NOTE ON A q-OPERATOR IDENTITY

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Abstract: In this paper, we obtain an interesting identity by applying two q-operator identities. From this identity, we can recover the terminating Sears' ${}_3\Phi_2$ transformation formulas and the Dilcher's identity and the Uchimura's identity. In addition, an interesting binomial identity can be concluded.
2000Msc: 05A30; 33D15; 33D60; 33D05
Keywords: q-Series; q-Derivative operator; Basic hypergeometric series

1. Introduction and Main Result

Following Gasper and Rahman [5], the q-shifted factorial of a is defined by

$$(a;q)_0 = 1, \quad (a;q)_n = \prod_{k=0}^{n-1}(1-aq^k), \quad n = 1,2,\cdots.$$

When $|q| < 1$, we have the infinite product expressions

$$(a;q)_\infty = \prod_{k=0}^{\infty}(1-aq^k) \quad \text{and} \quad (a;q)_\alpha = (a;q)_\infty/(aq^\alpha;q)_\infty, \tag{1}$$

where α is a complex number.

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The product and fraction forms of the shifted factorials are abbreviated throughout the paper respectively to

$$(a_1, a_2, \cdots, a_m; q)_n = (a_1; q)_n (a_2; q)_n \cdots (a_m; q)_n,$$
$$\begin{bmatrix} a_1, & a_2, & \cdots, & a_r \\ b_1, & b_2, & \cdots, & b_s \end{bmatrix}; q \Bigg]_n = \frac{(a_1, a_2, \cdots, a_r; q)_n}{(b_1, b_2, \cdots, b_s; q)_n}, \quad n = 0, 1, \cdots, \infty.$$

The basic hypergeometric series ${}_r\Phi_s$ is defined by:

$$ {}_r\Phi_s \left(\begin{matrix} a_1, & \cdots, & a_r \\ b_1, & \cdots, & b_s \end{matrix}; q, x \right) $$
$$ = \sum_{n=0}^{\infty} \left[\begin{matrix} a_1, & a_2, & \cdots, & a_r \\ q, & b_1, & \cdots, & b_s \end{matrix}; q \right]_n \left[(-1)^n q^{n(n-1)/2} \right]^{1+s-r} x^n. $$

The q-binomial coefficient is given by

$$\begin{bmatrix} n \\ k \end{bmatrix} = \frac{(q;q)_n}{(q;q)_k (q;q)_{n-k}}. \tag{2}$$

The q-derivative operator D_q and q-shifted operator η (cf. [1]), acting on the variable x, are defined by:

$$D_q \{f(x)\} = \frac{f(x) - f(xq)}{x} \quad \text{and} \quad \eta \{f(x)\} = f(xq).$$

Remark. The definition of D_q is different from the ordinary q-differential operator. Multiplying both sides of this definition by $1/(1-q)$, it should become the ordinary q-differential operator. The ordinary definition reduces to the ordinary differentiation for $q \to 1$. There are many people using this operator to obtain formulas of the q-series. The typical one is Cigler [2]. In [2], he applied this operator to give a system way of studying the q-series.

We can prove, by means of the induction principle, the following explicit formulae

$$D_q^n \left\{ \frac{(x\omega; q)_\infty}{(xs; q)_\infty} \right\} = s^n \frac{(\omega/s; q)_n (x\omega q^n; q)_\infty}{(xs; q)_\infty}, \tag{3}$$

$$D_q^n \{f(x)g(x)\} = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} q^{k(k-n)} D_q^k \{f(x)\} D_q^{n-k} \{g(xq^k)\}. \tag{4}$$

The following elegant identity

$$\sum_{k=1}^{n} \begin{bmatrix} n \\ k \end{bmatrix} (-1)^{k+1} \frac{q^{k(k-1)/2+kr}}{(1-q^k)^r} = \sum_{1 \le n_r \le n_{r-1} \le \cdots \le n_1 \le n} \prod_{i=1}^{r} \frac{q^{n_i}}{1-q^{n_i}} \tag{5}$$

was given by Dilcher [3, p. 91, Eq. 5.3]. Fu and Lascoux [4] presented an extension of it. Later, Prodinger [7], Zeng [8] applied different ways to prove the extension given by Fu and Lascoux [4].

In this paper, we apply the following q-operator identity

$$D_q^n\{f(x)\} = x^{-n}\sum_{k=0}^{n}\frac{(q^{-n};q)_k}{(q;q)_k}(q\eta)^k\{f(x)\} \tag{6}$$

to give an identity which contains (5) and (16) as its special cases. The identity (6) is simple but important. Recently, Chu [1] has given some applications of this operator identity. The main results of this paper are stated as:

Theorem 1. Let x, a_i, b_i be complex numbers, $i = 1, 2, \cdots, r$

$$\begin{aligned}
&{}_{r+1}\Phi_r\left(\begin{matrix} q^{-n}, & xb_1, & \cdots, & xb_r \\ & xa_1, & \cdots, & xa_r \end{matrix}; q, q\right) \\
&= (b_1x)^n\frac{(a_1/b_1;q)_n}{(xa_1;q)_n}\sum_{0\le k_{r-1}\le k_{r-2}\le\cdots\le k_1\le k_0=n} \\
&\times\prod_{i=1}^{r-1}\left[\begin{matrix} q^{-k_{i-1}}, & xb_i, & a_{i+1}/b_{i+1} \\ q, & b_iq^{1-k_{i-1}}/a_i, & xa_{i+1} \end{matrix}; q\right]_{k_i}\left(\frac{qb_{i+1}}{a_i}\right)^{k_i}.
\end{aligned} \tag{7}$$

Theorem 2. Let $a_i \neq 0, b_i \neq 0, x$ be complex numbers, $i = 1, 2, \cdots, r$

$$\begin{aligned}
&\sum_{k=0}^{n}\left[\begin{matrix} q^{-n}, & xb_1, & \cdots, & xb_r \\ q, & xa_1, & \cdots, & xa_r \end{matrix}; q\right]_k\left(\frac{q^na_1\cdots a_r}{b_1b_2\cdots b_r}\right)^k \\
&= \frac{(a_1/b_1;q)_n}{(xa_1;q)_n}\sum_{0\le k_{r-1}\le k_{r-2}\le\cdots\le k_1\le k_0=n} \\
&\times\prod_{i=1}^{r-1}\left[\begin{matrix} q^{-k_{i-1}}, & xb_i, & a_{i+1}/b_{i+1} \\ q, & b_iq^{1-k_{i-1}}/a_i, & xa_{i+1} \end{matrix}; q\right]_{k_i} q^{k_i}.
\end{aligned} \tag{8}$$

2. The Proof of the Theorems

Proof of Theorem 1
Taking

$$f(x) = \left[\begin{matrix} xa_1, & xa_2, & \cdots, & xa_r \\ xb_1, & xb_2, & \cdots, & xb_r \end{matrix}; q\right]_\infty$$

into (6), the right hand side equates to

$$x^{-n}f(x)\sum_{k=0}^{n}\begin{bmatrix} q^{-n}, & xb_1, & xb_2, & \cdots, & xb_r \\ q, & xa_1, & xa_2, & \cdots, & xa_r \end{bmatrix}; q\Bigg]_k q^k. \tag{9}$$

From (4), the left hand side of (6) can be rewritten as follows

$$\sum_{k_1=0}^{n}\begin{bmatrix} n \\ k \end{bmatrix} q^{k_1(k_1-n)}D_q^{n-k_1}\begin{bmatrix} xa_1q^{k_1} \\ xb_1q_1^{k} \end{bmatrix}; q\Bigg]_\infty$$

$$\times D_q^{k_1}\begin{bmatrix} xa_2, & xa_3, & \cdots, & xa_r \\ xb_2, & xb_3, & \cdots, & xb_r \end{bmatrix}; q\Bigg]_\infty$$

$$= \sum_{k_1=0}^{n}\frac{(q^{-n};q)_{k_1}}{(q;q)_{k_1}}(-1)^{k_1}q^{\binom{k_1+1}{2}}D_q^{n-k_1}\begin{bmatrix} xa_1q^{k_1} \\ xb_1q^{k_1} \end{bmatrix}; q\Bigg]_\infty$$

$$\times D_q^{k_1}\begin{bmatrix} xa_2, & \cdots, & xa_r \\ xb_2, & \cdots, & xb_r \end{bmatrix}; q\Bigg]_\infty$$

$$= b_1^n\frac{(a_1/b_1;q)_n(xa_1;q)_\infty}{(xa_1;q)_n(xb_1;q)_\infty}\sum_{k_1=0}^{n}\begin{bmatrix} q^{-n}, & xb_1 \\ q, & b_1q^{1-n}/a_1 \end{bmatrix}; q\Bigg]_{k_1}\left(\frac{q}{a_1}\right)^{k_1}$$

$$\times D_q^{k_1}\begin{bmatrix} xa_2, & \cdots, & xa_r \\ xb_2, & \cdots, & xb_r \end{bmatrix}; q\Bigg]_\infty.$$

Iterating the process above, by induction and using (3), the left hand side of (6) comes to

$$b_1^n\frac{(a_1/b_1;q)_n}{(xa_1;q)_n}f(x)\sum_{k_1=0}^{n}\begin{bmatrix} q^{-n}, & xb_1, & a_2/b_2 \\ q, & b_1q^{1-n}/a_1, & xa_2 \end{bmatrix}; q\Bigg]_{k_1}\left(\frac{qb_2}{a_1}\right)^{k_1}$$

$$\times\sum_{k_2=0}^{k_1}\begin{bmatrix} q^{-k_1}, & xb_2, & a_3/b_3 \\ q, & b_2q^{1-k_1}/a_2, & xa_3 \end{bmatrix}; q\Bigg]_{k_2}\left(\frac{qb_3}{a_2}\right)^{k_2}\cdots$$

$$\times\sum_{k_{r-1}=0}^{k_{r-2}}\begin{bmatrix} q^{-k_{r-2}}, & xb_{r-1}, & a_r/b_r \\ q, & b_{r-1}q^{1-k_{r-2}}/a_{r-1}, & xa_r \end{bmatrix}; q\Bigg]_{k_{r-1}}\left(\frac{qb_r}{a_{r-1}}\right)^{k_{r-1}}$$

$$= b_1^n\frac{(a_1/b_1;q)_n}{(xa_1;q)_n}f(x)\sum_{0\le k_{r-1}\le k_{r-2}\le\cdots\le k_1\le k_0=n}$$

$$\times\prod_{i=1}^{r-1}\begin{bmatrix} q^{-k_{i-1}}, & xb_i, & a_{i+1}/b_{i+1} \\ q, & b_iq^{1-k_{i-1}}/a_i, & xa_{i+1} \end{bmatrix}; q\Bigg]_{k_i}\left(\frac{qb_{i+1}}{a_i}\right)^{k_i}. \tag{10}$$

This proves the Theorem.■

Proof of Theorem 2

In (7), taking $q \to 1/q$, then replacing (x, a_i, b_i) by $(1/x, 1/a_i, 1/b_i)$ respectively, where $i = 1, 2, \cdots, r$. We can get the Theorem 2.■

3. Some special cases

Putting $r=2$ in (7), then replacing (a_1x, a_2x, b_1x, b_2x) by (a_1, a_2, b_1, b_2) respectively, we have the following Sears' transformation formula

Corollary 1. The terminating Sears' ${}_3\Phi_2$ transformation formula [5, p. 61, Eq. 3.2.2]

$$
{}_3\Phi_2\left(\begin{matrix} q^{-n}, & b_1, & b_2 \\ & a_1, & a_2 \end{matrix}; q, q\right) =
$$

$$
(b_1)^n \frac{(a_1/b_1;q)_n}{(a_1;q)_n} \sum_{k_1=0}^{n} \left[\begin{matrix} q^{-n}, & b_1, & a_2/b_2 \\ q, & b_1q^{1-n}/a_1, & a_2 \end{matrix}; q\right]_{k_1} \left(\frac{qb_2}{a_1}\right)^{k_1}. \quad (11)
$$

Setting $r=2$ in (8), then replacing (a_1x, a_2x, b_1x, b_2x) by (a_1, a_2, b_1, b_2) respectively, we have another Sears' transformation formula

Corollary 2. The terminating Sears' ${}_3\Phi_2$ transformation formula [5, p. 61, Eq. 3.2.5]

$$
{}_3\Phi_2\left(\begin{matrix} q^{-n}, & b_1, & b_2 \\ & a_1, & a_2 \end{matrix}; q, q^n a_1 a_2/b_1 b_2\right) =
$$

$$
\frac{(a_1/b_1;q)_n}{(a_1;q)_n} \sum_{k_1=0}^{n} \left[\begin{matrix} q^{-n}, & b_1, & a_2/b_2 \\ q, & b_1q^{1-n}/a_1, & a_2 \end{matrix}; q\right]_{k_1} q^{k_1}. \quad (12)
$$

Letting $a_i = qb_i$ and $a_1 = a_2 = \cdots = a_r$ in (8), where $i = 1, 2, \cdots, r$, then simplifying and putting $xb_i = x$, we have the following identity

Corollary 3. We have

$$
\sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} (-1)^k \frac{q^{\binom{k}{2}+kr}}{(1-xq^k)^r} = \frac{(q;q)_n}{(x;q)_{n+1}} \sum_{0 \le k_{r-1} \le k_{r-2} \le \cdots \le k_1 \le k_0 = n}
$$

$$
\times \prod_{i=1}^{r-1} \frac{q^{k_i}}{1-xq^{k_i}}. \quad (13)
$$

Putting $x = q$, then multiplying q^{r-1} on both sides of (13), we have the following finite extension of Uchimura's identity

Corollary 4. A finite extension of Uchimura's identity

$$
\sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} (-1)^k q^{\binom{k+2}{2}+(r-2)(k+1)} \frac{1}{(1-q^{k+1})^r}
$$

$$
= \frac{(q;q)_n}{(q;q)_{n+1}} \sum_{0 \le k_{r-1} \le k_{r-2} \le \cdots \le k_1 \le k_0 = n} \prod_{i=1}^{r-1} \frac{q^{k_i+1}}{1-q^{k_i+1}}. \quad (14)
$$

Taking $x = q$ and $n \to \infty$ in (14), we have the following extension of Uchimura's

Corollary 5. An extension of Uchimura's identity

$$\sum_{k=1}^{\infty}(-1)^{k-1}\frac{q^{\binom{k+1}{2}+k(r-2)}}{(q;q)_{k-1}(1-q^k)^r} = \sum_{1\le k_{r-1}\le k_{r-2}\le\cdots\le k_0=n}\prod_{i=1}^{r-1}\frac{q^{k_i}}{1-q^{k_i}}. \quad (15)$$

In (15), setting $r = 2$, we have Uchimura's identity (cf. [9])

$$\sum_{k=1}^{\infty}(-1)^{k-1}\frac{q^{\binom{k+1}{2}}}{(q;q)_{k-1}(1-q^k)^2} = \sum_{k_1=1}^{\infty}\frac{q^{k_1}}{1-q^{k_1}}. \quad (16)$$

In fact, this identity was known much earlier (cf. [6]).

Taking the limit as $q \to 1$ in (14), we have the following binomial identity

Corollary 6. We have

$$\sum_{k=0}^{n}\binom{n}{k}(-1)^k\frac{1}{(k+1)^r} = \frac{1}{n+1} \times \sum_{0\le k_{r-1}\le k_{r-2}\le\cdots\le k_1\le k_0=n}\frac{1}{(k_1+1)\cdots(k_{r-1}+1)}. \quad (17)$$

Remark. We can derive the identity (5) from (13) by using a slick trick: Let us take $x-1=\omega$ and set $k\neq 0, k_i\neq 0 \quad (i=1,\cdots,r-1)$. (13) can be rewritten as follows

$$\sum_{k=1}^{n}\begin{bmatrix}n\\k\end{bmatrix}(-1)^k\frac{q^{\binom{k}{2}+kr}}{(1-q^k)^r}\frac{(-1)^r\omega^r}{(1-\omega q^k/(1-q^k))^r} = \frac{1}{(1-\omega q/(1-q))\cdots(1-\omega q^n/(1-q^n))} \times \sum_{1\le k_{r-1}\le k_{r-2}\le\cdots\le k_1\le k_0=n}(-1)^{r-1} \times \prod_{i=1}^{r-1}\frac{q^{k_i}}{1-q^{k_i}}\frac{\omega}{(1-\omega q^{k_i}/(1-q^{k_i}))}. \quad (18)$$

Then we expand both sides of (18) into a power series of ω

$$\sum_{k=1}^{n}\begin{bmatrix} n \\ k \end{bmatrix}(-1)^{k+1}\frac{q^{\binom{k}{2}+kr}}{(1-q^k)^r}\omega^r\left(\sum_{j=0}^{\infty}\left(\frac{q^k}{1-q^k}\right)^j\omega^j\right)^r$$

$$=\sum_{m=1}^{n}\sum_{j=0}^{\infty}\left(\frac{q^m}{1-q^m}\right)^j\omega^j\sum_{1\le k_{r-1}\le k_{r-2}\le\cdots\le k_1\le k_0=n}$$

$$\times\prod_{i=1}^{r-1}\frac{q^{k_i}}{1-q^{k_i}}\omega\sum_{l=0}^{\infty}\left(\frac{q^{k_i}}{1-q^{k_i}}\right)^l\omega^l.$$

Thus we can get the Dilcher's identity (5) by comparing the coefficients of ω^r.

Acknowledgements

I am grateful for the referees for providing a detailed report offering many helpful criticisms and suggestions leading to an improvement of an earlier version of this paper. The author supported by Doctorial Program of ME of China 20060269011. The work also supported by Universities Natural Science Foundation of Jangsu (No. 09KJB110002) and National Natural Science Foundation of China (No. 10971078)

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