Some properties of the quasi-cross-cut partition and the dimension of bivariate continuous spline space*

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Abstract

In this paper, the author study the relation of vertices, edges and cells of the quasi-cross-cut partition. Moreover, the three-term recurrence relations of $\dim(S_d^0(\Delta))$ over the quasi-cross-cut partition and the triangulation are presented.
Keywords: bivariate splines, dimension, quasi-cross-cut partition, triangulation

1 Introduction

The bivariate spline spaces have been studied intensively in the past 40 years [3, 4]. In fact, spline has become a kind of fundamental tool for combinatorics, computational geometry, numerical analysis, approximation theory, and so on.

Let Ω be a simply connected domain in $\mathbb{R}^2$ and Δ be a *partition* of Ω given by a finite line segments. Denote by δ_i, $i = 1, \cdots, T$, all of the *cells* of Δ, where T is the number of cells of Δ. These line segments that form the boundary of each cell are called *edges*. Intersection points of the edges are called the *vertices*.

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Denote by $\mathbf{P}_d(\Delta) := \{p|p|_{\delta_i} \in \mathbf{P}_d, i = 1, 2, \cdots, T\}$ the collection of piecewise polynomials with total degree d, where $\mathbf{P}_d$ denotes the collection of bivariate polynomials with total degree d with real coefficients. For an integer $0 \leq d$, we say

$$S_d^0(\Delta) := \{s \in C^0(\Omega) \cap \mathbf{P}_d(\Delta)\}$$

is the *space of bivariate continuous piecewise polynomials* with degree d over Δ [3, 4].

The partition is called *triangulation* if it's the division of Ω into a set of triangles, with the restriction that each triangle side is entirely shared by two adjacent triangles. The partition is called *cross-cut partition* if Ω is divided by a finite straight lines, which are called cross-cut lines. Edge segments that starting from interior vertices and ending on boundary of Ω are called *rays*. We say a partition is a *quasi-cross-cut partition* if each edge is either a part of cross-cut line or a part of ray, denoted by Δ_{qc}. The explicit formulas on the dimension of $S_d^0(\Delta_t)$ and $S_d^0(\Delta_{qc})$ were presented by Billera [1], and Chui and Wang [2, 4] respectively.

The relationship of vertices, edges and cells of the partition is a very interesting problem for combinatorics, e.g., the famous Euler formula of triangulation. Unfortunately, there are few papers studied on the general partitions. In this short paper, we study the relations of vertices, edges and cells of the quasi-cross-cut partition. The idea is originally from [6], which presented a method to construct the Lagrange interpolation sets for the bivariate spline space over cross-cut partition. Furthermore, we give the three-term recurrences relations of the dimensions of $\dim\left(S_d^0(\Delta_t)\right)$ and $\dim\left(S_d^0(\Delta_{qc})\right)$.

2 Main results

Let L, R, T, V, V^o, E, E^o be the numbers of cross-cut lines, rays, cells, vertices, interior vertices, Edges, and interior edges of Δ_{qc} respectively. Denote by n_i the number of cross-cut lines and rays passing through the i-th interior vertex of $\Delta_{qc}, i = 1, \cdots, V^o$. The following theorem shows the relation of L, T and V^o.

Theorem 1. $T = L - V^o + 1 + \sum_{i=1}^{V^o} n_i$.

Proof. We prove the theorem by the mathematical induction on the number of rays R of Δ_{qc}.

(I) The statement is true for $R = 0$ since Δ_{qc} is a cross-cut partition, which has been proven in [6].

(II) Assume that the statement is true for $R = k$. To prove that the statement is also true for $R = k+1$. Suppose Δ'_{qc} is a quasi-cross-cut partition of Ω, $L, T', V^{o'}$ are the numbers of cross-cut lines, cells and interior vertices of Δ'_{qc} respectively, and Δ_{qc} is a partition given by adding a ray $l = 0$ to Δ'_{qc}. Suppose n_i is the number of cross-cut lines and rays passing through i-th interior vertex of $\Delta'_{qc}, i = 1, \cdots, V^{o'}$.

If $l = 0$ meets e interior edges of Δ'_{qc}, then there are e new interior vertices on Δ_{qc}, where each of them is the intersection of two lines on Δ_{qc}. If $l = 0$ passes through $t(t \leq V^{o'})$ interior vertices of Δ'_{qc}, denoted by $\{V_j\}_{j=k_1}^{k_t}$, then let $n_j = n_j + 1, j = k_1, \cdots, k_t$. Let $V^o = V^{o'} + e$, and for new interior vertices, $n_i = 2, i = V^{o'} + 1, \cdots, V^o$. Therefore, we have

$$
\begin{aligned}
T &= T' + e + t \\
&= L - V^{o'} + 1 + \sum_{i=1}^{V^{o'}} n_i + e + t \\
&= L - V^o + 1 + \sum_{i=1}^{V^o} n_i.
\end{aligned} \tag{1}
$$

Then the statement is true for $R = k+1$. This completes the proof. □

For the quasi-cross-cut partition, Xu and Wang obtained another relation of L and E^o.

Theorem 2. ([5]) $E^o = L + \sum_{i=1}^{V^o} n_i$.

Since $V - E = V^o - E^o$, and by Theorem 1 and 2, we have the following interesting result, which is similar to the famous Euler formula of triangulation.

Proposition 3.

$$T + V^o - E^o = 1, \tag{2}$$
$$T + V - E = 1. \tag{3}$$

Chui and Wang presented the explicit formula on the dimension of continuous piecewise polynomials in [2, 4].

Theorem 4. ([2, 4])

$$\dim(S_d^0(\Delta_{qc})) = \binom{d+2}{2} + L\binom{d+1}{2} + \sum_{i=1}^{V^o} q_d(n_i), \tag{4}$$

where

$$q_d(n_i) = \binom{d}{2} n_i - \binom{d+1}{2}, \quad i = 1, \cdots, V^o. \tag{5}$$

Theorem 5. *Let d, k, l be the natural numbers and $k \leq d$. Then*

$$\dim(S_{d+l}^0(\Delta_{qc})) - \dim(S_{d+l-k}^0(\Delta_{qc})) = \dim(S_d^0(\Delta_{qc})) - \dim(S_{d-k}^0(\Delta_{qc})) + Tkl, \tag{6}$$

where $d \geq k$ if Δ_{qc} has interior vertices, $d \geq k-1$ if Δ_{qc} has no interior vertices and $\dim(S_0^0(\Delta_{qc})) = 1$.

Proof. If Δ_{qc} has interior vertices, by (4), (5) and Theorem 1, we have

$$\begin{aligned} &\dim(S_{d+l}^0(\Delta_{qc})) - \dim(S_{d+l-k}^0(\Delta_{qc})) - \left(\dim(S_d^0(\Delta_{qc})) - \dim(S_{d-k}^0(\Delta_{qc}))\right) \\ &= \left(L - V^o + 1 + \sum_{i=1}^{V^o} n_i\right) kl \\ &= Tkl. \end{aligned}$$

Δ_{qc} with no interior vertices means that it is a cross-cut partition with no interior vertices. By conformality method of smoothing cofactor [3, 4], it's easy to obtain

$$\begin{aligned} &\dim(S_{d+l}^0(\Delta_{qc})) - \dim(S_{d+l-k}^0(\Delta_{qc})) - \left(\dim(S_d^0(\Delta_{qc})) - \dim(S_{d-k}^0(\Delta_{qc}))\right) \\ &= (L+1)kl \\ &= Tkl. \end{aligned}$$

□

By (6) and Theorem 1, the three-term recurrence relations of dimensions of $S_d^0(\Delta_{qc})$ is obtained, which is more easy to understand than (4).

Proposition 6.

$$\dim(S_0^0(\Delta_{qc})) = 1, \tag{7}$$

$$\dim(S_1^0(\Delta_{qc})) = T - V^o + 2, \tag{8}$$

$$\dim(S_d^0(\Delta_{qc})) = 2\dim(S_{d-1}^0(\Delta_{qc})) - \dim(S_{d-2}^0(\Delta_{qc})) + T, d \geq 2. \tag{9}$$

Proof. Since (9) can be deduced from (6) by setting $k = l = 1$, we only need to prove (8). By (4), (5) and $T = L - V^o + 1 + \sum_{i=1}^{V^o} n_i$, we get

$$\dim(S_1^0(\Delta_{qc})) = 3 + L + \sum_{i=1}^{V^o}(n_i - 2) = T - V^o + 2.$$

□

If Δ_{qc} is a triangulation, we have $V = \dim(S_1^0(\Delta_{qc})) = T - V^o + 2$. Then we can get some results using (2), which may be interesting for combinatorics.

Proposition 7. *If Δ_{qc} is a triangulation, then*

$$V = E^o - 2V^o + 3, \tag{10}$$
$$E = 2E^o - 3V^o + 3, \tag{11}$$
$$T = E^o - V^o + 1. \tag{12}$$

Let V, E, T be the numbers of vertices, edges and triangles of triangulation Δ_t. Billera gave the explicit formula on $\dim(S_d^0(\Delta_t))$ using the algebra [1].

Theorem 8. ([1])

$$\dim(S_d^0(\Delta_t)) = V\binom{d-1}{0} + E\binom{d-1}{1} + T\binom{d-1}{2}, \tag{13}$$

where $\binom{0}{0} = 1$.

Using the similar method of quasi-cross-cut partition, we get the following results.

Theorem 9. *Let d, k, l be the natural numbers and $k \le d$. Then*

$$\dim(S_{d+l}^0(\Delta_t)) - \dim(S_{d+l-k}^0(\Delta_t)) = \dim(S_d^0(\Delta_t)) - \dim(S_{d-k}^0(\Delta_t)) + Tkl, \tag{14}$$

where $d \ge k$ if Δ_t has interior vertices, $d \ge k - 1$ if Δ_t has no interior vertices and $\dim(S_0^0(\Delta_t)) = 1$.

Proposition 10.

$$\dim(S_0^0(\Delta_t)) = 1, \tag{15}$$
$$\dim(S_1^0(\Delta_t)) = V, \tag{16}$$
$$\dim(S_d^0(\Delta_t)) = 2\dim(S_{d-1}^0(\Delta_t)) - \dim(S_{d-2}^0(\Delta_t)) + T, d \ge 2. \tag{17}$$

Remark 11. *From Theorem 9 and Proposition 10, we find*

$$V + E = \dim(S_2^0(\Delta)) = 2\dim(S_1^0(\Delta)) - 1 + T = 2V - 1 + T,$$

which means $T + V - E = 1$. *It is the famous Euler formula for the triangulation.*

3 Conclusions

In this paper, the relation of vertices, edges and cells of the quasi-cross-cut partition is studied. The three-term recurrence relations of the dimension of bivariate continuous spline space $S_d^0(\Delta)$ over the quasi-cross-cut partition and the triangulation are presented. The author will study the problems for the arbitrary partition Δ and the bivariate continuous spline space $S_d^\mu(\Delta)$ with $d \geq \mu \geq 1$ in future.

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