On the metric dimension of Möbius ladders*

Murtaza Ali[2], Gohar Ali[2], Muhammad Imran[1], A. Q. Baig[3], Muhammad Kashif Shafiq[3]

[1] Center for Advanced Mathematics and Physics, National University of Science and Technology, Sector H-12, Islamabad, Pakistan
[2] Department of Mathematics, FAST-NU, Peshawar, Pakistan
[3] Department of Mathematics, GC University Faisalabad, Faisalabad, Pakistan
E-mail: {imrandhab, aqbaig1, kashif4v}@gmail.com, gohar.ali@nu.edu.pk, murtaza_psh@yahoo.com

2000 Mathematics Subject Classification: 05C12

Abstract. If G is a connected graph, the distance $d(u,v)$ between two vertices $u,v \in V(G)$ is the length of a shortest path between them. Let $W = \{w_1, w_2, ..., w_k\}$ be an ordered set of vertices of G and let v be a vertex of G. The representation $r(v|W)$ of v with respect to W is the k-tuple $(d(v,w_1), d(v,w_2), ..., d(v,w_k))$. If distinct vertices of G have distinct representations with respect to W, then W is called a resolving set or locating set for G. A resolving set of minimum cardinality is called a basis for G and this cardinality is the metric dimension of G, denoted by $dim(G)$.
A family $\mathcal{G}$ of connected graphs is a family with constant metric dimension if $dim(G)$ does not depend upon the choice of G in $\mathcal{G}$. In this paper, we are dealing with the study of metric dimension of Möbius ladders. We prove that Möbius ladder M_n constitute a family of cubic graphs with constant metric dimension and only three vertices suffice to resolve all the vertices of Möbius ladder M_n except when $n \equiv 2(mod\ 8)$. It is natural to ask for the characterization of regular graphs with constant metric dimension.

Keywords: *Metric dimension, basis, resolving set, Möbius ladder*

1 Notation and preliminary results

If G is a connected graph, the *distance* $d(u,v)$ between two vertices $u,v \in V(G)$ is the length of a shortest path between them. Let $W =$

* This research is supported by NUST Islambad, FAST-NU Peshawar and Higher Education Commission of Pakistan.

$\{w_1, w_2, ..., w_k\}$ be an ordered set of vertices of G and let v be a vertex of G. The *representation* $r(v|W)$ of v with respect to W is the k-tuple $(d(v, w_1), d(v, w_2), ..., d(v, w_k))$. If distinct vertices of G have distinct representations with respect to W, then W is called a *resolving set* or *locating set* for G [1]. A resolving set of minimum cardinality is called a *metric basis* for G and this cardinality is the *metric dimension* of G, denoted by $dim(G)$. The concepts of resolving set and metric basis have previously appeared in the literature (see [1-4, 6-23]).
For a given ordered set of vertices $W = \{w_1, w_2, ..., w_k\}$ of a graph G, the *ith* component of $r(v|W)$ is 0 if and only if $v = w_i$. Thus, to show that W is a resolving set it suffices to verify that $r(x|W) \neq r(y|W)$ for each pair of distinct vertices $x, y \in V(G) \backslash W$.
A useful property in finding $dim(G)$ is the following lemma [22]:

Lemma 1. *Let W be a resolving set for a connected graph G and $u, v \in V(G)$. If $d(u, w) = d(v, w)$ for all vertices $w \in V(G) \setminus \{u, v\}$, then $\{u, v\} \cap W \neq \emptyset$.*

Motivated by the problem of uniquely determining the location of an intruder in a network, the concept of metric dimension was introduced by Slater in [20, 21] and studied independently by Harary and Melter in [6]. Applications of this invariant to the navigation of robots in networks are discussed in [18] and applications to chemistry in [4] while applications to problems of pattern recognition and image processing, some of which involve the use of hierarchical data structures are given in [19].
By denoting $G + H$ the join of G and H a *wheel* W_n is defined as $W_n = K_1 + C_n$, for $n \geq 3$, a *fan* is $f_n = K_1 + P_n$ for $n \geq 1$ and *Jahangir graph* J_{2n}, $(n \geq 2)$ (also known as *gear graph*) is obtained from the *wheel* W_{2n} by alternately deleting n spokes. Buczkowski *et al.* [1] determined the dimension of *wheel* W_n, Caceres *et al.* [3] the dimension of *fan* f_n and Tomescu and Javaid [23] the dimension of *Jahangir graph* J_{2n}.

Theorem 1. *([1], [3], [23]) Let W_n be a wheel of order $n \geq 3$, f_n be fan of order $n \geq 1$ and J_{2n} be a Jahangir graph. Then*
(i) For $n \geq 7$, $dim(W_n) = \lfloor \frac{2n+2}{5} \rfloor$;
(ii) For $n \geq 7$, $dim(f_n) = \lfloor \frac{2n+2}{5} \rfloor$;
(iii) For $n \geq 4$, $dim(J_{2n}) = \lfloor \frac{2n}{3} \rfloor$.

The metric dimension of all these plane graphs depends upon the number of vertices in the graph.
On the other hand, we say that a family $\mathcal{G}$ of connected graphs is a family with constant metric dimension if dim(G) does not depend upon the choice of G in $\mathcal{G}$. In [4] Chartrand et al. proved that a graph has metric dimension 1 if and only if it is a *path*, hence paths on n vertices constitute a family of graphs with constant metric dimension. Similarly, *cycles* with $n(\geq 3)$

vertices also constitute such a family of graphs as their metric dimension is 2 and does not depend upon on the number of vertices n. Caceres *et al.* [2] proved that

$$dim(P_m \times C_n) = \begin{cases} 2, & \text{if } n \text{ is odd;} \\ 3, & \text{otherwise.} \end{cases}$$

Since *prisms* D_n are the cubic plane graphs obtained by the cross product of the path P_2 with a cycle C_n, hence they constitute a family of 3-*regular graphs* with constant metric dimension. Also Javaid *et al.* proved in [17] that the plane graph *antiprism* A_n constitutes a family of regular graphs with constant metric dimension as $dim(A_n) = 3$ for every $n \geq 5$.
A *Cartesian product* of two graphs G and H, denoted by $G\square H$, is the graph with vertex set $V(G)\square V(H)$, where two vertices (x, x') and (y, y') are adjacent if and only if $x = y$ and $x'y' \in E(H)$ or $x' = y'$ and $xy \in E(G)$. The metric dimension of the cartesian product of graphs has been studied in [2].
The metric dimension of some classes of *plane graphs* and convex *polytopes* has been studies in [7]-[14] while metric dimension of generalized Petersen graphs $P(n, 3)$ and some rotationally-symmetric graphs has been discussed in [15] and [16].
In this paper, we extend this study by considering the Möbius ladder M_n which is a cubic circulant graph with an even number of vertices, formed from an n-cycle by adding edges (called "rungs") connecting opposite pair of vertices in the cycle. We prove that Möbius ladder M_n are cubic circulant graphs having constant metric dimension 3 except when $n \equiv 2(mod\ 8)$. It is natural to ask for the characterization of regular graphs with constant metric dimension.

2 Möbius ladders

The Möbius ladder M_n is a cubic circulant graph with an even number of vertices, formed from an n-cycle by adding edges (called "rungs") connecting opposite pair of vertices in the cycle. It is so-named because (with the exception of $M_6 = K_{3,3}$) M_n has exactly $n/2$ 4-cycles which link together by their shared edges to form a topological Möbius strip. Möbius ladders can also be viewed as a prism with one twisted edge. Two different views of Möbius ladders M_{16} have been shown in Fig. 1. Möbius ladders have many applications in chemistry, chemical stereography, electronics and computer science.
For our convenience, we view the Möbius ladder M_n as an n-cycle by adding edges (called "rungs") connecting opposite pairs of vertices in the cycle.

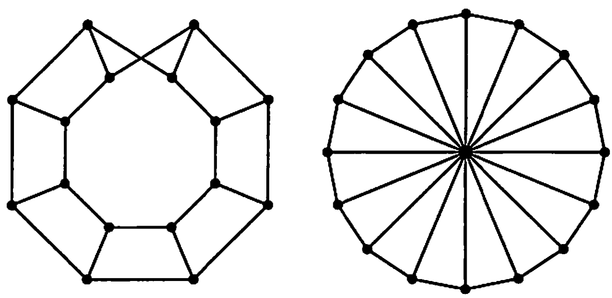

Fig. 1. Two views of Möbius ladder M_{16}

Suppose that the vertices of Möbius ladder M_n are numbered $\{v_1, ..., v_n\}$ counter clockwise.
In the next theorem, we prove that only three vertices suffice to resolve all the vertices of Möbius ladder M_n except when $n \equiv 2(mod\ 8)$. In that case, we prove that $3 \leq dim(M_n) \leq 4$. Note that the choice of an appropriate basis of vertices is the core of the problem.

Theorem 2. *Let M_n be the graph of Möbius ladder; then for every even positive integer $n \geq 8$, we have $dim(M_n) = 3$ when $n \not\equiv 2(mod\ 8)$ and $3 \leq dim(M_n) \leq 4$ otherwise.*

Proof. (a) Suppose that $n \not\equiv 2(mod\ 8)$. We will prove this case by double inequality. In this case we have the following subcases.
Case(i). When $n \equiv 0(mod\ 8)$
In this case, we can write $n = 8k, k \in \mathbf{Z}^+$. Let $W = \{v_1, v_2, v_{4k+1}\} \subset V(M_n)$, we show that W is a resolving set for M_n in this case. For this, we give the representation of any vertex of $V(M_n) \backslash W$ with respect to W.

$$r(v_{2i+1}|W) = \begin{cases} (2i, 2i-1, 2i+1), & 1 \leq i \leq k-1; \\ (2k, 2k-1, 2k), & i = k; \\ (4k-2i+1, 4k-2i+2, 4k-2i), & k+1 \leq i \leq 2k-1; \\ (2i-4k+1, 2i-4k, 2i-4k), & 2k+1 \leq i \leq 3k-1; \\ (2k, 2k, 2k), & i = 3k; \\ (8k-2i, 8k-2i+1, 8k-2i+1), & 3k+1 \leq i \leq 4k-1. \end{cases}$$

and

$$r(v_{2i}|W) = \begin{cases} (2i-1, 2i-2, 2i), & 2 \leq i \leq k; \\ (2k, 2k, 2k-1), & i = k+1; \\ (4k-2i+2, 4k-2i+3, 4k-2i+1), & k+2 \leq i \leq 2k; \\ (2i-4k, 2i-4k-1, 2i-4k-1), & 2k+1 \leq i \leq 3k; \\ (8k-2i+1, 8k-2i+2, 8k-2i+2), & 3k+1 \leq i \leq 4k. \end{cases}$$

We note that there are no two vertices having the same representations implying that $dim(M_n) \leq 3$.
On the other hand, we show that $dim(M_n) \geq 3$ by proving that there is no resolving set W such that $|W| = 2$. Suppose on contrary that $dim(M_n) = 2$, i.e., there exists a resolving set including exactly two vertices.
Without loss of generality, we can suppose that one resolving vertex is v_1. Suppose that the second resolving vertex is v_t $(2 \leq t \leq 4k+1)$. Then for $2 \leq t \leq 4k$, we have $r(v_n|\{v_1, v_t\}) = r(v_{4k+1}|\{v_1, v_t\}) = (1, t)$ and when $t = 4k+1$, $r(v_n|\{v_1, v_{4k+1}\}) = r(v_2|\{v_1, v_{4k+1}\}) = (1, 2)$, a contradiction.
We deduce that there is no resolving set with two vertices for $V(M_n)$, implying that $dim(M_n) = 3$ in this case.
Case(ii). When $n \equiv 4(mod\ 8)$
In this case, we can write $n = 8k+4, k \in \mathbf{Z}^+$. Let $W = \{v_1, v_2, v_{4k+3}\} \subset V(M_n)$, we show that W is a resolving set for M_n in this case. For this, we give the representation of any vertex of $V(M_n) \backslash W$ with respect to W.

$$r(v_{2i+1}|W) = \begin{cases} (2i, 2i-1, 2i+1), & 1 \leq i \leq k; \\ (2k+1, 2k+1, 2k), & i = k+1; \\ (4k-2i+3, 4k-2i+4, 4k-2i+2), & k+2 \leq i \leq 2k; \\ (2i-4k-1, 2i-4k-2, 2i-4k-2), & 2k+2 \leq i \leq 3k; \\ (8k-2i+4, 8k-2i+3, 8k-2i+3), & 3k+1 \leq i \leq 4k+1. \end{cases}$$

and

$$r(v_{2i}|W) = \begin{cases} (2i-1, 2i-2, 2i), & 2 \leq i \leq k; \\ (2k+1, 2k, 2k+1), & i = k+1; \\ (4k-2i+4, 4k-2i+5, 4k-2i+3), & k+2 \leq i \leq 2k+1; \\ (2i-4k-2, 2i-4k-3, 2i-4k-3), & 2k+2 \leq i \leq 3k+2; \\ (8k-2i+5, 8k-2i+6, 8k-2i+6), & 3k+3 \leq i \leq 4k+2. \end{cases}$$

We can see that there are no two vertices having the same representations implying that $dim(M_n) \leq 3$.
On the other hand, we show that $dim(M_n) \geq 3$ by proving that there is no resolving set W such that $|W| = 2$. Suppose on contrary that $dim(M_n) = 2$, i.e., there exists a resolving set including exactly two vertices.
Without loss of generality, we can suppose that one resolving vertex is v_1. Suppose that the other resolving vertex is v_t $(2 \leq t \leq 4k+3)$. Then for $2 \leq t \leq 4k+2$, we have $r(v_n|\{v_1, v_t\}) = r(v_{4k+3}|\{v_1, v_t\}) = (1, t)$ and when $t = 4k+1$, $r(v_n|\{v_1, v_{4k+3}\}) = r(v_2|\{v_1, v_{4k+3}\}) = (1, 2)$, a contradiction.
We deduce that there is no resolving set with two vertices for $V(M_n)$, implying that $dim(M_n) = 3$ in this case.
Case(iii). When $n \equiv 6(mod\ 8)$
In this case, we can write $n = 8k+6, k \in \mathbf{Z}^+$. For $n = 6$ we have $M_6 \cong K_{3,3}$, hence $dim(M_6) = 4$ because $dim(K_{n,n}) = 2n-2$. For every $n \geq 14$, let

$W = \{v_1, v_2, v_{4k+3}\} \subset V(M_n)$, we show that W is a resolving set for M_n in this case.. For this, we give the representation of any vertex of $V(M_n)\backslash W$ with respect to W.

$$r(v_{2i+1}|W) = \begin{cases} (2i, 2i-1, 2i+1), & 1 \leq i \leq k; \\ (2k+2, 2k+1, 2k+1), & i = k+1; \\ (4k-2i+4, 4k-2i+5, 4k-2i+3), & k+2 \leq i \leq 2k+1; \\ (2i-4k-2, 2i-4k-3, 2i-4k-3), & 2k+2 \leq i \leq 3k+2; \\ (8k-2i+6, 8k-2i+7, 8k-2i+7), & 3k+3 \leq i \leq 4k+2. \end{cases}$$

and

$$r(v_{2i}|W) = \begin{cases} (2i-1, 2i-2, 2i), & 2 \leq i \leq k+1; \\ (4k-2i+5, 4k-2i+6, 4k-2i+4), & k+2 \leq i \leq 2k+1; \\ (2i-4k-3, 2i-4k-4, 2i-4k-4), & 2k+3 \leq i \leq 3k+2; \\ (8k-2i+7, 8k-2i+8, 8k-2i+8), & 3k+3 \leq i \leq 4k+3. \end{cases}$$

Again we can note that there are no two vertices having the same representations implying that $dim(M_n) \leq 3$.
On the other hand, we show that $dim(M_n) \geq 3$ by proving that there is no resolving set W such that $|W| = 2$. Suppose on contrary that $dim(M_n) = 2$, i.e., there exists a resolving set including exactly two vertices.
Without loss of generality, we can suppose that one resolving vertex is v_1. Suppose that the other resolving vertex is v_t $(2 \leq t \leq 4k+4)$. Then for $2 \leq t \leq 4k+3$, we have $r(v_n|\{v_1, v_t\}) = r(v_{4k+4}|\{v_1, v_t\}) = (1, t)$ and when $t = 4k+1$, $r(v_n|\{v_1, v_{4k+4}\}) = r(v_2|\{v_1, v_{4k+4}\}) = (1, 2)$, a contradiction. We deduce that there is no resolving set with two vertices for $V(M_n)$, implying that $dim(M_n) = 3$ in this case.
(b) When $n \equiv 2 (mod\ 8)$
In this case, we can write $n = 8k+2, k \in \mathbf{Z}^+$. Let $W = \{v_1, v_2, v_{4k+2}, v_{6k+2}\} \subset V(M_n)$, we show that W is a resolving set for M_n in this case. For this, first we give the representation of any vertex of $V(M_n)\backslash U$ with respect to $U = \{v_1, v_2, v_{4k+2}\}$.

$$r(v_{2i+1}|U) = \begin{cases} (2i, 2i-1, 2i+1), & 1 \leq i \leq k; \\ (4k-2i-2, 4k-2i-3, 4k-2i-1), & k+2 \leq i \leq 2k; \\ (2i-4k+4, 2i-4k+3, 2i-4k+3), & 2k+1 \leq i \leq 3k; \\ (8k-2i-6, 8k-2i-5, 8k-2i-5), & 3k+1 \leq i \leq 4k. \end{cases}$$

and

$$r(v_{2i}|U) = \begin{cases} (2i-1, 2i-2, 2i), & 2 \leq i \leq k; \\ (2k+1, 2k, 2k), & i = k+1; \\ (4k-2i-1, 4k-2i, 4k-2i-2), & k+2 \leq i \leq 2k; \\ (2i-4k+3, 2i-4k+2, 2i-4k+2), & 2k+2 \leq i \leq 3k+1; \\ (8k-2i-5, 8k-2i-4, 8k-2i-4), & 3k+2 \leq i \leq 4k+1. \end{cases}$$

We note that the set U can distinguish all the vertices of M_n except the vertices v_{2k+2} and v_{2k+2}. As $r(v_{2k+2}|U) = r(v_{6k+2}|U) = (2k+1, 2k, 2k)$, it suggests that $W = U \cup \{v_{2k+2}\}$ is a resolving set for M_n in this case implying that $dim(M_n) \leq 4$.
On the other hand, we show that $dim(M_n) \geq 3$ by proving that there is no resolving set W such that $|W| = 2$. Suppose on contrary that $dim(M_n) = 2$, i.e., there exists a resolving set including exactly two vertices.
Without loss of generality, we can suppose that one resolving vertex is v_1. Suppose that the other resolving vertex is v_t $(2 \leq t \leq 4k+2)$. Then for $2 \leq t \leq 4k+1$, we have $r(v_n|\{v_1, v_t\}) = r(v_{4k+1}|\{v_1, v_t\}) = (1, t)$ and when $t = 4k+2$, $r(v_n|\{v_1, v_{4k+3}\}) = r(v_2|\{v_1, v_{4k+3}\}) = (1, 2)$, a contradiction. We deduce that there is no resolving set with two vertices for $V(M_n)$, implying that $dim(M_n) \geq 3$ in this case, which completes the proof.

□

3 Concluding remarks

In this paper, we have studied the metric dimension of Möbius ladder M_n which is a cubic circulant graph. We proved that only three vertices suffice to resolve all the vertices of Möbius ladder M_n except when $n \equiv 2(mod\ 8)$. In that case, we proved that $3 \leq dim(M_n) \leq 4$. It is natural to ask for the characterization of regular graphs with constant metric dimension.
Note that in [19] Melter and Tomescu gave an example of infinite regular graphs (namely the digital plane endowed with city-block and chessboard distances, respectively) having no finite metric basis. We close this section by by raising a question that naturally arises from the text.
Open Problem: *Find the exact value of metric dimension of Möbius ladder M_n when $n \equiv 2(mod\ 8)$.*

Acknowledgement

The authors are very grateful to the referee for his/her careful reading with corrections and useful comments.

References

1. P. S. Buczkowski, G. Chartrand, C. Poisson , P. Zhang, On k-dimensional graphs and their bases, *Periodica Math. Hung.*, 46(1)(2003), 9 – 15.
2. J. Caceres, C. Hernando, M. Mora, I. M. Pelayo, M. L. Puertas, C. Seara, D. R. Wood, On the metric dimension of cartesian product of graphs, *SIAM J. Disc. Math.*, 2(21), (2007), 423 – 441.

3. J. Caceres, C. Hernando, M. Mora, I. M. Pelayo, M. L. Puertas, C. Seara, D. R. Wood, On the metric dimension of some families of graphs. *Electronic Notes in Disc. Math.*, 22(2005), 129 – 133.
4. G. Chartrand, L. Eroh, M. A. Johnson, O. R. Oellermann, Resolvability in graphs and metric dimension of a graph, *Disc. Appl. Math.*, 105(2000), 99 – 113.
5. M. R. Garey, D. S. Johnson, *Computers and Intractability: A Guide to the Theory of NP- Completeness*, Freeman, New York, 1979.
6. F. Harary, R. A. Melter, On the metric dimension of a graph, *Ars Combin.*, 2(1976), 191 – 195.
7. M. Imran, A. Q. Baig, A. Ahmad, Families of plane graphs with constant metric dimension, *Utilitas Math.*, in press.
8. M. Imran, A. Q. Baig, M. K. Shafiq, A. Semaničová-Feňovčíková, Classes of convex polytopes with constant metric dimension, *Utilitas Math.*, in press.
9. M. Imran, S. A. Bokhary, A. Q. Baig, On families of convex polytopes with constant metric dimension, *Comput. Math. Appl.*, 60(9)(2010), 2629 – 2638.
10. M. Imran, F. Bashir, A. Q. Baig, S. A. Bokhary, A. Riyasat, I. Tomescu, On metric dimension of flower graphs $f_{n\times m}$ and convex polytopes, *Utilitas Math.*, in press.
11. M. Imran, S. A. Bokhary, Ali Ahmad, On the metric dimension of two classes of convex polytopes, *JCMCC*, 77(2011), 51 – 63.
12. M. Imran, A. Q. Baig, A special class of convex polytopes with constant metric dimension, *JCMCC*, 77(2011), 197 – 205.
13. M. Imran, A. Q. Baig, An infinite class of convex polytopes with constant metric dimension, *JCMCC*, in press.
14. M. Imran, A. Q. Baig, S. A. Bokhary, E. T. Baskoro, New classes of convex polytopes with constant metric dimension, *Utilitas Math.*, in press.
15. M. Imran, A. Q. Baig, M. K. Shafiq, I. Tomescu, On metric dimension of generalized Petersen graphs $P(n,3)$, *Ars Combin.*, in press.
16. M. Imran, A. Q. Baig, S. A. Bokhary, On the metric dimension of rotationally-symmetric graphs, *Ars Combin.*, in press.
17. I. Javaid, M. T. Rahim, K. Ali, Families of regular graphs with constant metric dimension, *Utilitas Math.*, 75(2008), 21 – 33.
18. S. Khuller, B. Raghavachari, A. Rosenfeld, Localization in graphs, Technical Report CS-TR-3326, University of Maryland at College Park, 1994.
19. R. A. Melter, I. Tomescu, Metric bases in digital geometry, *Computer Vision, Graphics, and Image Processing,* 25(1984), 113 – 121.
20. P. J. Slater, Leaves of trees, *Congress. Numer.*, 14(1975), 549 – 559.
21. P. J. Slater, Dominating and reference sets in graphs, *J. Math. Phys. Sci.*, 22(1998), 445 – 455.
22. I. Tomescu, M. Imran, On metric and partition dimensions of some infinite regular graphs, *Bull. Math. Soc. Sci. Math. Roumanie,* 52(100), 4(2009), 461 – 472.
23. I. Tomescu, I. Javaid, On the metric dimension of the Jahangir graph, *Bull. Math. Soc. Sci. Math. Roumanie,* 50(98), 4(2007), 371 – 376.