Block transitive $2-(v,13,1)$ Designs and Suzuki Groups *

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Abstract

This article is a contribution to the study of the automorphism groups of $2-(v,k,1)$ designs. Let $\mathcal{D}$ be $2-(v,13,1)$ design, $G \leq Aut(\mathcal{D})$ be block transitive and point primitive. If G is unsolvable, then $Soc(G)$, the socle of G, is not $Sz(q)$.

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1 Introduction

A $2-(v,k,1)$ design $\mathcal{D}=(\mathcal{P},\mathcal{B})$ is a pair consisting of a finite set $\mathcal{P}$ of v points and a collection $\mathcal{B}$ of $k-$subsets of $\mathcal{P}$, called blocks, such that any 2-subsets of $\mathcal{P}$ is contained in exactly one block. We will always assume that $2<k<v$.

Let $G \leq Aut(\mathcal{D})$ be a group of automorphisms of a $2-(v,k,1)$ design $\mathcal{D}$. Then G is said to be block transitive on $\mathcal{D}$ if G is transitive on $\mathcal{B}$ and is said to be point transitive(point primitive on $\mathcal{D}$ if G is transitive (primitive) on $\mathcal{P}$. A flag of $\mathcal{D}$ is a pair consisting of a point and a block through that point. Then G is flag transitive on $\mathcal{D}$ if G is transitive on the set of flags.

The classification of block transitive $2-(v,3,1)$ designs was completed about thirty years ago (see [1]). In [2], Camina and Siemons classified $2-(v,4,1)$ designs with a block transitive, solvable group of automorphisms. Li classified $2-(v,4,1)$ designs admitting a block transitive, unsolvable group of automorphisms (see [3]). Tong and Li [4] classified $2-(v,5,1)$ designs with a block transitive, solvable group of automorphisms. Han and

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Li [5] classified $2-(v,5,1)$ designs with a block transitive, unsolvable group of automorphisms. Liu [6] classified $2-(v,k,1)$(where $k = 6,7,8,9,10$) designs with a block transitive, solvable group of automorphisms. In [7], Han and Ma classified $2-(v,11,1)$ designs with a block transitive classical simple groups of automorphisms.

This article is a contribution to the study of the automorphism groups of $2-(v,k,1)$ designs. Let $\mathcal{D}$ be $2-(v,13,1)$ design, $G \leq Aut(\mathcal{D})$ be block transitive and point primitive. We prove that following theorem.

Main Theorem Let $\mathcal{D}$ be $2-(v,13,1)$ design, $G \leq Aut(\mathcal{D})$ be block transitive and point primitive. If G is unsolvable, then $Soc(G) \ncong Sz(q)$.

2 Preliminary Results

Let $\mathcal{D}$ be a $2-(v,k,1)$ design defined on the point set $\mathcal{P}$ and suppose that G is an automorphism group of $\mathcal{D}$ that acts transitively on blocks. For a $2-(v,k,1)$ design, as usual, b denotes the number of blocks and r denotes the number of blocks through a given point. If B is a block, G_B denotes the setwise stabilizer of B in G and $G_{(B)}$ is the pointwise stabilizer of B in G. Also, G^B denotes the permutation group induced by the action of G_B on the points of B, and so $G^B \cong G_B/G_{(B)}$.

The Suzuki groups $Sz(q)$ form an infinite family of simple groups of Lie type, and were defined in [8] and [9] as subgroups of $SL(4,q)$. Let $GF(q)$ be finite field of q elements, where $q = 2^{2n+1}$ for some positive integer $n \geq 1$ (in particular, $q \geq 8$). Set $t = 2^{n+1}$ so that $t^2 = 2q$.

Lemma 2.1 ([10]) Let $G = Sz(q)$, then every maximal subgroup of G is conjugate to one of the following:

(1) $Sz(a)$, $a^i = 2^{2n+1}$, i is a prime;

(2) QK, where Q is a Sylow 2-subgroup of G and K is a multiplicative group of $GF(q)$;

(3) $D_{2(q-1)}$;

(4) $Z_{q+\epsilon t+1} : Z_4$, where $\epsilon = \pm 1$.

Lemma 2.2 ([11]) Let $T = Sz(q)$ be an exceptional simple group of Lie type over $GF(q)$, and G be a group with $T \trianglelefteq G \leq Aut(T)$. Suppose that M is a maximal subgroup of G not containing T, then one of the following holds:

(1) $|M| < q^2|G:T|$;

(2) $T \cap M$ is a parabolic subgroup of T.

Lemma 2.3 ([7]) Let G and $\mathcal{D} = (\mathcal{P}, \mathcal{B})$ be a group and a design, and $G \leq Aut(\mathcal{D})$ be block transitive, point-primitive but not flag-transitive.

Let $Soc(G) = T$. Then

$$|T| \le \frac{v}{\lambda} \cdot |T_\alpha|^2 \cdot |G:T|,$$

where $\alpha \in \mathcal{P}$, λ is the length of the longest suborbit of G on $\mathcal{P}$.

3 Proof of the Main Theorem

Proposition 3.1 *Let $\mathcal{D}$ be $2-(v,13,1)$ design, G be block transitive, point primitive but not flag transitive, then $v = 156b_2 + 1$.*

Proof. Let $b_1 = (b,v)$, $b_2 = (b, v-1)$, $k_1 = (k,v)$, $k_2 = (k, v-1)$. Obviously,

$$k = k_1k_2,\ b = b_1b_2,\ r = b_2k_2,\ \ v = b_1k_1.$$

Since $k = 13$, we get $k_1 = 1$. Otherwise, $k \mid v$, by [11], G is flag transitive, a contradiction. Thus $v = k(k-1)b_2 + 1 = 156b_2 + 1$.

Proposition 3.2 *Let $\mathcal{D}$ be $2-(v,13,1)$ design, G be block transitive, point primitive but not flag transitive and $Soc(G) = T$ be even order. If G be unsolvable, then $|T| \le 79|T_\alpha|^2|G:T|$.*

Proof. Let $B = \{1, 2, \cdots, 13\} \in \mathcal{B}$. Since G is unsolvable, then the structure of G^B, the rank and subdegree of G do not occur:

Type of G^B	Rank of G	Subdegree of G
$\langle 1 \rangle$	157	$1,\overbrace{b_2,\cdots,b_2}^{156}$

Otherwise, $|G^B| = 1$ is odd and $|B| = 13$. We have $|G_B|$ and b_2 are odd. Since $v = 156b_2 + 1 = b_1$ and $b = b_1b_2$, then b is odd and $|G| = b|G_B|$ is also odd, a contradiction with $|T|$ be even. Thus $\lambda \ge 2b_2$. By Lemma 2.3,

$$\frac{|T|}{|T_\alpha|^2} \le \frac{v}{\lambda} \cdot |G:T| \le \frac{v}{2b_2} \cdot |G:T|.$$

By Proposition 3.1,

$$\frac{|T|}{|T_\alpha|^2} \le \frac{156b_2 + 1}{2b_2} \cdot |G:T| \le 79|G:T|.$$

Now we may prove our main theorem.

Suppose that $Soc(G) = Sz(q) = T$, then $Sz(q) \trianglelefteq G \le Aut(Sz(q))$. We have $G = T : \langle x \rangle$, where $x \in Out(T)$, the outer automorphisms group of T

which may be generated by an automorphism of field. We may assume that x is an automorphism of field. Set $o(x) = m$, then $m \mid (2n+1)$. Obviously, $|Sz(q)| = q^2(q^2+1)(q-1)$. By [12] and $k = 13$, G is not flag transitive. Since G is point primitive, G_α ($\alpha \in \mathcal{P}$) is the maximal subgroup of G, T is block transitive in $\mathcal{D}$. Hence $M = G_\alpha$ satisfies one of the two cases in Lemma 2.2. We will rule out these cases one by one.

Case (1) $|M| < q^2|G:T|$.

By Proposition 3.2, we have an upper bound of $|T|$,

$$|T| < 79|T_\alpha|^2|G:T| < 79q^4|G:T| = 79q^4m.$$

We get

$$q - 1 < 79(2n+1).$$

Let $2n+1 = s \geq 3$, then $2^s < 80s$. Thus $s = 3, 5, 7, 9$. Since $v = 156b_2 + 1$ is odd, then $2^s \mid |T_\alpha|$. Clearly T_α is contained in some maximal subgroups of T. By Lemma 2.1, $T_\alpha \cong QK_1$, where $K_1 \leq K$. We have

$$v - 1 = \frac{|T|}{|T_\alpha|} - 1 = \frac{2^{2s}(2^{2s}+1)(2^s-1)}{2^{2s}\cdot|K_1|} - 1 = \frac{(2^{2s}+1)(2^s-1)}{|K_1|} - 1.$$

Then $v-1$ is 64, 454, 1024, 31774, 16384, 2080894, 262144, 1835014, 19136584, 133956094. This conflicts with $v - 1 = 156b_2$, by Proposition 3.1.

Case (2) $T \cap M$ is a parabolic subgroup of T.

By Lemma 2.1, the parabolic subgroup of $Sz(q)$ is conjugate to QK. Then the order of parabolic subgroup is $q^2(q-1)$ and $v = q^2 + 1$. By Proposition 3.1, we have $q^2 = v - 1 = 156b_2$ and so $156 \mid q^2$, a contradiction.

This completes the proof the Main Theorem.

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