Distance-Dominating Cycles in P_3-Dominated Graphs*

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Abstract

Let G be a connected graph. For $x, y \in V(G)$ with $d(x, y) = 2$, we define $J(x, y) = \{u \in N(x) \cap N(y) \mid N[u] \subseteq N[x] \cup N[y]\}$ and $J'(x, y) = \{u \in N(x) \cap N(y) \mid \text{if } v \in N(u) \setminus (N[x] \cup N[y]) \text{ then } N(x) \cup N(y) \cup N(u) \subseteq N[v]\}$. A graph G is *quasi-claw-free* if $J(x, y) \neq \emptyset$ for each pair (x, y) of vertices at distance 2 in G. Broersma and Vumar introduced the class of P_3-dominated graphs defined as $J(x, y) \cup J'(x, y) \neq \emptyset$ for each $x, y \in V(G)$ with $d(x, y) = 2$. Let $\kappa(G)$ and $\alpha_l(G)$ be the connectivity of G and the maximum number of vertices that are pairwise at distance at least l in G, respectively. A cycle C is m-dominating if $d(x, C) = \min\{d(x, u) \mid u \in V(C)\} \leq m$ for all $x \in V(G)$. In this note, we prove that every 2-connected $\mathcal{P}_3$-dominated graph G has an m-dominating cycle if $\alpha_{2m+3}(G) \leq \kappa(G)$.
Keywords: Quasi-claw-free graph; P_3-dominated graph; m-dominating cycle.

1 Introduction

Throughout this note, we consider only finite, undirected and simple graphs. Let $G = (V, E)$ be a graph with vertex set $V = V(G)$ and edge set $E = E(G)$. The *open neighborhood* and the *closed neighborhood* of a vertex u are denoted by $N(u) = \{x \in V(G), xu \in E(G)\}$ and $N[u] = \{u\} \cup N(u)$, respectively. Let $\langle A \rangle$ denote the subgraph of G induced by the subset A of $V(G)$ and let $d(x, y)$ denote the distance between vertices x and y, i.e., the length of

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a shortest path from x to y in G. If H is a subgraph of G, then we define $d(x,H) = \min\{d(x,y) \mid y \in V(H)\}$ as the distance from x to H. A set $A \subset V(G)$ is *independent* if no two vertices in A are adjacent. The *independence number* $\alpha(G)$ of G is the cardinality of a maximum independent set in G, and the *connectivity* $\kappa(G)$ of G is the cardinality of a minimum cutset in G. Moreover, let $\alpha_l(G)$ be the maximum number of vertices of G that are pairwise at distance at least l in G. A cycle C is *m-dominating* if $d(x,C) \leq m$ for all $x \in V(G)$. Clearly, every 0-dominating cycle is hamiltonian. The following result on m-dominating cycles in graphs was proved by Fraisse.

Theorem 1 ([8]). *Let G be a 2-connected graph. If $\alpha_{2m+2}(G) \leq \kappa(G)$, then G has an m-dominating cycle.*

Let $C := c_0c_1\cdots c_{p-1}c_0$ be a cycle in G with an implicit orientation according to the increasing subscripts. For $i \neq j$, let $C[c_i,c_j]$ be the subpath $c_ic_{i+1}\cdots c_j$, the subscript is taken modulo p. We define $C(c_i,c_j] = C[c_{i+1},c_j]$, $C[c_i,c_j) = C[c_i,c_{j-1}]$ and $C(c_i,c_j) = C[c_{i+1},c_{j-1}]$. In each case we use the notation to refer both to the subpath and to its vertex set, depending on context. For any i, we put $c_i^+ = c_{i+1}$ and $c_i^- = c_{i-1}$. We use similar definitions for paths.

If $H \subset G$ is an induced subgraph of G isomorphic to the star $K_{1,r}$ ($r \geq 3$), then the only vertex of degree r in H is called the *center* of H and the other vertices of degree 1 in H are called *toes* of H. Whenever vertices of $K_{1,r}$ are listed, the center is always the first vertex of the list. Following Ainouche [1], we set $J(a,b) = \{u \in N(a) \cap N(b) \mid N[u] \subseteq N[a] \cup N[b]\}$ for each pair (a,b) of vertices at distance 2.

A graph G is said to belong to the class $\mathcal{CF}$ of *claw-free* graphs, if G does not contain an induced subgraph isomorphic to a claw — $K_{1,3}$. A large number of results have been obtained on claw-free graphs, while some interesting problems and some conjectures remain open [4]. During the last two decades several extensions of claw-free graphs have been introduced and many known results concerning matching and hamiltonicity on claw-free graphs have been extended to these classes. See [1], [3], [5], [7], [9], [10] and [11] for more details. We will repeat the definitions of some of these superclasses of claw-free graphs.

In 1994 [12], Ryjáček introduced the class $\mathcal{ACF}$ of *almost claw-free* graphs, and in 1998 [1], Ainouche introduced the class $\mathcal{QCF}$ of *quasi-claw-free* graphs. A graph G is in $\mathcal{ACF}$ if for any independent set A of G and for any $v \in A$, $N(v)$ contains two vertices x and y such that $N[v] \subseteq N[x] \cup N[y]$. A graph G is in $\mathcal{QCF}$, if $J(a,b) \neq \emptyset$ for each pair (a,b) of vertices at distance 2 in G. In [3], as a common generalization of almost claw-free and quasi claw-free graphs, Ainouche et al. introduced the class $\mathcal{DCT}$ of *dominated claw toes* graphs. A claw $\langle\{z,a_1,a_2,a_3\}\rangle$ with the claw center z is said to be *dominated* (*undominated*, resp.) if $\bigcup_{1\leq i<j\leq 3} J(a_i,a_j) \neq \emptyset$ ($\bigcup_{1\leq i<j\leq 3} J(a_i,a_j) = \emptyset$, resp.). A graph G belongs to $\mathcal{DCT}$, if every claw in G is dominated [2].

Recently, Broersma and Vumar [6] introduced a new class of graphs, namely

P_3-*dominated graphs*, which is a super class of quasi-claw-free graphs. They also extended some known results concerning hamiltonicity on $\mathcal{QCF}$ to this new class of graphs. The class $\mathcal{P3D}$ of P_3- dominated graphs is defined below.

Let (a, b) be a pair of vertices at distance 2 in G. We consider a common neighbor u of a and b with the following property.

$$\text{If } v \in N(u) \setminus \{a, b\} \text{ is adjacent neither to } a \text{ nor to } b \text{, then it is adjacent to all vertices of } N(a) \cup N(b) \cup N(u) \setminus \{a, b, v\}. \tag{1}$$

For a pair (a, b) of vertices at distance 2 in G, set $J'(a, b) = \{u \in N(a) \cap N(b) \mid u \text{ satisfies } (1)\}$. We say that G is a P_3-dominated graph if $J(a, b) \cup J'(a, b) \neq \emptyset$ for every pair (a, b) of vertices at distance 2 in G.

The following results are shown in [1] and in [6].

(i) $\mathcal{CF} \subset (\mathcal{QCF} \cap \mathcal{ACF})$, $\mathcal{QCF} \subset \mathcal{P3D}$, $(\mathcal{QCF} \cup \mathcal{ACF}) \subset \mathcal{DCT}$;

(ii) $\mathcal{QCF} \setminus \mathcal{ACF}$, $\mathcal{ACF} \setminus \mathcal{QCF}$, $(\mathcal{QCF} \cap \mathcal{ACF}) \setminus \mathcal{CF}$, $\mathcal{DCT} \setminus (\mathcal{QCF} \cup \mathcal{ACF})$, $\mathcal{P3D} \setminus \mathcal{QCF}$, $\mathcal{P3D} \setminus \mathcal{DCT}$ and $\mathcal{DCT} \setminus \mathcal{P3D}$ are infinite.

Chen et al. extend Theorem 1 in case of quasi claw-free graphs by showing the following.

Theorem 2 ([7]). *Let G be a 2-connected quasi claw-free graph. If $\alpha_{2m+3}(G) \leq \kappa(G)$, then G has an m-dominating cycle.*

In the present note we generalize the result in Theorem 2 to the class $\mathcal{P3D}$.

Theorem 3. *Let G be a 2-connected P_3-dominated graph. If $\alpha_{2m+3}(G) \leq \kappa(G)$, then G has an m-dominating cycle.*

The following sufficient condition for a 2-connected P_3-dominated graph to be hamiltonian follows immediately from Theorem 3.

Corollary 1 ([6]). *Let G be a 2-connected P_3-dominated graph. If $\alpha_3(G) \leq \kappa(G)$, then G is hamiltonian.*

We conclude this section with the following conjecture that was proposed in [7].

Conjecture 1 ([7]). *Every 2-connected DCT-graph G has an m-dominating cycle or $\alpha_{2m+3}(G) \geq \kappa(G) + 1$.*

2 Proof of Theorem 3

Let G be a 2-connected P_3-dominated graph. For each cycle C of G, define $F(C) = \{x \in V(G - C) \mid d(x, C) > m\}$. We prove the following result which implies the assertion of Theorem 3.

If G has no m-dominating cycle, then $\alpha_{2m+3}(G) > \kappa(G)$.

Let C be a cycle in G such that:

(a) $|F(C)|$ is as small as possible.

There is a component H of $G - C$ with $F(C) \cap V(H) \neq \emptyset$ because C is not an m-dominating cycle. We choose C such that:

(b) Subject to (a), $|H|$ is as small as possible;

(c) Subject to (a) and (b), C is as long as possible.

Let $x_0 \in F(C) \cap V(H)$. Let $A = \{a_1, a_2, \cdots, a_p\}$ be the set of vertices of C which are adjacent to vertices of H, assume that these vertices occur on C, in the order of their indices. Obviously, $p \geq \kappa(G)$ and there is a path $Q_{a_i a_j} := Q_{ij}$ between any pair a_i, a_j of A, whose internal vertices are all in H.

Let $S_i = C(a_i, a_{i+1})$. A vertex $u \in S_i$ is said to be *insertable* if there exist vertices $v, v^+ \in V(C) - S_i$ such that $uv, uv^+ \in E(G)$. Let I_i be the set of insertable vertices of S_i. For a cycle C' in G, we use $v_i(C')$ to denote the first vertex of C' on $C(a_i, a_{i+1}]$ (if any). Consider two indices i, j (not necessarily distinct) and let K_{ij} denote the set of cycles C' of G such that:

1. $V(C') \cap V(H) \neq \emptyset$.
2. $V(C) - V(C') \subseteq \left(C(a_i, v_i(C')) - I_i\right) \cup \left(C(a_j, v_j(C')) - I_j\right)$.

Now $C[a_{i+1}, a_i]Q_{i(i+1)}$ is a cycle and $S_i \subseteq V(G) - V(C[a_{i+1}, a_i]Q_{i(i+1)})$. It is easy to obtain a cycle C_i from $C[a_{i+1}, a_i]Q_{i(i+1)}$ and S_i such that $C[a_{i+1}, a_i]Q_{i(i+1)} \cup I_i \subseteq C_i$. Remark that C_i belongs to K_{ii}, and hence K_{ij} is not empty.

Let L_{ij} denote the subset of K_{ij}, defined as follows: a cycle C' belongs to L_{ij} if

3. $C(a_i, v_i(C')) \cup C(a_j, v_j(C'))$ is minimal for inclusion.

Notice that $L_{ii} \neq \emptyset$ because each cycle in K_{ii} corresponds to, in a sense of Condition 3, some cycle that belongs to L_{ii}.

Consider i as a fixed index, then, for each $j \in \{1, \cdots, p\}$ and for each cycle C' in K_{ij}, we have $F(C') - F(C) \neq \emptyset$, otherwise C' would contradict Condition (a) or (b) of the choice of C. It follows that there exists a vertex x_{ij} with $d(x_{ij}, C) \leq m$ and $d(x_{ij}, C') > m$. From Condition 2 in the definition of K_{ij}, we deduce that

4. There is a path of length at most m from x_{ij} to $C(a_i, v_i(C')) - I_i$ or to $C(a_j, v_j(C')) - I_j$ with no internal vertex in $V(C) \cup V(C')$.

For each $j \in \{1, \cdots, p\}$, let H_j^i be the set of all cycles C_{ij} in L_{ij} such that there is a vertex x_{ij} in $F(C_{ij}) - F(C)$ which is joined by a path with no internal vertex in $V(C) \cup V(C_{ij})$ to a vertex $u(x_{ij})$ of $C(a_i, v_i(C')) - I_i$ such that $|C(a_i, u(x_{ij}))|$ is minimum. Note $H_j^i \neq \emptyset$ because of the remark after Condition 2. Choose an index j, a cycle $C_{ij} \in H_j^i$ and a corresponding vertex x_{ij} in

$F(C_{ij}) - F(C)$ such that

5. $|C(a_i, v_i(C_{ij}))| = \min\{|C(a_i, v_i(C_{ij'}))| \mid 1 \le j' \le p \text{ and } C_{ij} \in H^i_{j'}\}$.

Then we redefine $x_i = x_{ij}$ and $u_i = u(x_{ij})$ for $1 \le i \le p$. We will show in the following that $x_0, x_1, x_2, \cdots, x_p$ are different vertices and of distance pairwisely at least $2m + 3$.

Claim 1. *(i) $x_i \notin V(H)$ for $1 \le i \le p$;*
(ii) $x_i \ne x_j$ and there is no path $P[x_i, x_j]$ such that $V(P(x_i, x_j)) \cap (V(C) \cup V(H)) = \emptyset$ for $1 \le i < j \le p$.

Proof. (i) If $x_i \in V(H)$ for some i, then, by the definition of x_i, there is a path P from x_i to $u_i \in C(a_i, v_i(C')) - I_i$ that has no internal vertex in $V(C) \cup V(C')$. Then, using the path P, we can construct a cycle $C'' \in L_{ij}$ such that $|C(a_i, v_i(C')) \cup C(a_j, v_j(C'))| > |C(a_i, v_i(C'')) \cup C(a_j, v_j(C''))|$, a contradiction.

(ii) Let x_i (x_j, resp.) be a corresponding vertex of C_{is} (C_{jt}, resp.) which is joined to $u_i \in (C(a_i, v_i(C_{is})) - I_i) \cup (C(a_s, v_s(C_{is})) - I_s)$ ($u_j \in (C(a_j, v_j(C_{jt})) - I_j) \cup (C(a_t, v_t(C_{jt})) - I_t)$, resp.). Suppose without loss of generality that $i < j, u_i \in C(a_i, v_i(C_{is})) - I_i$ and $u_j \in C(a_j, v_j(C_{jt})) - I_j$, and assume that (ii) is not true. Then, since $x_i, x_j \notin V(H)$, there is a path $P[u_i, u_j]$ which is internally disjoint from $C \cup H$. Setting $C_{ij} := a_i Q_{ij} C[a_j, u_i) P[u_i, u_j] C(u_j, a_i]$, and inserting the insertable vertices in $C(a_i, u_i)$ and in $C(a_j, u_j)$, we can construct a cycle C'_{ij} such that $C_{ij} \subseteq C'_{ij} \in H^i_j \subseteq L_{ij} \subseteq K_{ij}$ and

$$|C(a_i, v_i(C_{is}))| > |C(a_i, v_i(C'_{ij}))| \text{ and } |C(a_j, v_j(C_{jt}))| > |C(a_j, v_j(C'_{ij}))|.$$

And then Condition 4 is verified for C'_{ij}, contradicting Condition 5. □

From the proof of Claim 1, it is not difficult to obtain the following observation.

Observation 1. *There exists no path internally disjoint from $C \cup H$ that joins a vertex of $C(a_k, v_k(C_{is})]$ and a vertex of $C(a_q, v_q(C_{jt})]$, where $k \in \{i, s\}$ and $q \in \{j, t\}$.*

Claim 2. *$d(x_0, x_i) > 2m + 2$ for $1 \le i \le p$.*

Proof. By Claim 1 (i), every path from x_0 to x_i must contain a vertex of C. Let $P[x_0, x_i]$ be a shortest path from x_0 to x_i, and assume that $w \in V(P) \cap V(C)$ with $V(P[x_0, w]) \cap V(C) = \{w\}$. If $w \in C(a_q, v_q(C_{ij}))$, where $q \in \{i, j\}$ and C_{ij} is the cycle from which we define x_i, then the cycle $C[w, a_q]Q_{a_q,w}$ contradicts Condition 3 for C_{ij}. So we can assume that $w \in V(C) \cap V(C_{ij})$.

Let x and y be the predecessor and the successor of w on the path $P[x_0, x_i]$, respectively. By the minimality of the path, $xy \notin E(G)$, hence $d(x, y) = 2$. Since the graph is P_3-dominated, we have $J(x, y) \cup J'(x, y) \ne \emptyset$.

Case A. $J(x,y) = \emptyset$.

By definition, we have $J'(x,y) \neq \emptyset$ and hence there exist vertices $z \in N(x) \cap N(y)$ and $v \in N(z) \setminus (N[x] \cup N[y])$ such that $N(x) \cup N(y) \cup N(z) \subseteq N[v]$. Notice that the path $P \cup \{xz, zy\} - \{xw, wy\}$ from x_0 to x_i is of the same length as P, and z is also in $V(C) \cap V(C_{ij})$, for otherwise, by replacing xwy by xzy, we can construct a shortest path $P'[x_0, x_i]$ such that $V(P'(x_0, x_i)) \cap V(C) = \emptyset$ and $|P'[x_0, x_i]| = |P[x_0, x_i]|$, a contradiction. Notice again that $z^+x \notin E(G)$, for otherwise the cycle $xz^+C(z^+, z)zx$ would contradict the choice of C. If $z^+y \notin E(G)$, then $N(x) \cup N(y) \cup N(z) \subseteq N[z^+]$. Let x^- and y^+ be the predecessor and the successor of x and y on the path $P[x_0, x_i]$, respectively. Then $\{x^-z^+, z^+y^+\} \subset E(G)$ and therefore $|P[x_0, x^-]z^+P[y^+, x_i]| = |P[x_0, x_i]| - 2$, a contradiction. Thus we have $z^+y \in E(G)$. Similarly $z^-y \in E(G)$.

Clearly $y \in V(C)$, otherwise the cycle $yz^+C(z^+, z)zy$ contradicts Condition (c) for the choice of C. Moreover $y \in V(C) \cap V(C_{ij})$. Indeed, if $y \in C(a_q, v_q(C_{ij}))$, then, since $z^-y, zy, z^+y \in E(G)$ and $z \notin C(a_q, v_q(C_{ij}))$, we have $y \in I_q$. Therefore $y \in V(C_{ij})$. By the definitions of x_0 and x_i, we have $|P[x_0, x_i]| = |P[x_0, w)wyP(y, x_i]| = |P[x_0, w)| + |wy| + |P(y, x_i]| \geq m + 1 + 2 + m + 1 = 2m + 4$.

Case B. $J(x,y) \neq \emptyset$.

There exists a vertex $z \in N(x) \cap N(y)$ such that $N[z] \subseteq N[x] \cup N[y]$. Using the similar argument as in Case A, we have $z \in V(C) \cap V(C_{ij})$ and $z^+x \notin E(G)$. Since $z^+ \in N(z) \subseteq N[x] \cup N[y]$, we have $z^+y \in E(G)$. Analogously $z^-y \in E(G)$. And then this case can be settled in the similar manner as in Case A. □

Claim 3. *$d(x_i, x_j) > 2m + 2$ for $1 \leq i < j \leq p$.*

Proof. Let C_{is} and C_{jt} be the cycles from which we define x_i and x_j, respectively. By Claim 1 (ii), every path joining x_i and x_j has necessarily internal vertices on C. Let $P[x_i, x_j]$ be a shortest path from x_i to x_j and let y_i (y_j, resp.) denote the first vertex on $V(P[x_i, x_j]) \cap V(C)$ starting from x_i (x_j, resp.).

If y_i or $y_j \in V(C) \cap V(C_{is}) \cap V(C_{jt})$, then we settle this claim by using a similar argument as in the proof of Claim 2.

If $y_j \in C(a_q, v_q(C_{is}))$ $(q \in \{i, s\})$, then, since there is a path from x_j to u_j that is internally disjoint from $C \cup H$, we obtain a path form u_j to y_j with no internal vertices in $C \cup H$. This is contrary to Observation 1.

Now we can conclude that $y_j \in C(a_p, v_p(C_{jt}))$ $(p \in \{j, t\})$. Analogously $y_i \in C(a_r, v_r(C_{is}))$ $(r \in \{i, s\})$. By Observation 1, we have $V(P(y_i, y_j)) \cap V(C) \cap V(C_{is}) \cap V(C_{jt}) \neq \emptyset$. Then this proof can be completed by using a similar argument as in the proof of Claim 2. □

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