An Inequality on Laplacian Eigenvalues of Connected Graphs

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Abstract

Using Cioabă's inequality on the sum of the 3rd powers of the vertex degrees in connected graphs, we present an inequality on the Laplacian eigenvalues of connected graphs.

Keywords : Inequality, Laplacian Eigenvalue.

We consider only finite undirected graphs without loops and multiple edges. Notation and terminology not defined here follow that in [1]. Let G be a graph of order $n \geq 2$ with e edges. We assume $n \geq 2$ in order to avoid trivialities. We also assume that the vertices in G are ordered such that $d_1 \geq d_2 \geq ... \geq d_n$, where d_i, $1 \leq i \leq n$, is the degree of vertex v_i in G. We define $\Sigma_k(G)$ as $\Sigma_{i=1}^{n} d_i^k$. For each vertex v_i, $1 \leq i \leq n$, m_i is defined as the sum of degrees of vertices that are adjacent to v_i. Obviously, $\Sigma_{i=1}^{n} m_i = \Sigma_{i=1}^{n} d_i^2 = \Sigma_2(G)$. The Laplacian of a graph G is defined as $L(G) = D(G) - A(G)$, where $D(G)$ is the diagonal matrix of the degree sequence of G and $A(G)$ is the adjacency matrix of G. The eigenvalues $0 = \lambda_1(G) \leq \lambda_2(G) \leq ... \leq \lambda_n(G)$ of $L(G)$ are called the Laplacian eigenvalues of the graph G. $B_{n,t}$ is defined as the join between the complete graph K_t and the empty graph $\overline{K_{n-t}}$, where $1 \leq t \leq n$. Notice that $B_{n,t}$ is K_n when $t = n$ and $B_{n,t}$ is $K_{1,n-1}$ when $t = 1$.

The objective of this note is to prove the following theorem which is on an inequality for the Laplacian eigenvalues of graphs.

Theorem 1 Let G be a connected graph of order n with e edges and t triangles. Then each Laplacian eigenvalue λ_k, where $2 \leq k \leq n$, satisfies the following inequality

$$(2e-\lambda_k)^3+(n-2)^2\lambda_k^3$$
$$\leq (n-2)^2(\frac{2e-(d_1^2-d_n^2)+3n}{n}\Sigma_2(G)+\frac{2e(n-1)(d_1^2-d_n^2)}{n}-6t).$$

Moreover, if $3 \leq k \leq n-1$, then the equality holds if and only if G is K_n with $n \geq 2$; if $k=n$, then the equality holds if and only if G is K_n or $K_{\frac{n}{2},\frac{n}{2}}$ or $K_{1,n-1}$ with $n \geq 2$; if $k=2$, then the equality holds if and only if G is K_n with $n \geq 2$ or $B_{n,n-2}$ with $n \geq 3$.

The following result (Corollary 7 in [2]) obtained by Cioabă will be used in the proof of Theorem 1.

Theorem 2 [2] If G is a connected graph, then

$$\Sigma_3(G) \leq \frac{2e-(d_1^2-d_n^2)}{n}\Sigma_2(G)+\frac{2e(n-1)(d_1^2-d_n^2)}{n}.$$

Equality holds if and only if G is regular or $G=B_{n,t}$ for some t with $1 \leq t \leq n$.

The following four results will be used when we determine the graphs that make the equality happens in the inequality in Theorem 1.

Theorem 3 [5] Let G be a non-complete graph. Then $\lambda_2(G) \leq \kappa(G)$, where $\kappa(G)$ is the vertex connectivity of G.

Theorem 4 [6] Let G be a graph on n vertices and $2 \leq i \leq n$. Then $\lambda_i(G^C)=n-\lambda_{n-i+2}(G)$, where G^C is the complement of the graph G and $\lambda_i(G^C)$, $1 \leq i \leq n$, are the Laplacian eigenvalues of G^C.

Theorem 5 [7] Let G be a graph on n vertices. Then

$$\lambda_n(G) \leq 2+\sqrt{(d_1+d_2-2)(d_1+d_3-2)}.$$

If G is connected, the equality holds if and only if G is a regular bipartite graph or a path with three or four vertices.

Theorem 6 [8] Let G be a graph of order n. For any nontrivial subset X of the vertices of G, $X \neq \emptyset$, $X \neq V(G)$, we have

$$\lambda_2(G) \leq \frac{n\, e(X,X^C)}{|X||X^C|} \leq \lambda_n(G),$$

where X^C is $V(G)-X$ and $e(X,X^C)$ denotes the number of edges between X and X^C in G.

Theorem 3, Theorem 5, and Theorem 6 above were proved respectively by Fiedler in [5], Li and Zhang in [7], and Mohar in [8]. Theorem 4 can be found on Page 280 in [6].

Next we will prove Theorem 1.

Proof of Theorem 1. Since G is connected, we have $0 = \lambda_1 < \lambda_2 \leq ... \leq \lambda_n$. By the fact that

$$2e = \sum_{i=1}^{n} d_i = trace(L(G)) = \sum_{i=1}^{n} \lambda_i,$$

we have that, for any k with $2 \leq k \leq n$,

$$2e - \lambda_k = \sum_{i=2, i\neq k}^{n} \lambda_i.$$

By Hölder inequality, we have that

$$\sum_{i=2,i\neq k}^{n} \lambda_i \leq (\sum_{i=2,i\neq k}^{n} 1^{3/2})^{2/3} (\sum_{i=2,i\neq k}^{n} \lambda_i^{3/1})^{1/3},$$

Therefore

$$(2e - \lambda_k)^3 \leq (n-2)^2 \sum_{i=2,i\neq k}^{n} \lambda_i^3.$$

Notice that

$$\sum_{i=2,i\neq k}^{n} \lambda_i^3 = \sum_{i=2}^{n} \lambda_i^3 - \lambda_k^3 = \sum_{i=1}^{n} \lambda_i^3 - \lambda_k^3 = trace((L(G))^3) - \lambda_k^3.$$

Hence we have that

$$(2e - \lambda_k)^3 \leq (n-2)^2 (trace((L(G))^3) - \lambda_k^3), i.e.,$$

$$(2e - \lambda_k)^3 + (n-2)^2 \lambda_k^3 \leq (n-2)^2 trace((L(G))^3).$$

Now we evaluate $trace((L(G))^3)$. First notice that

$$(L(G))^3 = (D-A)^3 = D^3 - D^2A - DAD + DA^2 - AD^2 + ADA + A^2D - A^3.$$

It is easy to see that

$$trace(D^3) = \sum_{i=1}^{n} d_i^3, \qquad trace(D^2A) = 0,$$

$$trace(DAD) = 0, \qquad trace(DA^2) = \sum_{i=1}^{n} d_i^2,$$

$$trace(AD^2) = 0, \qquad trace(ADA) = \sum_{i=1}^{n} m_i,$$

$$trace(A^2D) = \sum_{i=1}^{n} d_i^2, \qquad trace(A^3) = 6t.$$

Recall that $\Sigma_{i=1}^{n} m_i = \Sigma_{i=1}^{n} d_i^2$, we have

$$trace((L(G))^3) = trace(D^3) - trace(D^2A) - trace(DAD)$$

$$+trace(DA^2) - trace(AD^2) + trace(ADA) + trace(A^2D) - trace(A^3)$$

$$= \sum_{i=1}^{n} d_i^3 + 3\sum_{i=1}^{n} d_i^2 - 6t.$$

From Theorem 2, we establish the desired inequality below

$$(2e - \lambda_k)^3 + (n-2)^2\lambda_k^3$$

$$\leq (n-2)^2(\frac{2e - (d_1^2 - d_n^2) + 3n}{n}\Sigma_2(G) + \frac{2e(n-1)(d_1^2 - d_n^2)}{n} - 6t).$$

Now we determine the graphs that make the inequality in Theorem 1 to become an equality when $3 \leq k \leq n-1$.

When G is K_n with $n \geq 2$, we have that $\lambda_k = n$ for each k with $2 \leq k \leq n$, $d_k = n-1$ for each k with $1 \leq k \leq n$, and $t = C(n,3) = n(n-1)(n-2)/6$. A simple computation shows that both sides of inequality in Theorem 1 are equal to $(n-2)^2(n-1)n^3$.

Now suppose that the inequality in Theorem 1 becomes an equality. Then

$$\sum_{i=2, i\neq k}^{n} \lambda_i \leq (\sum_{i=2, i\neq k}^{n} 1^{3/2})^{2/3}(\sum_{i=2, i\neq k}^{n} \lambda_i^{3/1})^{1/3},$$

must become an inequality. By the condition for an Hölder inequality becomes an equality, we have that $\lambda_2 = ... = \lambda_{k-1} = \lambda_{k+1} = ... = \lambda_n$. Let

v be the vertex in G such that v has the minimum degree. Set $X = \{v\}$ and apply Theorem 6, we can obtain that

$$\lambda_2 \leq \frac{nd(v)}{n-1} = \frac{n\delta}{n-1} \leq \lambda_n.$$

Suppose G is not a complete graph. We have by Theorem 3 that $\lambda_2 \leq \kappa \leq \delta$. Hence we have the following contradiction

$$\frac{n\delta}{n-1} \leq \lambda_n = \lambda_2 \leq \kappa \leq \delta.$$

Notice that when G is a complete graph, the inequality in Theorem 2 also becomes an equality.

Next we determine the graphs that make the inequality in Theorem 1 to become an equality when $k = n$.

When G is K_n with $n \geq 2$, we again have that both sides of inequality in Theorem 1 are the same. When G is $K_{\frac{n}{2},\frac{n}{2}}$, we have that $\lambda_k = \lambda_n = n$, $d_k = n/2$ for each k with $1 \leq k \leq n$, and $t = 0$. A simple computation shows that both sides of inequality in Theorem 1 are equal to $(n-2)^2 n^3 (n+6)/8$. When G is $K_{1,n-1}$ with $n \geq 2$, we have that $\lambda_k = \lambda_n = n$, $d_1 = n-1$ and $d_k = 1$ for each k with $2 \leq k \leq n$, and $t = 0$. A simple computation shows that both sides of inequality in Theorem 1 are equal to $(n-2)^2(n^3+n-2)$.

Now suppose that the inequality in Theorem 1 becomes an equality. Then

$$\sum_{i=2, i\neq k}^{n} \lambda_i \leq \Big(\sum_{i=2, i\neq k}^{n} 1^{3/2}\Big)^{2/3} \Big(\sum_{i=2, i\neq k}^{n} \lambda_i^{3/1}\Big)^{1/3},$$

must become an inequality. By the condition for an Hölder inequality becomes an equality, we have that $\lambda_2 = \lambda_3 = ... = \lambda_{n-1}$. If $\lambda_2 = \lambda_3 = ... = \lambda_{n-1} = \lambda_n$, then by a similar argument as in the case that $3 \leq k \leq n-1$, we can show that G is K_n with $n \geq 2$. Now we assume that $\lambda_2 = \lambda_3 = ... = \lambda_{n-1} < \lambda_n$. This assumpution implies that G is not complete since for each complete graph we have that $\lambda_2 = \lambda_3 = ... = \lambda_{n-1} = \lambda_n$. Notice that now the inequality in Theorem 2 must also become an equality. By Theorem 2, we have G is regular or $G = B_{n,t}$ for some t with $1 \leq t \leq n$.

We first consider the case that G is a regular graph and assume that the degree of each vertex in G is d. Since G is not complete, Theorem 3 implies that $\lambda_2 = \lambda_3 = \ldots = \lambda_{n-1} \le \kappa \le d$. From the fact that

$$\sum_{i=1}^{n} d_i = \sum_{i=1}^{n} \lambda_i,$$

we have that $nd = \sum_{i=1}^{n} \lambda_i \le (n-2)d + \lambda_n$. Thus $2d \le \lambda_n$. From Theorem 5, we have that

$$2d \le \lambda_n \le 2 + \sqrt{(d_1 + d_2 - 2)(d_1 + d_3 - 2)} = 2d.$$

Thus $\lambda_n = 2d$ and the inequality in Theorem 5 becomes an equality. Since G is connected, by Theorem 5 again, we have that G is a regular bipartite graph or a path with three or four vertices. Obviously, G cannot be a path with three or four vertices since G is a regular graph. Let X and Y be the partition of $V(G)$ in the regular bipartite G so that each edge in G has one end in X and another end in Y. Then $n = |V(G)| = |X| + |Y| \ge d + d = 2d$. From the fact that

$$\sum_{i=1}^{n} (d_i + d_i^2) = trace((L(G))^2) = \sum_{i=1}^{n} \lambda_i^2,$$

we have that $nd + nd^2 = (n-2)\lambda_2^2 + \lambda_n^2 \le (n-2)d^2 + 4d^2$. Thus $n \le 2d$. Hence $n = 2d$. Recall that $n = |V(G)| = |X| + |Y| \ge d + d = 2d$. Therefore $|X| = |Y| = d = n/2$. So G is $K_{\frac{n}{2}, \frac{n}{2}}$ with $n \ge 2$.

We now consider the case that $G = B_{n,t}$ for some t with $1 \le t \le n$. Notice that the Laplacian eigenvalues of $B_{n,t}$ are 0 with multiplicity 1, t with multiplicity $n - t - 1$, and n with multiplicity t. Clearly, $t < n$ otherwise $G = B_{n,t}$ is a complete graph. Since $\lambda_2 = \lambda_3 = \ldots = \lambda_{n-1}$, we must have that $t = 1$. Thus $G = B_{n,t}$ is $K_{1,n-1}$ with $n \ge 2$.

Finally we determine the graphs that make the inequality in Theorem 1 to become an equality when $k = 2$.

When G is K_n with $n \ge 2$, we, as before, have that both sides of inequality in Theorem 1 are the same. When G is $B_{n,n-2}$ with $n \ge 3$, we have that $\lambda_k = \lambda_2 = n - 2$, $d_k = n - 1$ for each k with $1 \le k \le n-2$, $d_k = n - 2$ for $n - 1 \le k \le n$, and $t = C(n,3) - (n-2) = \frac{n(n-1)(n-2)}{6} - (n-2)$. A simple computation shows that both sides of inequality in Theorem 1 are equal to $(n-2)^3((n-2)^2 + n^3)$.

Now suppose that the inequality in Theorem 1 becomes an equality. Then

$$\sum_{i=2, i\ne k}^{n} \lambda_i \le \Big(\sum_{i=2, i\ne k}^{n} 1^{3/2}\Big)^{2/3} \Big(\sum_{i=2, i\ne k}^{n} \lambda_i^{3/1}\Big)^{1/3},$$

must become an inequality. By the condition for an Hölder inequality becomes an equality, we have that $\lambda_3 = ... = \lambda_{n-1} = \lambda_n$. If $\lambda_2 = \lambda_3 = ... = \lambda_{n-1} = \lambda_n$, then by a similar argument as in the case that $3 \leq k \leq n-1$, we can show that G is K_n with $n \geq 2$. Now we assume that $\lambda_2 < \lambda_3 = ... = \lambda_{n-1} = \lambda_n$. This assumpution implies that G is not complete since for each complete graph we have that $\lambda_2 = \lambda_3 = ... = \lambda_{n-1} = \lambda_n$. Notice that now the inequality in Theorem 2 must also become an equality. By Theorem 2, we have G is regular or $G = B_{n,t}$ for some t with $1 \leq t \leq n$.

We again first consider the case that G is a regular graph and assume that the degree of each vertex in G is d. Since G is not complete, $d \leq n-2$. From Theorem 4, we have that $0 \leq n-\lambda_n = n-\lambda_{n-1} = ... = n-\lambda_3 < n-\lambda_2$ are the Laplacian eigenvalues of G^C, the complement of G. Next we consider the following two subcases.

Suppose first that G^C is connected. Notice that G^C is non-complete and regular. Apply the arguments in the subcase that G is a regular graph when we determine the graphs that make the inequality in Theorem 1 to become an equality if $k = n$, we have that G^C is $K_{\frac{n}{2},\frac{n}{2}}$ or $K_{1,n-1}$ with $n \geq 2$. We therefore arrive at a contradiction that G cannot be a connected graph. So this case cannot occur.

Suppose now that G^C is not connected. Then its algebraic connectivity is equal to zero. Namely, $n = \lambda_n$. So $n = \lambda_3 = ... = \lambda_{n-1} = \lambda_n$. From the fact that

$$\sum_{i=1}^{n} d_i = \sum_{i=1}^{n} \lambda_i,$$

we have that $n(n-2) \geq nd = n(n-2) + \lambda_2$. Thus $\lambda_2 \leq 0$, contradicting to the assumption that G is connected. So this case cannot occur either.

We now consider the case that $G = B_{n,t}$ for some t with $1 \leq t \leq n$. Notice that the Laplacian eigenvalues of $B_{n,t}$ are 0 with multiplicity 1, t with multiplicity $n-t-1$, and n with multiplicity t. Clearly, $t < n$ otherwise $G = B_{n,t}$ is a complete graph. Since $\lambda_3 = \lambda_4 = ... = \lambda_n$, we must have that $n-t-1 = 1$. Thus $n = t+2$. So $G = B_{n,n-2}$ with $n \geq 3$. QED.

Remark. Note that several tight upper bounds for $\sum_2(G)$ are available. From Theorem 1 proved by De Caen in [4], we have that

$$e(\frac{2e}{n-1} + n - 2)$$

is an upper bound for $\sum_2(G)$. From Theorem 4.1, Theorem 4.2, and Theorem 4.3 proved by Das in [3], we have, respectively, that

$$e(\frac{2e}{n-1} + \frac{n-2}{n-1}d_1 + (d_1 - d_n)(1 - \frac{d_1}{n-1})),$$

$$\frac{2e(2e + (d_1 - d_n)(n-1))}{n + d_1 - d_n}, \text{ and}$$

$$2e(d_1 + d_n) - nd_1d_n$$

are three upper bounds for $\sum_2(G)$. Thus if we use those upper bounds for $\sum_2(G)$ to replace $\sum_2(G)$ in the inequality in Theorem 1, we can get inequalities for λ_k, where $2 \leq k \leq n$. Using the formula for the roots of cubic equations, we can solve those inequalities and find bounds for λ_k, where $2 \leq k \leq n$. But we will have very complicated mathematical expressions. So we leave the main result in this note in the inequality form.

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References

[1] J. A. Bondy and U. S. R. Murty, Graph Theory with Applications, Macmillan, London and Elsevier, New York (1976).

[2] S. M. Cioabă, Sums of powers of the degrees of a graph, *Discrete Math.* 306 (2006) 1959 - 1964.

[3] K. Ch. Das, Maximizing the sum of the squares of the degrees of a graph, *Discrete Math.* 285 (2004) 57 - 66.

[4] D. de Caen, An upper bound on the sum of squares of degrees in a graph, *Discrete Math.* 185 (1998) 245 - 248.

[5] M. Fiedler, Algebraic connectivity of graphs, *Czechoslovak Mathematical Journal* 23 (98) (1973) 298 - 305.

[6] C. Godsil and G. Royle, Algebraic Graph Theory, Springer - Verlag New York (2001).

[7] J. Li and X. Zhang, A new upper bound for eigenvalues of the Laplacian matrix of a graph, *Linear Algebra and Its Applications* 265 (1997) 93 - 100.

[8] B. Mohar, Laplace eigenvalues of graphs - a survey, *Discrete Math.* 109 (1992) 171 - 183.