Signed edge majority total domination numbers in graphs

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Abstract

We initiate the study of signed edge majority total domination in graphs. The open neighborhood $N_G(e)$ of an edge e in a graph G is the set consisting of all edges having a common vertex with e. Let f be a function on $E(G)$, the edge set of G, into the set $\{-1, 1\}$. If $\sum_{x \in N_G(e)} f(x) \geq 1$ for at least a half of the edges $e \in E(G)$, then f is called a signed edge majority total dominating function of G. The value $\min \sum_{e \in E(G)} f(e)$, taking the minimum over all signed edge majority total dominating function f of G, is called the signed edge majority total domination number of G and denoted by $\gamma'_{smt}(G)$. Obviously, $\gamma'_{smt}(G)$ is defined only for graphs G which have no connected components isomorphic to K_2. In this paper we establish lower bounds on the signed edge majority total domination number of forests.

Keywords: signed edge dominating function; signed edge majority total dominating function; signed edge majority total domination number

1 Introduction

Let G be a graph with the vertex set $V(G)$ and the edge set $E(G)$. We use [2] for terminology and notation which are not defined here and consider simple graphs only. The *line graph* of a graph G, written $L(G)$, is the graph whose vertices are the edges of G, with $ee' \in E(L(G))$ when $e = uv$ and $e' = vw$ in G. It is easy to see that $L(C_n) = C_n$ and $L(P_n) = P_{n-1}$. For every nonempty subset E' of $E(G)$, the subgraph of G whose vertex set is the set of vertices of the edges in E' and whose edge set is E', is called the subgraph of G induced by E' and denoted by $G[E']$.

Two edges e_1, e_2 of G are called *adjacent* if they are distinct and have a common vertex. The *open neighborhood* $N_G(e)$ of an edge $e \in E(G)$ is the set of all edges adjacent to e. Its *closed neighborhood* is $N_G[e] = N_G(e) \cup \{e\}$. For a function $f : E(G) \longrightarrow \{-1, 1\}$ and a subset S of $E(G)$ we define $f(S) = \sum_{e \in S} f(e)$. The *edge-neighborhood* $E_G(v)$ of a vertex $v \in V(G)$ is the set of all edges at vertex v. For each vertex $v \in V(G)$ we also define $f(v) = \sum_{e \in E_G(v)} f(e)$. A function $f : E(G) \longrightarrow \{-1, 1\}$ is called a *signed edge majority total dominating function* (SEMTDF) of G, if $f(N_G(e)) \geq 1$ for at least a half of the edges $e \in E(G)$. It is clear that there exists an SEMTDF only for graphs G which have no connected components isomorphic to K_2. Throughout this paper we assume G is a simple graph in which the order of each component of G is at least 3. The *signed edge majority total domination number* (SEMTDN) of a graph G is $\gamma'_{smt}(G) = \min\{\sum_{e \in E} f(e) \mid f$ is an SEMTDF on $G\}$. The signed edge majority total dominating function f of G with $f(E(G)) = \gamma'_{smt}(G)$ is called $\gamma'_{smt}(G)$*-function.*

A *signed majority total dominating function* (SMTDF) is a function $f : V \longrightarrow \{-1, +1\}$ such that $\sum_{u \in N(v)} f(u) \geq 1$ for at least a half of the vertices $v \in V$. The *signed majority total domination number* (SMTDN) of a graph G is $\gamma^t_{maj}(G) = \min\{\sum_{v \in V} f(v) \mid f$ is an SMTDF on $G\}$. The signed majority total domination number was introduced by Xing and Chen in [3].

A function $f : E(G) \longrightarrow \{-1, 1\}$ is called a *signed edge total dominating function* (SETDF) of G, if $f(N_G(e)) \geq 1$ for each edge $e \in E(G)$. The *signed edge total domination number* (SETDN) of a graph G is $\gamma'_{st}(G) = \min\{\sum_{e \in E} f(e) \mid f$ is an SETDF on $G\}$. The signed edge total domination number was introduced by Zelika in [6].
Here are some well-known results on $\gamma^t_{maj}(G)$ and $\gamma'_{st}(G)$.

Theorem A. [3] For any path P_n $(n \geq 2)$, $\gamma^t_{maj}(P_n) = -1$ if n is odd and $\gamma^t_{maj}(P_n) = 0$ if n is even.

Theorem B. [3] For any cycle C_n $(n \geq 3)$, $\gamma^t_{maj}(C_n) = 3$ if n is odd and $\gamma^t_{maj}(C_n) = 0$ if n is even.

Theorem C. [3] If G is a k-regular graph of order n, then $\gamma^t_{maj}(G) \geq (1-k)n/2k$ if k is odd and $\gamma^t_{maj}(G) \geq (2-k)n/2k$ if k is even.

Theorem D. [1] For every tree T of size $m \geq 2$, $\gamma'_{st}(T) \geq 2 - m/3$.

We make use of the following terminology and notation in this paper. A graph G with an SEMTDF f of G, denoted by (G, f), is called a *signed edge majority total graph* (SEMTG). For simplicity, an edge e is said to be a $+1$ edge of (G, f) if $f(e) = 1$. Similarly, an edge e is said to be a -1 edge of (G, f) if $f(e) = -1$. Similar to Theorem 1 of [3] we have:

Theorem 1. A signed edge majority total dominating function f of a graph G is a $\gamma'_{smt}(G)$-function only if for every edge $e \in E$ with $f(e) = 1$, there exists an edge $e' \in N(e)$ with $f(N(e')) \in \{1, 2\}$.

Proof. Let f be a $\gamma'_{smt}(G)$-function and assume that there is an edge e such that $f(e) = 1$ and $f(N(e')) \notin \{1, 2\}$ for any $e' \in N(e)$. Define a new function $g : E \longrightarrow \{-1, 1\}$ by $g(e) = -1$ and $g(e') = f(e')$ for all $e' \neq e$. Then for all $e' \in N(e)$ either $f(N(e')) \leq 0$, in which case $g(N(e')) = f(N(e')) - 2 \leq -2$, or $f(N(e')) \geq 3$, in which case $g(N(e')) \geq 1$. For $e' \notin N(e)$ we have $g(N(e')) = f(N(e'))$. Thus g is a signed edge majority total dominating function and $g(E(G)) < f(E(G))$, which is a contradiction. □

Obviously, every signed edge total dominating function is also a signed edge majority total dominating function. Thus we have:

Theorem 2. For any graph G, $\gamma'_{smt}(G) \leq \gamma'_{st}(G)$.

The proof of the following theorem is straightforward and therefore omitted.

Theorem 3. For any graph G of order $n \geq 3$, $\gamma'_{smt}(G) = \gamma^t_{maj}(L(G))$.

Theorem 3 together with Theorems A, B and C lead to:

Corollary 4. For any path P_n of order $n \geq 3$, $\gamma'_{smt}(P_n) = 0$ if n is odd and $\gamma'_{smt}(P_n) = -1$ if n is even.

Corollary 5. For any cycle C_n of order $n \geq 3$, $\gamma'_{smt}(C_n) = 3$ if n is odd and $\gamma'_{smt}(C_n) = 0$ if n is even.

Corollary 6. If $k \geq 2$ and G is a k-regular graph of order $n \geq 3$, then

$$\gamma'_{smt}(G) \geq \frac{nk(2-k)}{4(k-1)}.$$

Furthermore, this bound is sharp when $k = 2$ and $G = C_{2n}$.

2 A lower bound for SEMTDN of forests

In this section we study the signed edge majority total domination number of forests. We first find a sharp lower bound for the SEMTDN of forests whose connected components are only P_3, P_4 or $K_{1,3}$. Then we establish a lower bound for the SEMTDN of forests without K_1 and K_2-components and with a component of size at least 4.

Lemma 7. For every forest F of size m whose connected components are only P_3, P_4 or $K_{1,3}$, $\gamma'_{smt}(F) \geq -\lfloor \frac{m}{2} \rfloor$ with equality if and only if $m = 4k$ or $4k+3$ for some $k = 3x + 2y + 3z$, where x, y, z are nonnegative integers and F consists of x A_2-components, y A_4-components, z A_7-components, and $\frac{1}{2}\lceil \frac{m}{2} \rceil$ A_8-components.

Proof. The proof is by induction on m . The statement is obviously true for forests of size less than 6. Assume $m \geq 6$ and that the statement holds for all forests of size less than m whose connected components are only P_3, P_4 or $K_{1,3}$. Suppose f is a $\gamma'_{smt}(F)$-function. We claim that the SEMTDG (F, f) cannot contain a connected component isomorphic to a path $x_1x_2x_3x_4$ with $f(x_1x_2) = f(x_3x_4) = 1$ and $f(x_2x_3) = -1$. Otherwise, we define $g : E(F) \longrightarrow \{-1, 1\}$ by $g(x_1x_2) = g(x_3x_4) = -1$, $g(x_2x_3) = 1$ and $g(e) = f(e)$ for $e \in E(F) \setminus \{x_1x_2, x_2x_3, x_3x_4\}$. Then g is an SEMTDF, which contradicts the fact that f is a $\gamma'_{smt}(F)$-function. Similarly, the SEMTDG (F, f) cannot have a connected component isomorphic to a path $x_1x_2x_3x_4$ with $f(x_1x_2) = f(x_2x_3) = 1$ and $f(x_3x_4) = -1$ or a star on 4 vertices x_1, x_2, x_3, x_4 with $f(x_1x_2) = f(x_1x_3) = -1$ and $f(x_1x_4) = 1$. Hence, each connected component of the SEMTDG (F, f) must have one of the following forms:

Let s_i be the number of A_i-components of the SEMTDF (F, f). First assume $s_1 \neq 0$. Let F' be obtained from F by deleting one of the A_1-components and adding a new component $P_3 = xyz$. Define $g : E(F') \longrightarrow \{-1, +1\}$ by

$$g(xy) = 1,\ g(yz) = -1 \ \text{ and } \ g(e) = f(e) \ \text{ if } \ e \in E(F) \cap E(F').$$

Obviously, g is an SEMTDF of F'. Hence, $g(E(F')) \geq -\lfloor \frac{m-1}{2} \rfloor$ by the inductive hypothesis. Thus

$$\gamma'_{smt}(F) = f(E(F)) = g(E(F')) + 1 \geq -\lfloor \frac{m-1}{2} \rfloor + 1 > -\lfloor \frac{m}{2} \rfloor. \quad (1)$$

Now assume $s_1 = 0$. If $s_5 \neq 0$ and F'' is obtained from F by deleting one of the A_5-components, then obviously $f|_{F''}$ is an SEMTDF of F''. Hence, by the inductive hypothesis we have

$$f(E(F)) = f(E(F'')) \geq -\lfloor \frac{m-2}{2} \rfloor > -\lfloor \frac{m}{2} \rfloor. \quad (2)$$

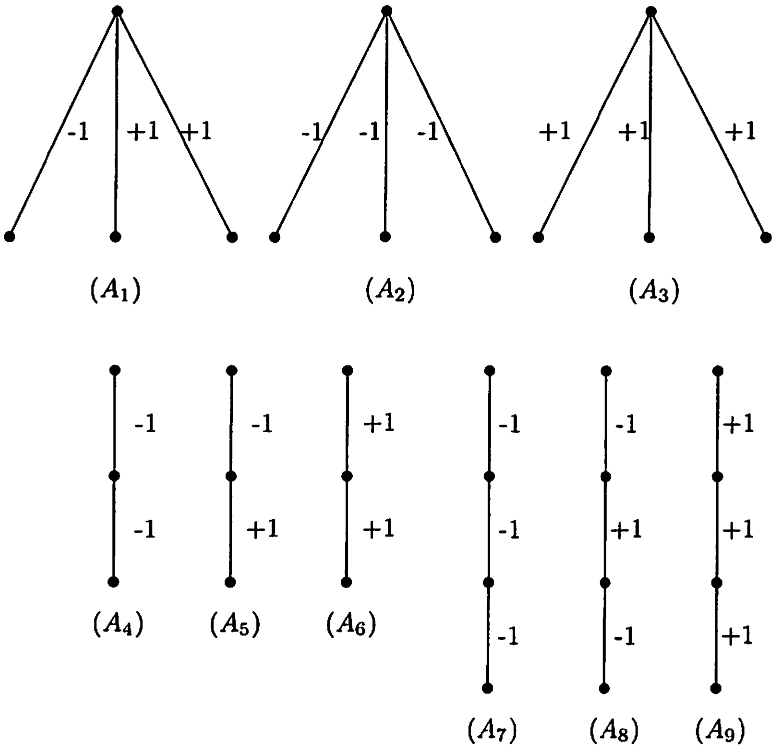

Figure 1: The connected components of (F, f)

Now let $s_5 = 0$ and define $P = \{e \in E(F) \mid f(e) = 1\}$. The fact that f is an SEMTDF leads to $3s_3 + 2s_6 + 2s_8 + 3s_9 \geq \lceil \frac{m}{2} \rceil$. Since $m = 3(s_2 + s_3 + s_7 + s_8 + s_9) + 2(s_4 + s_6)$ and $m \geq 6$, we have

$$
\begin{aligned}
f(E(F)) = |P| - |P^c| &= 3(s_3 - s_2) + 2(s_6 - s_4) + 3(s_9 - s_7) - s_8 \\
&= -m + (6s_3 + 4s_6 + 6s_9 + 2s_8) \\
&\geq -m + \lceil \tfrac{m}{2} \rceil + (3s_3 + 2s_6 + 3s_9) \\
&= -\lfloor \frac{m}{2} \rfloor + (3s_3 + 2s_6 + 3s_9).
\end{aligned}
\tag{3}
$$

If $m = 4k$ or $4k + 3$ for some $k = 3x + 2y + 3z$ and F consists of x A_2-components, y A_4-components, z A_7-components and $\frac{1}{2}\lceil \frac{m}{2} \rceil$ A_8-components, then $\gamma'_{smt}(F) = -\lfloor \frac{m}{2} \rfloor$ by (3). Now let F be a forest of size m whose connected components are only P_3, P_4 or $K_{1,3}$ and $\gamma'_{smt}(F) = -\lfloor \frac{m}{2} \rfloor$.

If $m \leq 5$, then obviously $F = P_4$. Let $m \geq 6$. By (1), (2) and (3) we have $s_1 = s_3 = s_5 = s_6 = s_9 = 0$. Then $m = 3s_2 + 2s_4 + 3s_7 + 3s_8$. Now we have $-\lfloor \frac{m}{2} \rfloor = \gamma'_{smt}(F) = -3s_2 - 2s_4 - 3s_7 - s_8 = -m + 2s_8$, and hence $2s_8 = \lceil \frac{m}{2} \rceil$. Therefore $m = 4k$ or $4k + 3$ for some nonnegative integer k. Now since $\lfloor \frac{m}{2} \rfloor = 3s_2 + 2s_4 + 3s_7 + s_8$, we obtain $k = \lfloor \frac{m}{2} \rfloor - \frac{1}{2}\lceil \frac{m}{2} \rceil = 3s_2 + 2s_4 + 3s_7$. This completes the proof. □

Lemma 8. Let F be a forest of size m without K_1 and K_2-components and with a component T of size at least 4 which satisfies the following conditions:

1. $\gamma'_{smt}(F) < 2 - 2m/3$;
2. with respect to Condition (1), F has as few edges as possible and has maximum number of connected components.

Let f be a $\gamma'_{smt}(F)$-function and $X = \{e \in E(F) \mid \sum_{e' \in N(e)} f(e') \geq 1\}$. Then there is no vertex u in T with $\deg(u) \geq 3$ which satisfies the following conditions:

1. $e = uv$ is a pendant edge with $f(e) = -1$ and $e \in X$;
2. $e' = uw$ is an edge with $f(e') = 1$ and $e' \notin X$;
3. if we split T at u into T_1 and T_2 such that T_2 contains every edge at u except e and e', then $T_2 \neq K_2$.

Proof. Assume such a vertex u exists. Define $F' = (F - T) \cup T_1 \cup T_2$. By assumption on vertex u we see that f is an SEMTDF of F' and hence $\gamma'_{smt}(F') \leq \gamma'_{smt}(F) < 2 - 2m/3$. On the other hand, $|E(F')| = |E(F)|$ and $\omega(F') = \omega(F) + 1$. (Recall that $\omega(F)$ is the number of connected components of F.) This contradicts the assumption on F. □

Theorem 9. For every forest F of size m without K_1 and K_2-components and with a component of size at least 4, $\gamma'_{smt}(F) \geq 2 - 2m/3$.

Proof. We use the method of contradiction and the notation in the proof of Lemma 7. Let F be a forest without K_1 and K_2-components, with a component of size at least 4 and $\gamma'_{smt}(F) < 2 - 2m/3$. Choose such a forest with as few edges as possible and with maximum number of connected components. Let $T_1, \ldots, T_k$ be the connected components of F. Suppose that $T_1, \ldots, T_r$ are the components with at most three edges and $T_{r+1}, \ldots, T_k$ are the components with at least four edges. Assume f is a $\gamma'_{smt}(F)$-function. Since F is a forest with a component of size at least 4 and does not have

K_1 and K_2-components, $m > 2\omega(F)$. Define $M = \{e \in E(F) \mid f(e) = -1\}$ and $X = \{e \in E(F) \mid \sum_{e' \in N(e)} f(e') \geq 1\}$.

Claim 1. If $T \in \{T_{r+1}, \dots, T_k\}$ and $e = uv \in E(T) \cap M$, then one of the connected components of $T - e$ is K_1 or K_2.

Proof of Claim 1. Without loss of generality we may assume that $T = T_{r+1}$. Let T^1_{r+1} and T^2_{r+1} be the connected components of $T - e$ containing u and v, respectively. Let, to the contrary, $|E(T^1_{r+1})| \geq 2$ and $|E(T^2_{r+1})| \geq 2$. First suppose that $e \in X^c$. Let T'_{r+1} be obtained from T^1_{r+1} by adding a pendant edge uu'. Let F' be a forest consists of $T_1, \dots, T_r, T'_{r+1}, T^2_{r+1}, T_{r+2}, \dots, T_k$. Define $g : E(F') \longrightarrow \{-1, +1\}$ by

$$g(uu') = -1 \text{ and } g(e) = f(e) \text{ if } e \neq uu'.$$

Obviously, g is an SEMTDF of F'. Since $|E(F')| = |E(F)|$ and $\omega(F') = \omega(F) + 1$, by assumption on F we have $f(E(F)) = g(E(F')) \geq 2 - 2m/3$, a contradiction. Now let $e \in X$. Since $f(u) + f(v) + 2 = \sum_{e' \in N(uv)} f(e') \geq 1$, it follows $f(u) \geq 0$ or $f(v) \geq 0$. Without loss of generality we assume $f(u) \geq 0$. Let T'_{r+1} be obtained from T^1_{r+1} by adding a pendant edge uu'. As before, it is easy to verify that this leads to a contradiction. □

By Claim 1, each $e = uv \in M$ is either a pendant edge or adjacent to a pendant edge vw in which $\deg(v) = 2$. In the later case, if $f(vw) = 1$, then the connected component of F containing e has at least four edges (see Figure 1). Without loss of generality, we may assume this connected component is T_{r+1}. Now split T_{r+1} at u into T' and T'' such that $E(T') = \{uv, vw\}$. Define $F' = (F - T) \cup T' \cup T''$. Then f is an SEMTDF of F'. Since $|E(F')| = |E(F)|$ and $\omega(F') = \omega(F) + 1$, by assumption on F we have $f(E(F)) = f(E(F')) \geq 2 - 2m/3$, a contradiction. Hence, $f(vw) = -1$.

Define $L_1 = \{e = uv \in M \mid e \text{ is a pendant edge whose support vertex is of degree 2 and is adjacent to a } -1 \text{ edge}\}$, $L_2 = \{e = uv \in M \setminus L_1 \mid e \text{ is a pendant edge}\}$ and $L_3 = M \setminus (L_1 \cup L_2)$. Then each edge of L_3 is adjacent to an edge in L_1.

Claim 2. If $T \in \{T_{r+1}, \dots, T_k\}$, $v \in V(T)$ and $\deg(v) \geq 3$, then $f(v) \geq 0$.

Proof of Claim 2. Let, to the contrary, $f(v) \leq -1$. Since $\deg(v) \geq 3$, there exist at least two -1 edges at v. First let there exist two -1 pendant edges at v, say e, e'. Split T at v into T' and T'' such that $E(T') = \{e, e'\}$. Define $F' = (F - T) \cup T' \cup T''$. Obviously, f is an SEMTDF of F'. Since $|E(F')| = |E(F)|$ and $\omega(F') = \omega(F) + 1$, by assumption on F we have $f(E(F)) = g(E(F')) \geq 2 - 2m/3$, a contradiction. Now assume there exists an edge $e = vu \in L_3$ at v. Then $\deg(u) = 2$ and u is adjacent to a Leaf, say w. Split T at v into T' and T'' such that $E(T') = \{uv, uw\}$. Define $F' = (F - T) \cup T' \cup T''$ and proceed as before to see a contradiction.
□

Claim 3. If $T \in \{T_{r+1}, \dots, T_k\}$, then $E(T) \setminus L_1 \subseteq X$.

Proof of Claim 3. Let $e = uv \in E(T) \setminus L_1$. First assume $e \in L_2$. Without loss of generality we may assume $\deg(v) = 1$. If $\deg(u) \geq 3$, then $e \in X$ by Claim 2. If $\deg(u) = 2$ and $uw \in E(T)$, then $f(uw) = 1$ because $e \notin L_1$. Thus $e \in X$. Now assume $e \in L_3$ and $e' = uw \in L_1$ in which $f(e') = -1$. Let, to the contrary, $e \notin X$. Split T into T' and T'' such that $E(T') = \{e, e'\}$. Define $F' = (F - T) \cup (T' \cup T'')$ and proceed as before to see a contradiction. Hence, $L_3 \subseteq X$. Finally, assume $e \in E(T) \setminus (L_1 \cup L_2 \cup L_3)$, hence $f(uv) = 1$. If $\deg(u) \leq 2$, then obviously $f(u) \geq 0$. If $\deg(u) \geq 3$, then $f(u) \geq 0$ by Claim 2. Similarly, $f(v) \geq 0$. Let, to the contrary, $e \notin X$. Then $f(u) + f(v) \leq 2$ and e is adjacent to an edge, say e', with $f(e') = -1$. Without loss of generality we may assume $e' = uw$. Consider two cases.

Case 1. $\deg(w) = 1$. If $\deg(v) = 1$, we apply Lemma 8 with vertex u and edges uv and uw to see a contradiction. Hence, $\deg(v) \geq 2$. First assume $\deg(u) \geq 3$ and H is the connected component of $T - \{e, e'\}$ containing u. If $H = uz$, then $f(uz) = 1$ and $uz \notin X$. Apply Lemma 8 with vertex u and edges uw and uz to see a contradiction. If H has at least two edges, we apply Lemma 8 with vertex u and edges uv and uw to see a contradiction.

Now let $\deg(u) = 2$. We consider two subcases.

Subcase 1.1 $E(v) \cap M = \emptyset$. Since $e \notin X$, $\deg(v) = 2$. Let $vv' \in E(T)$. If $vv' \notin X$, we split T at v into T' and T'' such that $E(T') = \{uw, uv\}$ to see a contradiction with the assumption on F. Assume $vv' \in X$. If $T = P_5 = wuvv'v''$, then $f(v'v'') = 1$. Now split T at v to see a contradiction. If $T \neq P_5$, we proceed as follows. If $E(v') \cap M = \emptyset$, we split T at v to see a contradiction. If $E(v') \cap M \neq \emptyset$ and $e' \in M \cap E(v')$, we split T at v' into T' and T'' such that $E(T') = \{uw, uv, vv'\}$. Define $F' = (F - T) \cup T' \cup T''$ and $g : E(F') \longrightarrow \{-1, 1\}$ by

$$g(vv') = -1, g(e') = 1 \text{ and } g(e) = f(e) \text{ if } e \in E(F) \setminus \{vv', e'\}.$$

Obviously, g is an SEMTDF of F' with $g(E(F')) = f(E(F'))$ which leads to a contradiction.

Subcase 1.2 $E(v) \cap M \neq \emptyset$. First assume there exists a pendant edge vv' for which $f(vv') = -1$. Split T at v into T' and T'' such that $E(T'') = \{uw, uv, vv'\}$. If $|E(T')| \geq 2$, we define $F' = (F - T) \cup T' \cup T''$. Obviously, f is an SEMTDF of F' with $f(E(F')) < 2 - 2m/3$. This contradicts the assumption on F. If $E(T) = \{vv''\}$, split T at v into T' and T'' such that $E(T'') = \{wu, uv\}$. Now it is easy to see a contradiction.

Suppose there is no -1 pendant edge at v. Then there exists a path $vv'v''$ for which $\deg(v') = 2$ and $f(vv') = f(v'v'') = -1$. If there is another path vzz' with $f(vz) = f(zz') = -1$, we proceed as follows. Define $T' = x_1x_2x_3$ (x_1, x_2, x_3 are new vertices), $T'' = T[E(T) - \{v'v'', zz'\}]$ and $F' = (F - T) \cup T' \cup T''$. Define $g : E(F') \longrightarrow \{-1, 1\}$ by

$$g(x_1x_2) = g(x_2x_3) = -1 \text{ and } g(e) = f(e) \text{ if } e \in E(F) \setminus \{zz', v'v''\}.$$

Obviously, g is an SEMTDF of F' with $g(E(F')) = f(E(F))$, which leads to a contradiction. Finally, let the only -1 edge at v be vv'. Sine $e = uv \notin X$ and e has exactly two -1 edges in its neighborhood, $\deg(v) = 3$ or 4. First assume $\deg(v) = 3$. Define $T' = T[E(T) \setminus \{uw, uv, v'v''\}]$, $T'' = P_4 = w_1u_1v_1v_2$ (w_1, u_1, v_1, v_2 are new vertices) and $F' = (F - T) \cup T' \cup T''$. Define $g : E(F') \longrightarrow \{-1, 1\}$ by

$$g(w_1u_1) = g(v_1v_2) = -1,\ g(u_1v_1) = 1, \text{ and } g(e) = f(e) \text{ otherwise.}$$

Obviously, g is an SEMTDF of F' with $g(E(F')) = f(E(F'))$, which leads to a contradiction. If $\deg(v) = 4$, we split T at v into T' and T'' such that $E(T'') = \{vv', v'v''\}$. Suppose that $F' = (F - T) \cup T' \cup T''$. Obviously, F is an SEMTDF of F', which leads to a contradiction.

Case 2. $\deg(w) \geq 2$. Then $uw \in L_3 \subseteq X$, $\deg(u) \geq 3$, $\deg(w) = 2$ and uw is adjacent to a pendant edge, say ww', for which $f(ww') = -1$. Let H be the connected component of $T - \{uv, uw\}$ containing u. If $H = uz$, we define $T' = T[E(T) \setminus \{uz, ww'\}]$, $T'' = z_1z_2z_3$ and $F' = (F - T) \cup T' \cup T''$. Define $g : E(F') \longrightarrow \{-1, 1\}$ by

$$g(z_1z_2) = 1, g(z_2z_3) = -1 \text{ and } g(e) = f(e) \text{ if } e \in E(F) \setminus \{uz, ww'\}.$$

Obviously, g is an SEMTDF of F' with $g(E(F')) = f(E(F))$, which contradicts the assumption on F. Therefore, the size of H is greater then 1. Consider two subcases.

Subcase 2.1 $\deg(v) = 1$. Let $T' = H$, $T'' = z_1z_2z_3z_4$ and $F' = (F - T) \cup T' \cup T''$. Define $g : E(F') \longrightarrow \{-1, 1\}$ by $g(z_1z_2) = g(z_3z_4) = -1$, $g(z_2z_3) = 1$ and $g(e) = f(e)$ if $e \in E(F) \setminus \{uv, uw, ww'\}$. Obviously, g is an SEMTDF of F' with $g(E(F')) = f(E(F))$, which contradicts the assumption on F.

Subcase 2.2 $\deg(v) \geq 2$. By the facts $uv \notin X$ and $f(u) \geq 0$ there exists an edge vv' with $f(vv') = -1$. If vv' is a pendant edge, then apply Case 1 with v and v' instead of u and w'. Now assume there exits a path vv_1v_2 in which $f(vv_1) = f(v_1v_2) = -1$ and $\deg(v_1) = 2$. Let $T' = T[E(T) \setminus \{v_1v_2, ww'\}]$, $T'' = z_1z_2z_3$ and $F' = (F - T) \cup T' \cup T''$. Define $g : E(F') \longrightarrow \{-1, 1\}$ by

$$g(z_1z_2) = g(z_2z_3) = -1 \text{ and } g(e) = f(e) \text{ if } e \in E(F) \setminus \{v_1v_2, ww'\}.$$

Obviously, g is an SEMTDF of F' with $g(E(F')) = f(E(F))$, which contradicts the assumption on F. This completes the proof of Claim 3. □

Define $T_i' = T_i \setminus L_1$ for each $r + 1 \leq i \leq k$. By Claim 3, $f|_{T_i'}$ is a signed edge total dominating function on T_i' for each $r + 1 \leq i \leq k$. Thus,

$f|_{T_i'}(E(T_i')) \geq 2 - m_i/3$ by Theorem D, where $m_i = |E(T_i')|$. Recall that s_i is the number of A_i-components of (F, f) (see Figure 1). Now we have

$$|X| = \sum_{i=r+1}^{k} m_i + s_1 + 3s_3 + s_5 + 2s_6 + 2s_8 + 3s_9 \geq \lceil \frac{m}{2} \rceil$$

and

$$|X^c| = |L_1| + 2s_1 + 3s_2 + 2s_4 + s_5 + 3s_7 + s_8 \leq \lfloor \frac{m}{2} \rfloor.$$

On the other hand,

$$\sum_{i=r+1}^{k} f|_{T_i'}(E(T_i')) \geq \sum_{i=r+1}^{k} (2 - m_i/3) = 2(k-r) - (1/3) \sum_{i=r+1}^{k} m_i.$$

Therefore,

$$\begin{aligned}
f(E(F)) &= \textstyle\sum_{i=r+1}^{k} f|_{T_i'}(E(T_i')) - |L_1| + s_1 - 3s_2 + 3s_3 - 2s_4 + 2s_6 \\
&\quad -3s_7 - s_8 + 3s_9 \\
&\geq 2(k-r) - (1/3) \textstyle\sum_{i=r+1}^{k} m_i + 3s_1 + 3s_3 + s_5 + 2s_6 \\
&\quad +3s_9 - |X^c| \\
&= 2(k-r) - (1/3)|X| + \frac{10s_1}{3} + 4s_3 + \frac{4s_5}{3} + \frac{8s_6}{3} + \frac{2s_8}{3} \\
&\quad +4s_9 - |X^c| \\
&\geq 2(k-r) - \frac{1}{3}(|X| + |X^c|) - \frac{2}{3}|X^c| \\
&\geq 2(k-r) - \frac{m}{3} - \frac{2}{3}\lfloor \frac{m}{2} \rfloor \geq 2 - 2m/3.
\end{aligned}$$

This is a contradiction. □

We conclude this paper with the following observation. Let $k \geq 0$. If a forest F of size m consists of $4k$ (or $4k+1$) components each isomorphic to P_4, then $\gamma'_{smt}(F) = 1 - 2m/3 + (2k-1)$ (or $\gamma'_{smt}(F) = 1 - 2m/3 + 2k$, respectively).

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