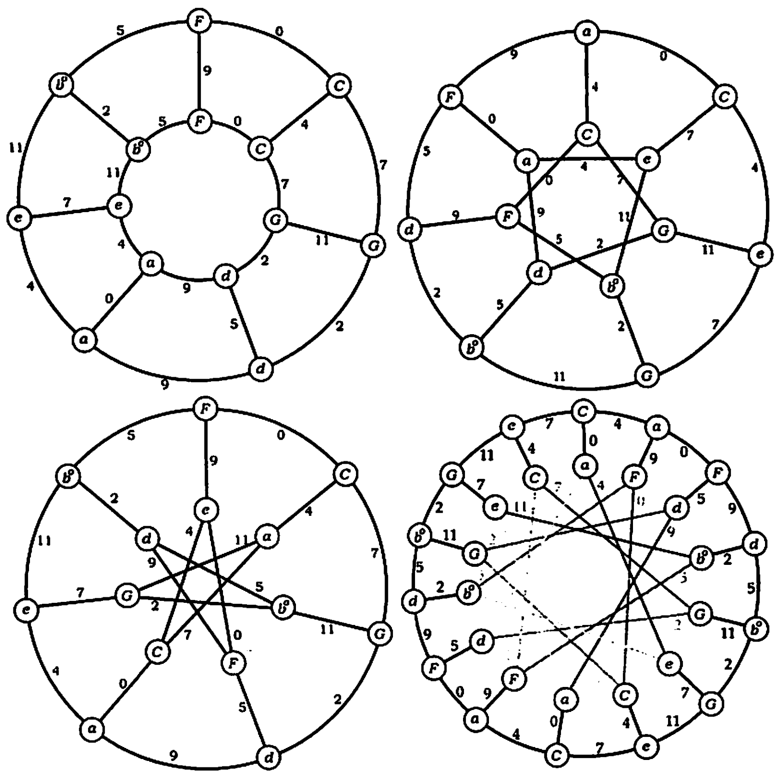

FIGURE 5. $P(7,1)$ $P(7,2)$ $P(7,3)$ $P(14,5)$ diatonic

so each edge corresponds to a pc in two triads in the listing. Since each pc must be counted an even number of times, and each pc is already written three times, each pc must appear at least once more in the listing of the triads appearing on the $n-7$ vertices left over.

If $n = 8$, then the eighth vertex will be a repeated triad. However, each of the seven pcs would have to be a label of an edge incident with the last vertex. This is a contradiction to the fact that G is cubic.

Let $n = 10$. In listing the ten triads, there are thirty pcs. There are three vertices and at most nine edges yet to be determined, and all seven pcs must appear at least one more time. Thus, there are nine pcs not yet listed. From this information, we deduce the system of equations:

$$4a + 6b = 30\ ,\ a + b = 7,$$

where a represents the number of pcs in $\mathbb{Z}_{12}$ listed four times, and b six times. We find that $a = 6$, and $b = 1$.

Thus, the three remaining vertices are incident to a set of edges in which each of the seven pcs appear in the labeling, six pcs appear four times, and one pc appears six times. Each has already appeared three times, so one pc must appear three more times. Choose one pc to be in all three triads, then those triads are determined, and by (D3) not all seven pcs will appear. Hence there are no solutions to label G.

Let $n = 12$. Then there are five vertices, v_0, v_1, v_2, v_3, and v_4, and at most fifteen edges not determined, and once again all seven pcs must appear at least one more time.

If one pc appears ten times in our list of the twelve triads, then that pc appears in three of the first seven triads and seven more triads. However, there are only five triads not yet listed. Thus, no pc appears ten times.

Likewise, if a pc appears eight times in this list of twelve triads, then that pc appears in the five triads not yet listed. By (D3), there are two pcs that will not appear in these five triads. Thus, no pc appears eight times. From this, we deduce the system of equations:

$$4a + 6b = 36 \ , \ a + b = 7,$$

where a represents the number of pcs in $\mathbb{Z}_{12}$ listed four times, and b six times. We find that $a = 3$ and $b = 4$.

Thus, the five remaining vertices are incident to each of the seven pcs, three pcs appear four times, and four pcs appear six times. Hence four pcs need to appear three more times, and three pcs need to appear one more time. Without loss of generality, say 0 appears three more times.

Case 1: Each 0-triad appears in the five remaining vertices.

Then, without loss of generality, $v_0 = \{0,4,7\}, v_1 = \{9,0,4\}$, and $v_2 = \{5,9,0\}$. Then 4 appears a total of five times in our list, and thus 4 must appear one more time. The 4-triads are $\{0,4,7\}, \{9,0,4\}$, and $\{4,7,11\}$. Now 0 cannot appear again. Thus $v_3 = \{4,7,11\}$. However, both 7 and 9 appear five times in our list, and they must both appear again. However, $9 \notin \mathcal{D}_7$ and $7 \notin \mathcal{D}_9$, and only one vertex lacks a label. Hence, not every 0-triad appears.

Case 2: The same 0-triad appears on v_0 and v_1.

Say that $v_0 = v_1 = \{0, x, y\}$, then x and y must appear once more. Recall that 0 appears on v_2. Assume that neither x nor y appear at v_2. Then either x and y appears at the same vertex or one appears at v_3 and the other at v_4. Assume the prior, that is x and y appear at v_3. There is another pc z that must appear three more times, so z appears at v_2, v_3, and v_4. Now v_2 must have another pc w so $v_2 = \{0, z, w\}$, and so $\mathcal{D}_z = \{0, x, y, w\}$. Since $0, x, y$ have appeared three times already, w must be at v_4. However, now w has appeared five times in our listing, and this cannot happen. Thus x appears at v_3, and y appears at v_4. Once again, z must appear three times, so z appears at v_2, v_3, and v_4. This contradicts Lemma 1.4.

Hence, either x or y appear at v_2. Say x appears at v_2, and y appears at v_3. Then, there is a pc z that appears three more times, and so z appears at v_2, v_3 and v_4. To complete the triad at v_3, there is another pc w that appears at v_3. Then $\mathcal{D}_z = \{0, x, y, w\}$. Since 0, x and y have appeared three times already, w must appear with z at v_4. However, w appears an odd number of times. Once again, this cannot occur, and so the same 0-triad cannot appear at v_0 and v_1.

Case 3: The same 0-triad appears on v_0, v_1, and v_2.

Obviously then v_0, v_1, and v_2 contain the same pcs. These three pcs show up a total of six times. There must be one more pcs that appears six times, which implies that this pc appears three times in v_3 and v_4. This cannot happen without contradicting (D1). Thus there is no diatonic graph on 12 vertices. □

In light of the previous theorem, the only possible Platonic Solid candidate to be diatonic is the dodecahedron. In fact:

Theorem 2.6. *The dodecahedron is diatonic.*

Proof. See the labelings of edges in Figure 6. □

Examples of diatonic graphs are not limited to the Platonic Solids. There are numerous examples. As previously illustrated, Generalized Petersen Graphs can be labeled with a diatonic harmony. First, the Generalized Petersen Graph is defined as follows:

Definition 2.7. Let $n \geq 3$ and $1 \leq k \leq n-1$. The *Generalized Petersen Graph* $P(n,k)$ is defined as follows:
Vertex set: $V = \{v_0, v_2, \ldots, v_{n-1}\} \bigcup \{w_0, w_2, \ldots, w_{n-1}\}$

$$\text{Edge Set: } E = \begin{cases} \{v_i \sim v_{i+1}\} = e_i & \text{for all } i \\ \{v_i \sim w_i\} = s_i & \text{for all } i \\ \{w_i \sim w_{i+k}\} = f_i & \text{for all } i \end{cases}$$

where subscript arithmetic is modulo n.

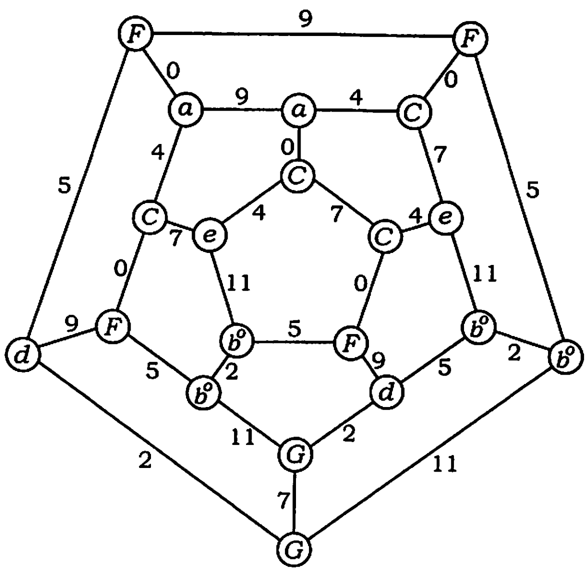

FIGURE 6. the dodecahedron with a diatonic harmony

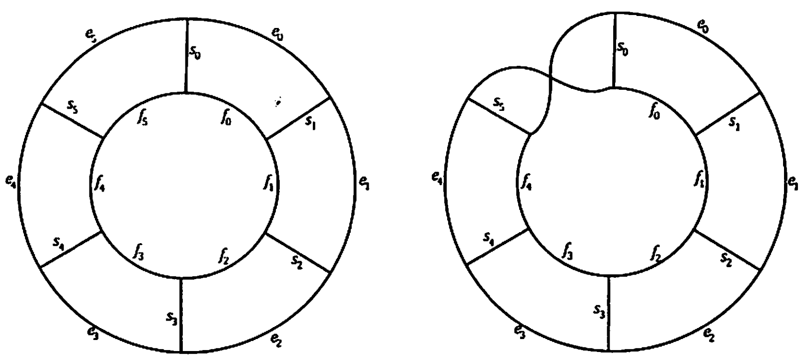

FIGURE 7. Cylinder $P(6,1)$ and Möbius Ladder M_6

Definition 2.8. If $k = 1$, $P(n,k) = P(n,1)$ is called a *cylinder.*
The *Möbius Ladder* M_n is defined like $P(n,1)$ but with $v_n \sim w_1, w_n \sim v_1$ rather than $v_n \sim v_1, w_n \sim w_1$.

The cylinder $P(6,1)$ and the Möbius Ladder M_6 are shown in Figure 7. See Figure 8 for $P(14,1)$ and $P(7,2)$.

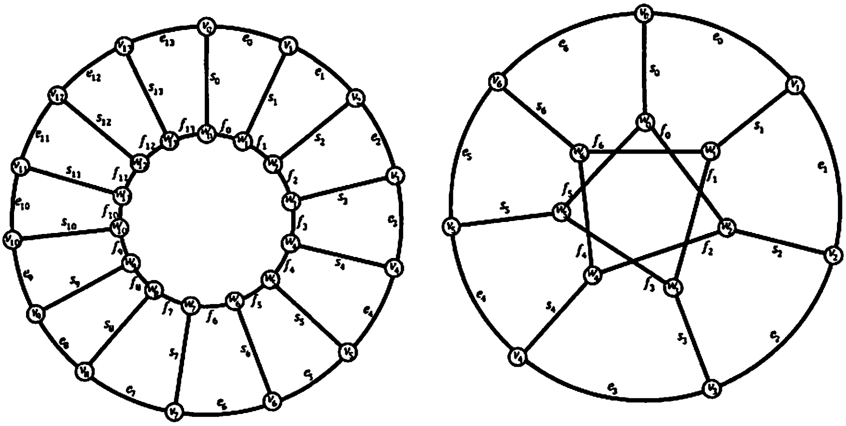

FIGURE 8. $P(14,1)$ and $P(7,2)$ with notation

subscript mod 7	0	1	2	3	4	5	6
s_i label	9	4	11	5	0	7	2
e_i label	0	7	2	9	4	11	5
f_i label	0	7	2	9	4	11	5

TABLE 2. edge labels for $P(7n,1)$

subscript mod 7	0	1	2	3	4	5	6
v_i and w_i labels	5,9,0	0,4,7	7,11,2	2,5,9	9,0,4	4,7,11	11,2,5
vertex triads	F	C	G	d	a	e	b°

TABLE 3. triad labels of $P(7n,1)$

Using these conventions, the edges of $P(7n,1)$ will be labeled with pcs of $\mathbb{Z}_{12}$ as in Table 2.

Note that the edge labels are precisely the pcs in the diatonic scale in the key of C. With these labels on the edges, each vertex is incident with three edges with labels.

Also note that the triads in Table 3 are precisely the 7 triads constituting the key of C. Hence $P(7n,1)$ is diatonic. Figure 9 shows the labeling of $P(14,1)$ with vertex v_0 at the north pole.

In the figure, edges are shown with their $\mathbb{Z}_{12}$ labels and vertices are given their appropriate triad labels. Note that the labels on the e_i's, on the f_i's,

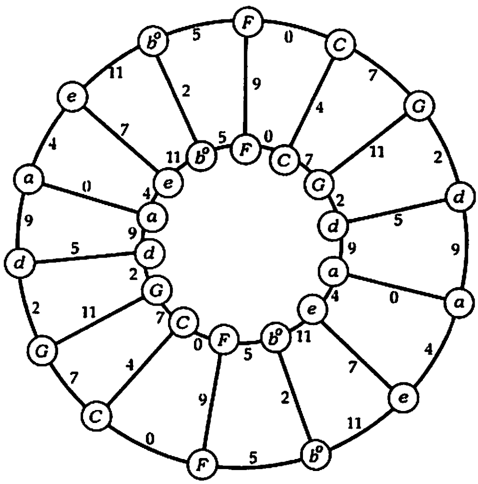

FIGURE 9. $P(14,1)$ diatonic

subscript mod 7	0	1	2	3	4	5	6
s_i label	4	7	11	2	5	9	0
e_i label	0	4	7	11	2	5	9
f_i label	7	11	2	5	9	0	4
triad at v_i	a	C	e	G	b°	d	F
triad at w_i	C	e	G	b°	d	F	a

TABLE 4. labels for $P(7n,2)$

subscript mod 7	0	1	2	3	4	5	6
s_i label	9	4	11	5	0	7	2
e_i label	0	7	2	9	4	11	5
f_i label	0	7	2	9	4	11	5
v_i triad	F	C	G	d	a	e	b°
w_i triad	a	e	b°	F	C	G	d

TABLE 5. labels for $P(7n,3)$

and on the s_i's respectively progress by fifths with one step a diminished fifth. By a similar method, it can be shown that the graphs $P(7n,2)$ and $P(7n,3)$ are diatonic. The details are shown in the tables 4 and 5.

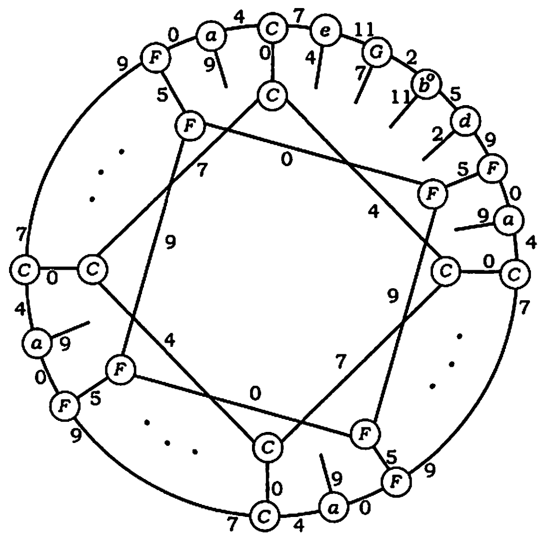

FIGURE 10. $P(28,7)$ diatonic

Note that the edge labelings shown in Table 4 progress in steps of major and minor thirds. So $P(7n,2)$ is diatonic.

Table 5 shows the label scheme for $P(7n,3)$. Once again, the edge labeling here progresses in steps of fifths as was the case for $P(7n,1)$. So $P(7n,3)$ is diatonic.

It is well known that for $1 \leq k < n$, $P(n,k)$ is isomorphic to $P(n,n-k)$ [11]. Hence, the knowledge that $P(7,1)$, $P(7,2)$, and $P(7,3)$ are diatonic implies that $P(7,6)$, $P(7,5)$, and $P(7,4)$ are also. Similarly, in what follows, it will always suffice to consider $1 \leq k < \frac{n}{2}$. The case $k = \frac{n}{2}$, when n is even, is not relevant, since, in this case $P(n,k)$ is not cubic.

Now, labelings of $P(7,k)$ for $1 \leq k \leq 3$ have been shown in Tables 2, 3, 4, and 5 to be extendable to labelings of $P(7n,k)$. In an entirely parallel way, labelings of $P(7n,j)$ can be constructed from labelings of $P(7,j)$ for $4 \leq j \leq 6$. Hence it follows that $P(7n,k)$ is diatonic for $1 \leq k \leq 6$.

Using the tables presented for $P(7n,k)$ for $1 \leq k \leq 3$ and the tables which exist but are not presented here for $4 \leq k \leq 6$, it will be shown that for all $n \geq 1$ and $1 \leq k < n$ with $k \not\equiv 0 \bmod 7$, $P(7n,k)$ is diatonic.

Theorem 2.9. *Let $n \geq 1$. Suppose $1 \leq k \leq n-1$ with $k \not\equiv 0 \bmod 7$. Then $P(7n,k)$ is diatonic.*

subscript mod 7	0	1	2	3	4	5	6
s_i label	9	4	11	5	0	7	2
e_i label	0	7	2	9	4	11	5
f_i label i odd	5	0	7	2	9	4	11
f_i label i even	0	7	2	9	4	11	5
v_i, w_i triad	F	C	G	d	a	e	b°

TABLE 6. labels for $P(7n, 7)$ when $n = 2k, k \geq 2$

Proof. Let $1 \leq r \leq 6$ with $k \equiv r \bmod 7$. We consider the graph $P(7n, r)$, which is diatonic according to the labeling given in one of the six tables. Interpret the table of $P(7n, r)$ as a labeling of the edges of $P(7n, k)$. Note that in this interpretation, the edges e_i and s_i play exactly the same role in $P(7n, r)$ as in $P(7n, k)$, that is, e_i joins v_i to v_{i+1}, and s_i joins v_i to w_i. However, the edge f_i in $P(7n, k)$ joins w_i and w_{i+k}, whereas the edge of the same name joins w_i and w_{i+r} in the graph $P(7n, r)$.

The labeling of edges in $P(7n, r)$ is known to give every triad of the diatonic scale. It must be shown that the same is true for $P(7n, k)$ in the interpretation above. First, consider vertex v_i. The edges incident with v_i are e_{i-1}, e_i, and s_i. These edges are identical in $P(7n, k)$ and $P(7n, r)$, so the vertices v_i have the same triads in $P(7n, k)$ as in $P(7n, r)$. Now consider the vertices w_i. The edges incident with w_i in $P(7n, k)$ are s_i, f_i, and f_{i-k}. In $P(7n, r)$, the edges incident with w_i are s_i, f_i, and f_{i-r}. But $(i - r) - (i - k) = k - r \equiv 0 \bmod 7$ since $k \equiv r \bmod 7$. It follows that the label on f_{i-k} in $P(7n, k)$ is identical to the label on f_{i-r} in $P(7n, r)$. Hence w_i has the same triad in both labelings, and indeed $P(7n, k)$ is diatonic. □

This theorem raises the interesting question of the labelings of $P(7n, k)$ when $k \equiv 0 \bmod 7$. Now $P(14, 7)$ is not cubic and thus it need not be considered. For $P(7n, 7)$ with $n > 2$, the f_i's induce disjoint n-cycles.

Theorem 2.10. *Let $n = 2k, k \geq 2$. Then $P(7n, 7)$ is diatonic.*

Proof. Label edges according to table 6.

This ensures that w_j is a triad. Thus $P(7n, 7)$ is diatonic when $n = 2k, k \geq 2$. Consider for example $P(28, 7)$ as in Figure 10. □

Using Theorem 2.10, it is possible to show that still more of the $P(7n, 7r)$ are diatonic.

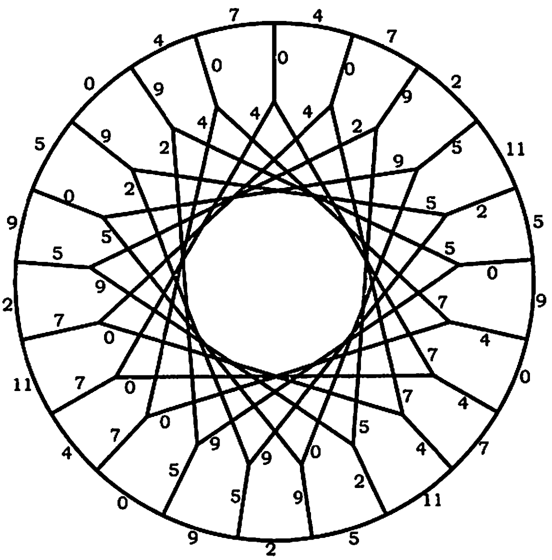

FIGURE 11. $P(21,7)$ diatonic

Lemma 2.11. *Using standard notational conventions in $P(m,k)$, let $d_{m,k} =$ $\gcd(m,k)$, c be the size of mutually disjoint cycles in the subgraph of $P(m,k)$ induced by the w vertices, and C be the number of such cycles. Then*

(i) $c = \frac{m}{d_{m,k}}$;
(ii) $C = d_{m,k}$.

Proof. Let C' be a cycle of induced by the w vertices such that $w_0 \in V(C')$. Then $C' = w_0, w_k, w_{2k}, \ldots, w_{ck}$ where c is the least positive integer such that $ck \equiv 0(\bmod m)$. As $\gcd(k/d_{m,k}, m/d_{m,k}) = 1$, $c(k/d_{m,k}) \equiv 0(\bmod m/d_{k,m})$. Hence $c \equiv 0(\bmod m/d_{k,m})$, and c is the smallest such positive integer. Thus $c = m/d_{k,m}$.

We can find the number of cycles induced by the w vertices by dividing the number of w vertices by the length of such a cycle. Hence $C = \dfrac{m}{m/d_{k,m}} =$ $d_{m,k}$. □

Theorem 2.12. *Let n, r be integers with $r \neq \frac{1}{2}n$. Suppose that for some positive exponent q, $2^q | n$ but $2^q \nmid r$. Then $P(7n, 7r)$ is diatonic.*

Proof. The cycle size c among the w_i vertices is $\frac{n}{\gcd(n,r)}$, which is even by the previous lemma and our assumption on n and r. That is, 2^q divides the

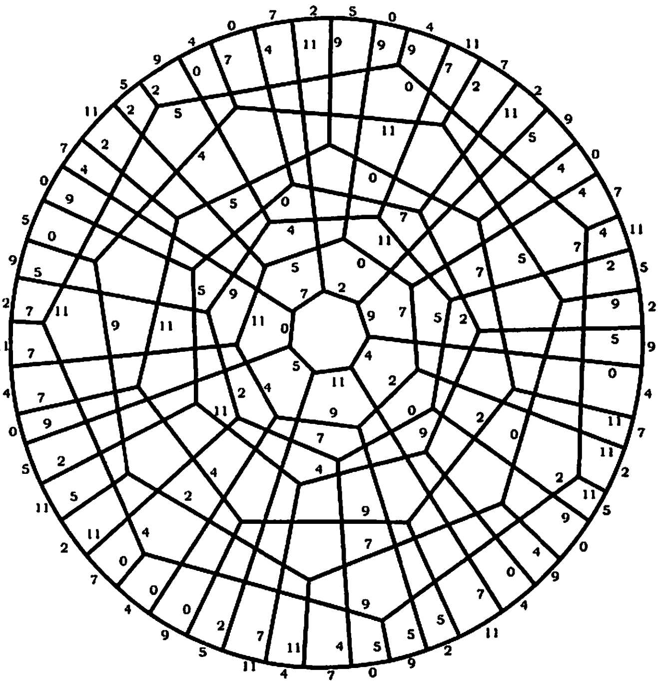

FIGURE 12. $P(49,7)$ diatonic

numerator in the expression for c, but 2^q does not divide the denominator. Even cycles may be edge-colored in two colors, and this allows use of the labeling scheme of table 6 following Theorem 2.10. □

Note that not every $P(7n,k)$ is covered by the preceding results. For example, $P(21,7)$, $P(35,7)$, and $P(49,7)$ are not covered. However, there are examples of diatonic labelings of $P(21,7)$ and $P(49,7)$ as in Figure 11 and 12. At the time of this writing, the status of $P(35,7)$ is not known.

We can further generalize the diatonic harmonies of the cylinders as the following theorem demonstrates.

Theorem 2.13. *If $n \geq 7$, then $P(n,1)$ is diatonic.*

Proof. The graphs $P(n,1)$ with n divisible by 7 have already been shown diatonic. Suppose now that $n = 7m + k$ with $1 \leq k \leq 6$. Then $P(n,1) = P(7m+k,1)$ can be visualized as $Q_{7m} \bigcup Q_k$ with the addition of four edges as shown in Figure 13 with the four new edges dotted.

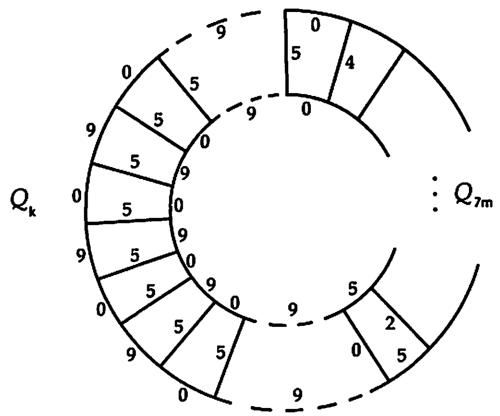

FIGURE 13. $P(n, 1)$ the case when k is even

Note that Q_{7m} is a spanning subgraph of $P(7m, 1)$, so we borrow the edge labeling of $P(7m, 1)$ from Theorem 2.9 for every edge of Q_{7m} except for the first and last spokes. In Figure 13, the labels of six of the edges are indicated. If $m \geq 2$, then all seven triads of the diatonic scale appear on the vertices of Q_{7m}. If $m = 1$, then the only missing triad of $\mathcal{D}$ is the F major triad $\{5, 9, 0\}$. In Q_k, as pictured in Figure 13, the edges will be referred to "inner", "outer", and "spokes". The spokes of Q_k will be given label 5.

If k is even, both the inner and outer edges will be labeled alternately 0 and 9, beginning at the "bottom" in Figure 13. There are four edges joining Q_{7m} with Q_k. These edges will labeled 9. Finally the end spokes of Q_{7m} are given labels 5 at the "top" and 0 at the "bottom". With these labels, $P(7m + 7, 1)$ has all triads of the diatonic scale. Note that the vertices at the ends of Q_k and at the ends of Q_{7m} and all the vertices of Q_k have the same F triad $\{5, 9, 0\}$.

If k is odd, the construction is slightly different. The inner and outer edges alternate with 9 and 0 beginning at the "bottom" with 9. The spokes of Q_k are again labeled 5. The top end spoke of Q_{7m} is labeled 5, and the bottom spoke is labeled 9. The top edges joining Q_k to Q_{7m} are labeled 9. The bottom edges joining Q_k to Q_{7m} are labeled 0. Again, the F triads makes multiple appearances, and $P(7m + k, 1)$ is diatonic. □

Corollary 2.14. *Möbius Ladders M_n for $n \geq 7$ are diatonic.*

Proof. In the labelings of $P(n, 1)$ exhibited in the previous theorem, corresponding inner and outer edges have the same label. □

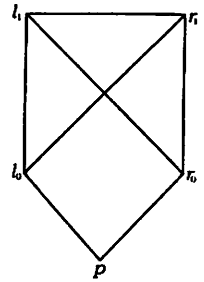

FIGURE 14. W with vertices labeled

After characterizing numerous graphs with harmonies, natural curiosity leads to the search for examples that do not satisfy such schemes. As mentioned, we can easily find examples that are not tonal, but cubic graphs with no diatonic labelings on 14 or more vertices are not easily demonstrated.

Definition 2.15. If H is a graph with $\Delta(H) = 3$ and if no diatonic graphs contain H as a subgraph, then H is said to be a *diatonic clash.*

Definition 2.16. W is a graph with vertex set $V = \{p, l_0, l_1, r_0, r_1\}$ and edge set $E = \{pl_0, pr_0, l_0l_1, r_0r_1, l_1r_1, l_0r_1, r_0l_1\}$ as in Figure 14.

Theorem 2.17. *W is a diatonic clash.*

Proof. Let G be a cubic graph containing W as a subgraph, let $e_1, e_2, e_3, e_4, e_5, e_6, e_7$ be the edges $pl_0, l_0l_1, l_1r_1, r_1r_0, r_0p, r_0l_1$, and l_0r_1 respectively, and let s be the edge incident with p where $s \notin E(W)$. We can be certain that s exists since G is cubic. In order for G to be diatonic, W must be colored with elements of $\mathbb{D}_{12}$ so that each vertex is a triad. Assume such a coloring exists. Then a triad appears at the vertex of l_1, and without loss of generality this triad is the C $\{0, 4, 7\}$. This leads to six cases.

Case 1: Let $e_3 = 0, e_6 = 4$, and $e_2 = 7$.

Then e_7 is incident with 0 and 7. Thus e_7 has labeled 4. Now r_1 is incident with 0 and 4, so e_4 may be labeled 7 or 9. Then r_0 is incident with either 4 and 7 or 4 and 9, thus e_5 is either labeled 0 or 11. Also, e_1 is incident with edges labeled 7 and 4, so e_1 may be labeled either 0 or 11. There is no available label for s, and Case 1 fails.

Case 2: Let $e_3 = 0, e_6 = 7$, and $e_2 = 4$.

Then e_4 is incident with edges labeled 0 and 7, so e_4 is labeled 4. Now e_7 is incident with edges labeled 0 and 4, so e_7 may be labeled 7 or 9. Now e_5 may be labeled either 0 or 11 as well as e_1. Now s cannot be labeled. Case 2 fails.

Case 3: Let $e_3 = 4, e_6 = 0$, and $e_2 = 7$.

Then e_7 is incident with edges labeled 4 and 7, so e_7 may be labeled either 0 or 11. Assume e_7 is labeled 0, then e_4 is incident with edges labeled 4 and 0 so e_4 must be either labeled 7 or 9. Thus e_5 must be labeled 4 or 5. Now e_1 is incident with edges labeled 0 and 7, and so e_1 must be labeled 4. Hence, there is no possible label for s. Thus e_7 must be labeled 11. Now e_4 is incident with edges labeled 11 and 4, so e_4 must be labeled 7, and so e_5 must be labeled 4. Now e_1 is incident edges labeled 7, 4, and 11. By Lemma 1.4, e_1 has no available label. Thus Case 3 fails.

Case 4: Let $e_3 = 7, e_6 = 0$, and $e_2 = 4$.

Then e_4 is incident with edges labeled 0 and 7, so e_4 must be labeled 4. Now e_7 is incident with edges labeled 4 and 7, then e_7 may be labeled either 0 or 11. Now e_1 and e_5 may be labeled either 7 or 9, and there is no available label for s. Thus Case 4 fails.

Case 5: Let $e_3 = 7, e_6 = 4$, and $e_2 = 0$.

Then e_7 is incident with edges labeled 0 and 7, so e_7 must be labeled 4. Then e_1 may be labeled either 9 or 11. Now e_4 is incident with edges labeled 7 and 4, so e_4 may be labeled either 0 or 11. Then e_5 is incident with edges labeled either 0 and 4 or 11 and 4, so e_5 may be labeled either 7 or 9. Once again, there is no available label for s, and so Case 5 fails.

Case 6: Let $e_3 = 4, e_6 = 7$, and $e_2 = 0$

Then e_7 is incident with edges labeled 0 and 4, so e_7 may be labeled either 7 or 9. Assume that e_7 is labeled 7, then e_1 must be labeled 4, and e_4 may be labeled either 0 or 11. Then e_5 is incident with edges labeled either 0, 4, and 7 or 11, 4, and 7. By Lemma 1.4 e_5 cannot be labeled. Thus e_7 must be labeled 9, then e_4 must be labeled 0. Then e_5 is incident with edges labeled 0 and 7, so e_5 must be labeled 4. By Lemma 1.4, there is no available label for e_1. Thus Case 6 fails.

We have then exhausted all possible cases for the labeling of the subgraph W, and so G is not diatonic. Thus W is a clash. □

With the knowledge that W is a diatonic clash, the existence of graphs that have no diatonic labelling is assured.

Theorem 2.18. *If n is even and $n \geq 4$, there is a cubic graph G with n vertices such that G has no diatonic labelling.*

Proof. The smallest cubic graph has four vertices. By Theorem 2.5, there are in fact no diatonic graphs with fewer than 14 vertices. For non-diatonic

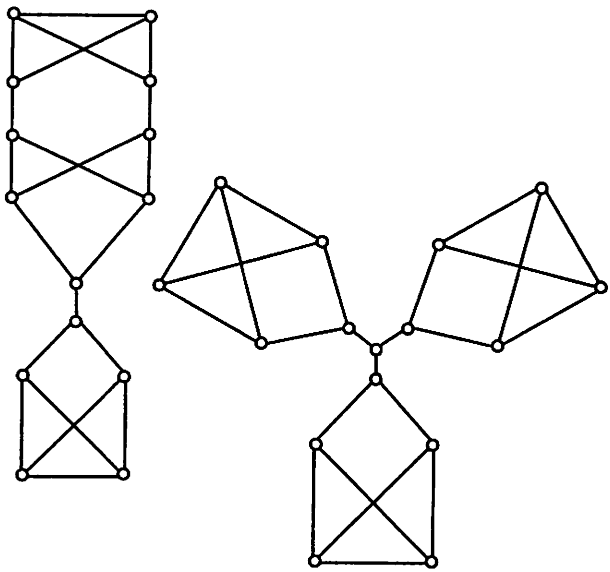

FIGURE 15. Graphs with 14 and 16 vertices that have no diatonic labeling

graphs with 14 and 16 vertices, see Figure 15. These graphs have no diatonic labelling because of the presence of W.

Now suppose $n \geq 18$. Let H be a cubic graph with $n - 14$ vertices. Delete one vertex leaving a subgraph J with $n - 15$ vertices, of which three have degree 2. Attach W to each of those three vertices in the obvious way. The resulting cubic graph G has n vertices and is non-diatonic due to the clashes. □

There are many interesting questions to pursue. Are there clashes other than W? If so, is there a finite collection of clashes whose exclusion guarantees that a cubic graph is diatonic? We are also pursuing questions similar to those above for the class of consonant graphs. Many interesting questions remain in both of these classes.

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