Automorphism groups of tetravalent Cayley graphs on regular 5-groups

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Abstract

In [2] it is proved that if $X = \text{Cay}(G, S)$ is a connected tetravalent Cayley graph on a regular p-group G (for $p \neq 2, 5$), then the right regular representation of G is normal in the automorphism group of X. In this paper we prove that a similar result holds, for $p = 5$, under a slightly stronger hypothesis. Some remarkable examples are presented.

Keywords: Cayley graph, regular p-group

1 Introduction

Let G be a group and S a subset of G such that $1 \notin S$. The *Cayley graph* on G with *connection set* S, denoted $\text{Cay}(G, S)$, is the digraph with vertex set G and edge set $\{(g, h) \mid hg^{-1} \in S\}$. We recall that $\text{Cay}(G, S)$ is connected if and only if $G = \langle S \rangle$. Note that if $S = S^{-1}$, then $\text{Cay}(G, S)$ can be viewed as an undirected graph by identifying two oppositely directed edges.

A Cayley graph $X = \text{Cay}(G, S)$ on G is said to be *normal* on (G, S) if the right regular representation $R(G)$ of G, i.e. the permutation representation of G acting by right multiplication, is a normal subgroup of $\text{Aut}(X)$. In particular, it was noticed in [4] that the digraph X is normal on (G, S) if and only if $\text{Aut}(X)$ is the semidirect product of $R(G)$ with $\text{Aut}(G, S)$, where $\text{Aut}(G, S) = \{\alpha \in \text{Aut}(G) \mid S^\alpha = S\}$, i.e. $\text{Aut}(X) = R(G) \rtimes \text{Aut}(G, S)$.

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When the set S is clear from the context we simply say that X is normal on G.

We recall the following definition, see [5] and [6].

Definition 1 *A finite p-group G is regular if for every $x, y \in G$ there exists $c_{xy} \in (\gamma_2(\langle x, y\rangle))^p$ such that $(xy)^p = x^p y^p c_{xy}$.*

In here $\gamma_2(G)$ denotes the second term of the lower central series of G, i.e. the commutator subgroup of G. Also, G^p denotes the group $\langle g^p \mid g \in G\rangle$, i.e the subgroup generated by the pth powers of the elements of G. We note that the class of regular p-groups was introduced by P.Hall as a generalization of the class of abelian p-groups. We refer the interested reader to [6] for an overview on regular p-groups and their applications. In particular we recall that the class of regular p-groups is very rich (for example any p-group of nilpotency class at most $p-1$ or exponent p is a regular p-group) and is a very useful tool for studying p-groups of maximal class, or more generally p-groups of bounded coclass. Moreover, we recall that regular p-groups and abelian p-groups have several common properties, see [6].

It was proved in [1] the following interesting theorem on Cayley graphs on abelian groups.

Theorem ([1], Theorem 1.1) *Let $X = \mathrm{Cay}(G, S)$ be a connected undirected Cayley graph on an abelian group G with valency 4. Then either X is normal on G or X is one of a known list of exceptions. In particular, the only exception, when G is a finite odd abelian p-group, is $X = K_5$.*

In [2], using some remarkable similarities between abelian p-groups and regular p-groups, it was proved the following generalization of Theorem 1.1 in [1].

Theorem ([2], Theorem 3.1) *Any connected tetravalent undirected Cayley graph on a regular p-group G, with $p \neq 2, 5$, is normal on G.*

It is worth noticing that a regular 2-group is abelian (see [5]) and so the classification of the connected tetravalent undirected Cayley graphs on regular 2-groups was achieved in [1]. In this article, we point out that a result similar to Theorem 3.1 in [2] holds, for $p = 5$, under a slightly stronger hypothesis.

In this paragraph we introduce some notation. If G is a group and φ is an automorphism of G, then we denote by g^φ the image of the group element g via φ, i.e. we let φ act on G on the right. In particular, if $V = \mathbb{F}_p^n$ is the row-vector space over $\mathbb{F}_p$ of dimension n, v is a row-vector of V and

A is a matrix in $\mathrm{GL}(n, \mathbb{F}_p)$, then vA is the image of v via the isomorphism induced by A on V.

Let T be the 5-group with presentation

$$T = \langle e_1, e_2, e_3, x \mid e_i^5 = [e_i, e_j] = 1, x^5 = e_3, e_1^x = e_1e_2, e_2^x = e_2e_3, e_3^x = e_3 \rangle.$$

In particular, T is a non-split extension of the elementary abelian 5-group $\langle e_1, e_2, e_3 \rangle$ by the element x, where the action of x is given by the matrix

$$A = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

(note that, by our convention, the matrix A acts on the right on $\langle e_1, e_2, e_3 \rangle$).

In Section 2, we prove the following theorem.

Theorem 1 *Let G be a regular 5-group with no image isomorphic to T. If $X = \mathrm{Cay}(G, S)$ is a connected tetravalent undirected Cayley graph on G, then either X is normal on G or $X = K_5$.*

We point out that the proof of Theorem 1 uses the proof of Theorem 3.1 in [2].

2 Proof of Theorem 1

In the sequel we use the same notation as in [2]. We denote by $\Phi(G)$ the Frattini subgroup of the p-group G, i.e. $\Phi(G) = G^p\gamma_2(G)$. We recall that $G/\Phi(G)$ is an elementary abelian p-group and that the minimal number of generators of G is the number of elements in a basis of $G/\Phi(G)$, see [5].

Proposition 1 *Let G be a regular 5-group with no image isomorphic to T and let S be a subset of G with $S = S^{-1}$, $|S| = 4$ and $G = \langle S \rangle$. Let H be a non-trivial elementary abelian normal subgroup of G. If $\overline{X} = \mathrm{Cay}(G/H, SH/H) = K_5$, then $X = \mathrm{Cay}(G, S)$ is normal on (G, S).*

Proof. Since $|V(\overline{X})| = 5$, we have that H has index 5 in G and so $|V(X)| \geq 5^2$. Now, by Theorem 1.1 in [1], if G is abelian, then X is normal on (G, S). Therefore we may assume that G is not abelian and so, in particular, $|H| \geq 5^2$. Since G is generated by S, we have that G is 2-generated.

We claim that there are exactly five non-abelian 2-generated regular 5-groups G with no image isomorphic to T and with an elementary abelian maximal subgroup H.

Set $|H| = 5^k$. Let x be an element in $G \backslash H$. The element x acts, by conjugation, on H as an automorphism ι_x. Fix $\mathfrak{B}$ a basis of H such that the matrix M corresponding to ι_x, with respect to the basis $\mathfrak{B}$, is in Jordan form. Assume that M has t Jordan blocks of size $n_1, \ldots, n_t$. We may assume that $n_1 \leq \cdots \leq n_t$. In particular,

$$M = \begin{pmatrix} J_{n_1} & & \\ & \ddots & \\ & & J_{n_t} \end{pmatrix}$$

where

$$J_{n_i} = \begin{pmatrix} 1 & 1 & & \\ & \ddots & \ddots & \\ & & 1 & 1 \\ & & & 1 \end{pmatrix}$$

denotes a Jordan block of size n_i.

Now, we label the elements of $\mathfrak{B}$ as $h_{1,1}, \ldots, h_{1,n_1}, \ldots, h_{t,1}, \ldots, h_{t,n_t}$ so that $h_{i,1}, \ldots h_{i,n_i}$ are the row-vectors corresponding to the ith Jordan block of M. For example, $h_{1,1}, \ldots, h_{1,n_1}$ are the first n_1 elements of the basis $\mathfrak{B}$ and $h_{2,j}$ is the $(n_1 + j)$th element of $\mathfrak{B}$, for $1 \leq j \leq n_2$. In particular, $h_{i,j}^x = h_{i,j} h_{i,j+1}$ for $1 \leq j < n_i$ and $h_{i,n_i}^x = h_{i,n_i}$.

From the previous paragraph we get $[H, x] = \langle h_{1,2}, \ldots, h_{1,n_1}, \ldots, h_{t,2}, \ldots, h_{t,n}$
Furthermore, the elements $x, h_{1,1}, \ldots, h_{t,1}$ generate G and commute modulo $[H, x]$. Therefore, $G/[H, x]$ is abelian. Hence, $\gamma_2(G) = [H, x]$.

Since $G/\gamma_2(G)$ is abelian, we have $(ab)^5 \equiv a^5 b^5 \mod \gamma_2(G)$ for every a, b in G. As H has exponent 5 and $G = \langle x \rangle H$, we get that $\Phi(G) = G^5 \gamma_2(G) = \langle x^5 \rangle \gamma_2(G)$. Now, in order to prove that there are exactly five non-abelian 2-generated regular 5-groups G with no image isomorphic to T and with an elementary abelian maximal subgroup H, we consider two cases.

Case A: x^5 lies in $\gamma_2(G)$.
In this case $\Phi(G) = \gamma_2(G) = [H, x]$ and $|\Phi(G)| = |[H, x]| = 5^{k-t}$. The group G is 2-generated, so $5^2 = |G/\Phi(G)| = 5^{k+1-(k-t)} = 5^{1+t}$. Therefore $t = 1$ and M is a Jordan block of size k. Now, the isomorphism class of G depends only on the order of x. If x has order 5, then $G = \langle x \rangle \ltimes H$, where the action of x on H is given by $M = J_k$.

Now assume $x^5 \neq 1$. We have $x^5 \in H$ and x^5 commutes with every element of G. Therefore $x^5 \in \langle h_{1,k} \rangle$, so, without loss of generality, we may assume that $x^5 = h_{1,k}$. Therefore, $G = \langle x \rangle \cdot H$, where $x^5 = h_{1,k}$ and the action of x on H is given by $M = J_k$. In particular, since G is non-abelian and G has no image isomorphic to T, we have $k = 2$.

In either case ($x^5 = 1$ or $x^5 \neq 1$), if $k \leq 4$, then $|G| \leq 5^5$. Thus, G has nilpotency class at most 4. Therefore G is a regular 5-group. It is routine to check that if $k \geq 5$, then $G = \langle x \rangle \ltimes H$ is not regular, see Satz 10.3(d) in [5]. This gives us four groups.

CASE B: x^5 does not lie in $\gamma_2(G)$.
In this case $\Phi(G) = \langle x^5 \rangle \gamma_2(G) = \langle x^5 \rangle [H, x]$ and $|\Phi(G)| = 5|[H, x]| = 5^{k-t+1}$. The group G is 2-generated, so $5^2 = |G/\Phi(G)| = 5^{k+1-(k-t+1)} = 5^t$. Therefore $t = 2$ and M has two Jordan blocks. We claim that $n_1 = 1$ and $n_2 = k - 1$. Since $x^5 \in H$, we have $x^5 = \prod_{i=1}^{n_1} h_{1,i}^{r_i} \prod_{j=1}^{n_2} h_{2,j}^{s_j}$ for some r_i, s_j. Further, since $x^5 \notin \gamma_2(G) = [H, x]$, we have $(r_1, s_1) \neq (0, 0)$. Now, x^5 commutes with x. Therefore,

$$\begin{aligned}
\prod_{i=1}^{n_1} h_{1,i}^{r_i} \prod_{j=1}^{n_2} h_{2,j}^{s_j} &= x^5 = (x^5)^x = \prod_{i=1}^{n_1} (h_{1,i}^{r_i})^x \prod_{j=1}^{n_2} (h_{2,j}^{s_j})^x \\
&= \prod_{i=1}^{n_1-1} (h_{1,i} h_{1,i+1})^{r_i} h_{1,n_1}^{r_{n_1}} \prod_{j=1}^{n_2-1} (h_{2,j} h_{2,j+1})^{s_j} h_{2,n_2}^{s_{n_2}} \\
&= h_{1,1}^{r_1} \prod_{i=2}^{n_1} h_{1,i}^{r_{i-1}+r_i} h_{2,1}^{s_1} \prod_{j=2}^{n_2} h_{2,j}^{s_{j-1}+s_j}.
\end{aligned}$$

This yields $r_i = r_{i-1} + r_i$ for $2 \leq i \leq n_1$ and $s_j = s_{j-1} + s_j$ for $2 \leq j \leq n_2$. So, $r_i = 0$ for $i = 1, \ldots, n_1 - 1$ and $s_j = 0$ for $j = 1, \ldots, n_2 - 1$. If $n_1 > 1$, then $r_1 = s_1 = 0$ and $x^5 \in [H, x]$, a contradiction. This yields $n_1 = 1$, $n_2 = k - 1$ and $s_j = 0$ for $j = 1, \ldots, n_2 - 1$. In particular, we get that $x^5 = h_{1,1}^{r_1} h_{2,k-1}^{s_{k-1}}$ for some r_1, s_{k-1} and $r_1 \neq 0$. Now, up to replacing the basis $h_{1,1}, h_{2,1}, \ldots, h_{2,k-1}$ of H with $h_{1,1}^{r_1} h_{2,k-1}^{s_{k-1}}, h_{2,1}, \ldots, h_{2,k-1}$, we may assume that $x^5 = h_{1,1}$. In particular, we have that $G = \langle x \rangle \cdot H$, where $x^5 = h_{1,1}$ and the action of x on H is given by

$$M = \begin{pmatrix} 1 & \\ & J_{k-1} \end{pmatrix}.$$

Since G is non-abelian, we have $k \geq 3$. The group G has an image isomorphic to T if and only if $k \geq 4$. In fact, if $k \geq 4$, then $T \cong G/N$ where $N = \langle h_{1,1} h_{2,3}, h_{2,4}, \ldots, h_{2,k-1} \rangle$. For $k = 3$, the group G has nilpotency class 2, thus G is a regular 5-group with no image isomorphic to T. Hence Case B yields only one group.

Therefore, as claimed, there are exactly five non-abelian 2-generated regular 5-groups with no image isomorphic to T and containing an elementary abelian maximal subgroup.

The rest of the proof of this lemma is entirely computational and is left to the reader. Indeed, since the group structure of G is determined and since we are dealing with a finite number of relatively small groups, we can check Proposition 1 using the very efficient package GRAPE of GAP written by L.Soicher and the package nauty written by B.McKay, see [7] and [8]. As a matter of completeness, in the next paragraph, we show how this can be done using as a guiding example the group $G = \langle x \rangle \ltimes H$, where $|H| = 5^3$ and the action of x on H is given by $M = J_3$.

Clearly, $G = \langle x, h_{1,1} \rangle$ and $\Phi(G) = \langle h_{1,2}, h_{1,3} \rangle$. We recall that if $F = \langle X \mid R \rangle$ is a finite group, defined by the generators X and by the relations R, and $\varphi : F \to F$ is a function mapping the generating set X of F to another generating set of F and preserving the defining relations R of F, then φ is an automorphism of F. Now, the group G has presentation

$$G = \langle x, h_{1,1}, h_{1,2}, h_{1,3} \mid x^5 = h_{1,i}^5 = [h_{1,i}, h_{1,j}] = 1, \\ h_{1,1}^x = h_{1,1}h_{1,2}, h_{1,2}^x = h_{1,2}h_{1,3}, h_{1,3}^x = h_{1,3} \rangle.$$

We claim that for any $v \in H$ and for any $w \in \Phi(G)$, there exists an automorphism φ of G such that $x^\varphi = xv$ and $h_{1,1}^\varphi = h_{1,1}w$. Set $h_{1,2}^\varphi = [h_{1,1}^\varphi, x^\varphi]$ and $h_{1,3}^\varphi = [h_{1,2}^\varphi, x^\varphi]$. By the given presentation of G and by the above remark, to prove our claim it suffices to show that $x^\varphi, h_{1,1}^\varphi$ are generators of G and that φ preserves the defining relations of G. Since $w \in \Phi(G)$, we have $\langle xv, h_{1,1}w \rangle = \langle xv, h_{1,1} \rangle = G$. So, $x^\varphi, h_{1,1}^\varphi$ are generators of G. Also, since G has exponent 5, we have $(h_{1,i}^\varphi)^5 = 1$ and $(x^\varphi)^5 = 1$. All the remaining relations can be checked similarly.

This shows that if S is a subset of G such that $|S| = 4$, $S = S^{-1}$, $G = \langle S \rangle$ and $\mathrm{Cay}(G/H, SH/H) = K_5$, then S is $\mathrm{Aut}(G)$-conjugate to the following set

$$J = \{x, x^{-1}, (xh_{1,1})^2, (xh_{1,1})^{-2}\}.$$

Therefore, to prove Proposition 1 for the group G, it is enough to check that the Cayley graph $X = \mathrm{Cay}(G, J)$ is normal on (G, J). Now, the built-in command `CayleyGraph` in GRAPE allows us to compute X. Then, the command `AutomorphismGroup` allows us to compute $A = \mathrm{Aut}(X)$. Finally, we can check that $R(G)$ is normal in A.

The remaining four groups can be checked similarly. □

PROOF OF THEOREM 1. Let G be a regular 5-group with no image isomorphic to T and $X = \mathrm{Cay}(G, S)$ a connected tetravalent undirected Cayley graph. We have to prove that either X is normal on (G, S) or $X = K_5$. We argue by induction on $|V(X)|$. If $|V(X)| = 5$, then $X = K_5$ and so there is nothing to prove.

We use the proof of Theorem 3.1 in [2] to conclude the induction. Indeed, the proof of Theorem 3.1 in [2] uses the hypothesis that $p \neq 5$ *only* on page 358 (lines 11–20). So, we aim to use the the hypothesis "G has no image isomorphic to T" to adjust the proof of Theorem 3.1 to deal with the case $p = 5$. In the following paragraph we sketch the proof of Theorem 3.1, up to page 358 line 20, to point out to the reader where the hypothesis $p \neq 5$ is used (we suggest the use of [2] as a crib for the reader).

Let A be $\mathrm{Aut}(X)$ and N a minimal normal subgroup of A. Let $\mathcal{B}$ be the set of orbits of N on $V(X)$, $m = |\mathcal{B}|$ and K the kernel of A on its natural action on $\mathcal{B}$. Let $\overline{X}$ be the quotient graph corresponding to the blocks in $\mathcal{B}$. Using the classification of the finite simple groups the authors of [2] proved that, if $m > 1$ and $\overline{X}$ has valency 4, then N is a non-trivial elementary abelian p-group, $N \subseteq R(G)$ and $K = N$ (lines 11–17 page 358). In particular $N = R(H)$, where H is a non-trivial elementary abelian normal subgroup of G. From this the authors of [2] proved that $\overline{X}$ is the tetravalent Cayley graph on the regular p-group G/H with connection set SH/H, i.e. $\overline{X} = \mathrm{Cay}(G/H, SH/H)$. In particular, $A/R(H) \subseteq \mathrm{Aut}(\overline{X})$. Now, if $p \neq 2, 5$, using induction and Theorem 1.1 in [1], we have that the Cayley graph $\overline{X}$ is normal on $(G/H, SH/H)$. Therefore, the group $R(G/H) = R(G)/R(H)$ is a normal subgroup of $\mathrm{Aut}(\overline{X})$. Thus $R(G)/R(H)$ is a normal subgroup of $A/R(H)$, i.e. $R(G)$ is a normal subgroup of A. Hence X is normal on (G, S) (lines 18–20).

Clearly, the previous argument cannot be used in the case $p = 5$, as in general $\overline{X}$ is not normal on $(G/H, SH/H)$, (for instance K_5 is not normal).

Now, assume in the previous argument $p = 5$. If $\overline{X}$ is normal on $(G/H, SH/H)$, then X is normal on (G, S) (use the same argument as in the case $p \neq 5$).

Therefore, we may assume that $\overline{X}$ is not normal on $(G/H, SH/H)$. Since G/H is a regular 5-group with no image isomorphic to T and $|V(\overline{X})| < |V(X)|$, by induction we get $\overline{X} = K_5$. Since H is a non-trivial elementary abelian subgroup of G, Proposition 1 yields that X is normal on (G, S).

In particular, in either case ($\overline{X}$ is normal on G/H or $\overline{X}$ is not normal on G/H) the Cayley graph X is normal on (G, S).

The rest of the proof of Theorem 3.1 in [2] (i.e. the case $m = 1$ and the case $\overline{X}$ is not tetravalent) does not use the hypothesis that $p \neq 5$. Therefore, the rest of the proof of Theorem 3.1 in [2] yields Theorem 1. □

3 Some remarkable examples

Let $n \geq 1$ be a natural number and $G_n = \mathrm{PSL}(2, \mathbb{Z}/5^n\mathbb{Z})$. Consider

$$i = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, t = \begin{bmatrix} -(y+1) & y \\ y+1 & y \end{bmatrix} \text{ and } H_n = \langle i, t \rangle,$$

where $2y^2 + 2y + 1 = 0$ and $y \equiv 3 \mod 5$ (note that y exists by Hensel's lemma). The element i has order 2 and the element t has order 3. Set

$j = i^t$ and $k = j^t$. We have

$$j = \begin{bmatrix} 0 & 2y+1 \\ 2y+1 & 0 \end{bmatrix}, k = \begin{bmatrix} 2y+1 & 0 \\ 0 & -(2y+1) \end{bmatrix} \text{ and } k = ij.$$

Therefore the group H_n is a subgroup of G_n isomorphic to Alt(4). We consider the Sylow 5-subgroup of G_n defined by

$$P_n = \left\{ \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \mid a_{11}, a_{22} \equiv 1 \bmod 5, a_{21} \equiv 0 \bmod 5, a_{11}a_{22} - a_{12}a_{21} = 1 \right\}.$$

Also, we consider the action of G_n on the right cosets $\Omega_n = G_n/H_n$. Since the order of P_n and H_n are coprime, we have $P_n \cap H_n = 1$. Further, $|G_n| = |P_n||H_n|$. Therefore P_n acts regularly on Ω_n. Since the only subgroup of H_n normal in G_n is the trivial group, we get that G_n acts faithfully on Ω_n.

Since P_n is a regular subgroup of G_n, we identify the elements of Ω_n with the elements of P_n. Indeed, every point of Ω_n is of the form $H_n x$, for a unique x in P_n. So, we identify the point $H_n x$ with the group element x of P_n. Further, if $\omega \in \Omega_n$ and $g \in G_n$, then we denote by $\omega \cdot g$ the action of g on ω, to distinguish it from the matrix multiplication ωg.

We claim that the element t of H_n fixes the point

$$\alpha = \begin{bmatrix} 1 & 2y+2 \\ 0 & 1 \end{bmatrix}$$

of Ω_n. Indeed,

$$\alpha t = \begin{bmatrix} y & y-1 \\ y+1 & y \end{bmatrix} = \begin{bmatrix} y & y \\ y+1 & -(y+1) \end{bmatrix} \alpha$$

and

$$\begin{bmatrix} y & y \\ y+1 & -(y+1) \end{bmatrix} = t^{-1} i$$

lies in H_n. Therefore $H_n \alpha t = H_n t^{-1} i \alpha = H_n \alpha$, i.e. $\alpha \cdot t = \alpha$.

It follows that the H_n-orbit S_n of the point α contains four elements. Namely $S_n = \{\alpha, \alpha \cdot i, \alpha \cdot j, \alpha \cdot k\}$. We leave it to the reader to check that

$$\begin{aligned} \beta = \alpha \cdot i &= \begin{bmatrix} y+1 & -(y+1) \\ -(3y+1) & y+1 \end{bmatrix}, \\ \alpha \cdot j &= \beta^{-1}, \\ \alpha \cdot k &= \alpha^{-1}. \end{aligned}$$

So, $S_n = \{\alpha, \alpha^{-1}, \beta, \beta^{-1}\}$.

We note that identifying the points of Ω_n with the elements of P_n we get that G_n is a subgroup of the automorphism group of $\Gamma_n = \mathrm{Cay}(P_n, S_n)$.

Furthermore, since $S_n = S_n^{-1}$, we have that Γ_n is a graph and not a digraph. Therefore Γ_n is a tetravalent undirected Cayley graph. Since P_n is not a normal subgroup of G_n, we have that Γ_n is not normal on (P_n, S_n). We note that $P_2 \cong T$.

We recall that P_n is 2-generated and that

$$\Phi(P_n) = \left\{ \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \middle| \begin{array}{l} a_{11}, a_{22} \equiv 1 \mod 5, a_{12} \equiv 0 \mod 5, \\ a_{21} \equiv 0 \mod 25, a_{11}a_{22} - a_{12}a_{21} = 1 \end{array} \right\},$$

see Section 17 of [5]. Now, since α, β generate P_n modulo $\Phi(P_n)$, we have that $P_n = \langle S_n \rangle$. Therefore Γ_n is connected.

This construction gives rise to an infinite family, namely $\{\Gamma_n\}_{n \geq 1}$, of non-normal connected tetravalent undirected Cayley graphs.

In the rest of this section we prove that the group P_n is a regular 5-group. We need to fix some notation. For $1 \leq i \leq n$, let $- : \mathbb{Z}/5^n\mathbb{Z} \to \mathbb{Z}/5^i\mathbb{Z}$ be the natural projection and $\pi_{n,i} : G_n \to G_i$ the natural homomorphism induced by $-$, i.e.

$$\pi_{n,i} : \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \longmapsto \begin{bmatrix} \overline{a_{11}} & \overline{a_{12}} \\ \overline{a_{21}} & \overline{a_{22}} \end{bmatrix}.$$

We denote by $V_{n,i}$ the kernel of $\pi_{n,i}$. We have

$$V_{n,i} = \left\{ \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \middle| \begin{array}{l} a_{11}, a_{22} \equiv 1 \mod 5^i, a_{12}, a_{21} \equiv 0 \mod 5^i, \\ a_{11}a_{22} - a_{12}a_{21} = 1 \end{array} \right\}.$$

Clearly, $V_{n,i+1} \subseteq V_{n,i}$ and $V_{n,n} = 1$. Also, $\Phi(P_n) \subseteq V_{n,1}$ and $V_{n,1}$ is a maximal subgroup of P_n. Furthermore, if $n \geq 2$, then the group $V_{n,n-1}$ is an elementary abelian 5-group of order 5^3, see Section 17 in [5]. Also, since $\pi_{n,i}$ is surjective, we have $P_n/V_{n,i} \cong P_i$ and $V_{n,k}/V_{n,i} \cong V_{i,k}$, for $1 \leq k \leq i \leq n$.

We recall the following result on regular p-groups.

Proposition 2 ([5], Satz 10.14) *Let G be a p-group. If G has no normal subgroup N of exponent p with $|N| \geq p^{p-1}$, then G is a regular p-group.*

We need the following lemma.

Lemma 1 *Let g be in P_2. The element g has order 5 if and only if $g \in V_{2,1}$.*

PROOF. Since $V_{2,1}$ is an elementary abelian 5-group, we have that every element in $V_{2,1}$ has order 5.

We have $|P_2| = 5^4$. So, P_2 has nilpotency class at most 3. Therefore P_2 is a regular 5-group. Thus the set O of elements of P_2 of order at most 5 is a subgroup of P_2. Now, the element

$$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

lies in P_2 and has order 25. In particular, since $x \notin O$ and $V_{2,1} \subseteq O$, we have $O = V_{2,1}$. Thus the lemma is proved. □

Finally, using Proposition 2 and Lemma 1 we prove that P_n is a regular 5-group.

Lemma 2 *The group P_n is a regular 5-group.*

PROOF. The group P_1 has order 5. So, we may assume that $n \geq 2$.

Let N be a normal subgroup of P_n of exponent 5. We claim that $|N| \leq 5^3$. In particular, by Proposition 2, we have that P_n is a regular 5-group.

We argue by contradiction. Assume $|N| \geq 5^4$. We claim that if $1 \leq i \leq n-2$ and $x \in V_{n,i} \backslash V_{n,i+1}$, then $x^5 \in V_{n,i+1} \backslash V_{n,i+2}$. Let x be in $V_{n,i} \backslash V_{n,i+1}$. We have

$$x = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + 5^i \begin{bmatrix} a & b \\ c & d \end{bmatrix} = I + 5^i M, \text{ where } a, b, c, d \text{ are not all divisible by 5,}$$

(I denotes the identity matrix). By the binomial formula, we get

$$x^5 = I + 5^{i+1}M + 2 \cdot 5^{2i+1}M^2 + 2 \cdot 5^{3i+1}M^3 + 5^{4i+1}M^4 + 5^{5i}M^5.$$

Now, since $2i+1, 3i+1, 4i+1, 5i \geq i+2$, we have that $x^5 \in V_{n,i+2}$ if and only if the entries of $5^{i+1}M$ are divisible by 5^{i+2}. By hypothesis not all the entries of M are divisible by 5, thus $x^5 \in V_{n,i+1} \backslash V_{n,i+2}$. So the claim is proved.

In particular, from the previous claim we have that, if $x \in V_{n,1}$ has order 5, then $x \in V_{n,n-1}$. Therefore $V_{n,n-1} = \{x \in V_{n,1} \mid x^5 = 1\}$. Since $|N| \geq 5^4$ and $V_{n,1}$ is a maximal subgroup of P_n, we have $|V_{n,1} \cap N| \geq 5^3$. Since N has exponent 5, we have $N \cap V_{n,1} = V_{n,n-1}$.

We claim that no element in $P_n \backslash V_{n,1}$ has order 5. Let g be an element of order 5 in P_n. Then $\pi_{n,2}(g)$ has order 5 in P_2. Therefore, by Lemma 1, we get $\pi_{n,2}(g) \in V_{2,1}$, i.e. $gV_{n,2} \in V_{n,1}/V_{n,2}$. Thus, $g \in V_{n,1}$.

In particular, since N has exponent 5, we get $N \subseteq V_{n,1}$. Thus $N = N \cap V_{n,1} = V_{n,n-1}$ and $|N| = 5^3$. Therefore, there exists no normal subgroups of P_n of exponent 5 with $|N| \geq 5^4$. Thus P_n is a regular 5-group. □

Proposition 3 *There exist infinitely many non-normal connected tetravalent undirected Cayley graphs on regular 5-groups.*

PROOF. The graph $\Gamma_n = \mathrm{Cay}(P_n, S_n)$ is a connected tetravalent undirected Cayley graph on P_n. Also, Γ_n is not normal on (P_n, S_n). Finally, by Lemma 2, the group P_n is a regular 5-group. □

We point out that all other examples known to the author of this paper of non-normal connected tetravalent Cayley graphs are on non-regular p-groups (for $p \neq 2$). We remark that, for any $n \geq 2$, the graph Γ_n is a cover of Γ_{n-1}.

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