Some Extremal Problems for Edge-Regular Graphs

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Abstract

We consider the class $\mathrm{ER}(n, d, \lambda)$ of edge-regular graphs for some $n > d > \lambda$, i.e., graphs regular of degree d on n vertices, with each pair of adjacent vertices having λ common neighbors. It has previously been shown that for such graphs with $\lambda > 0$ we have $n \geq 3(d - \lambda)$ and much has been done to characterize such graphs when equality holds.

Here we show that $n \geq 3(d-\lambda)+1$ if $\lambda > 0$ and d is odd and contribute to the characterization of the graphs in $\mathrm{ER}(n,d,\lambda)$, $\lambda > 0$, $n = 3(d-\lambda)+1$ by proving some lemmas about the structure of such graphs, and by classifying such graphs that satisfy a strong additional requirement, that the number $t = t(u,v)$ of edges in the subgraph induced by the λ common neighbors of any two adjacent vertices u and v is positive, and independent of u and v. The result is that there are exactly 4 such graphs: K_4 and 3 strongly regular graphs.

1 Introduction

For integers $n > d > \lambda \geq 0$, let $\mathrm{ER}(n,d,\lambda)$ denote the collection of (simple) graphs $G = (V,E) = (V(G),E(G))$ such that $|V| = n$, G is regular of degree d, and if $uv \in E$, then $|N(u,v)| = \lambda$, where $N(u,v)$ denotes the subset of V of vertices adjacent to both u and v. (Our notation is fairly standard; we follow [11].) The notation "ER" is chosen because in [1] such graphs are called "edge-regular." These graphs might also be called "nearly strongly regular." A strongly regular graph is a graph $G \in \mathrm{ER}(n,d,\lambda)$, for some n, d, and λ, neither complete nor empty, such that for some integer $\mu > 0$, any two distinct non-adjacent vertices in G have exactly μ common neighbors. Let the collection of such strongly regular graphs be denoted $\mathrm{SR}(n,d,\lambda,\mu)$.

If $G \in \mathrm{ER}(n,d,\lambda)$, then $p = n - 2d + \lambda$ is the number of mutual non-neighbors of any two adjacent vertices in G. It has proven unexpectedly effective to approach the analysis of the edge-regular graphs with the indices n, λ, and p in various categories. The first result of this approach, in [3], is that the edge-regular graphs with $p = 0$ are the regular Turán graphs. In [4] it is shown that the edge-regular graphs with $p = 1$ are strongly regular; they are, in fact, the complements of the famous and mysterious Moore graphs, which may constitute an infinite family, although only three are known to exist. In [5] the edge-regular graphs with $p = 2$ and $\lambda = 0$ are characterized; except for two small non-bipartite graphs, these turn out to be regular bipartite graphs of the appropriate degree, $n/2 - 1$. In [7] this result is generalized: If $\lambda = 0 < p$ and n is sufficiently large ($n > 5p$), then if p is odd, $\mathrm{ER}(n, \frac{n-p}{2}, 0)$ is empty, and if p is even, and $G \in \mathrm{ER}(n, \frac{n-p}{2}, 0)$, then G is bipartite.

In [5] it is noted that if $\lambda > 0$ and $p = 2$ then $n \leq 3\lambda+6$ if $\mathrm{ER}(n, \frac{n-2+\lambda}{2}, \lambda)$ is non-empty, and in [6] the graphs in $\mathrm{ER}(n, \frac{n-2+\lambda}{2}, \lambda)$ satisfying $n = 3\lambda+6$ are completely characterized: there is only one, obtained by removing the edges of a 2-factor consisting of K_3's from the complete tripartite graph $K_{\lambda+2,\lambda+2,\lambda+2}$. There is an intriguing formal connection of this result with the results mentioned above on the case $\lambda = 0$: every regular non-empty bipartite graph is obtainable by removing 1-factors from a complete bipartite graph.

In [8] it is shown that if $\lambda > 0$ and $\mathrm{ER}(n, \frac{n-p+\lambda}{2}, \lambda)$ is non-empty, then $n \leq 3\lambda + 3p$. (We shall reprove this result in passing, in the next section.) Further, a result analogous to the result above, when $p = 2$, is proven for $p > 0$, even: if λ is sufficiently large (depending on p), $n = 3p + 3\lambda$, and $\mathrm{ER}(n, \frac{n-p+\lambda}{2}, \lambda)$ is non-empty, then $p/2$ divides λ and there is only one (unlabelled) graph in $\mathrm{ER}(n, \frac{n-p+\lambda}{2}, \lambda)$, obtained by removing the edges of a p-factor consisting of $K_{p/2,p/2,p/2}$'s from $K_{\lambda+p,\lambda+p,\lambda+p}$.

In fact, in the result quoted above, it is shown that $\mathrm{ER}(n, \frac{n-p+\lambda}{2}, \lambda) = \mathrm{ER}(3\lambda + 3p, 2\lambda + p, \lambda)$ is non-empty only if p is even. Our work here departs from this fact, to be reproved in the next section: if $\lambda > 0$, p is odd, and $\mathrm{ER}(n, \frac{n-p+\lambda}{2}, \lambda)$ is non-empty, then $n \leq 3\lambda + 3p - 2$. Our main interest is in the case $n = 3\lambda + 3p - 2$, whether p is odd or even.

2 Results

A *clique* of G is a subset of vertices that are mutually adjacent. A clique of size 3 is a triangle, a clique of size 4 induces a subgraph isomorphic to the complete graph K_4 on 4 vertices and shall be simply called 'a K_4' in what follows.

Lemma 1 *Let $G \in \mathrm{ER}(n, d, \lambda)$ for some n, d, and λ such that $\lambda > 0$. Then*

1. $n \geq 3(d - \lambda)$.
2. $n = 3(d - \lambda)$ *if and only if every vertex of G is adjacent to either* 1 *or* 2 *vertices of every triangle in G.*
3. *If $n = 3(d - \lambda)$ then d must be even.*
4. *If $n = 3(d - \lambda) + 1$ then every triangle belongs to at most one K_4.*

Proof : As $\lambda > 0$ the graph contains at least one triangle. Consider a triangle T and denote by n_i the number of vertices of $V - T$ adjacent to exactly i vertices of T. Then

$$\begin{array}{rcrcrcrcl} n_0 & + & n_1 & + & n_2 & + & n_3 & = & n - 3, \\ & & n_1 & + & 2n_2 & + & 3n_3 & = & 3(d-2), \\ & & & & n_2 & + & 3n_3 & = & 3(\lambda - 1). \end{array} \qquad (1)$$

Indeed, the first equation simply counts the number of elements of $V - T$ in two ways, the second equation counts the number of adjacent pairs (a, b) with $a \in T$, $b \in V - T$, and the third equation counts the number of triangles $aa'b$ with $a, a' \in T$, $b \in V - T$.

Adding the first and last equation and subtracting the second, we obtain

$$n_0 + n_3 = n - 3(d - \lambda), \tag{2}$$

which yields statement 1 of this lemma, because $n_0, n_3 \geq 0$. Note that equality occurs if and only if $n_0 = n_3 = 0$, whence statement 2.

The number of edges of G is equal to $nd/2$ and hence nd must be even. Note that the subgraph induced on the neighbors of a vertex is a regular graph of order d and degree λ, hence also $d\lambda$ must be even. When $n = 3(d-\lambda)$ we therefore find that $nd = 3d^2 - 3d\lambda$ must be even, and hence d^2 must be even, yielding statement 3.

Finally, if $n = 3(d - \lambda) + 1$ then by (2) $n_0 + n_3 = 1$ and hence $n_3 \leq 1$, yielding statement 4. ■

Note that the inequality $n \geq 3(d - \lambda)$ can also be expressed in terms of n, p and λ, yielding $n \leq 3(\lambda + p)$. Similarly, if $n = 3(d - \lambda)$ then p and d have the same parity, also $n = 3(d-\lambda)+1$ is equivalent to $n = 3(\lambda+p)-2$. Lemma 1 therefore gives an alternative proof to some of the results of [8].

As explained in the introduction, a good start has been made towards describing the edge-regular graphs with $\lambda > 0$, $n = 3(d-\lambda)$. Our aim here is to begin work on the apparently more difficult problem of describing edge-regular graphs with $\lambda > 0$, $n = 3(d - \lambda) + 1$. By Lemma 1 these graphs are "extremal" for the case of odd degree. These graphs are also "extremal" in case p is odd, for if $n = 3(d - \lambda)$ then d is even, by Lemma 1, so n and λ have the same parity, and so $p = n - 2d + \lambda$ is even.

Lemma 2 *Let $G \in \mathrm{ER}(n, d, \lambda)$ for some n, d, and λ such that $\lambda > 0$. Assume G contains at least one K_4. Then*

1. $n \geq 4(d + 1) - 6\lambda$.
2. $n = 4(d+1) - 6\lambda$ *if and only if every vertex of G is adjacent to either 1 or 2 vertices of every K_4 of which it is not an element.*
3. *If $n = 4(d + 1) - 6\lambda$, then $\lambda \leq d/3 + 1$.*
4. $n = 4(d + 1) - 6\lambda$ *and $\lambda = d/3 + 1$ if and only if every vertex of G is adjacent to exactly 2 vertices of every K_4 of which it is not an element.*

Proof : Let K denote a K_4 in G. Denote by N_i the number of vertices of $V - K$ adjacent to exactly i vertices of K. Then, using similar counting arguments as in the proof of Lemma 1. we obtain

$$\begin{array}{rcrcrcrcrcl} N_0 & + & N_1 & + & N_2 & + & N_3 & + & N_4 & = & n - 4, \\ & & N_1 & + & 2N_2 & + & 3N_3 & + & 4N_4 & = & 4(d-3), \\ & & & & N_2 & + & 3N_3 & + & 6N_4 & = & 6(\lambda - 2). \end{array} \tag{3}$$

and then

$$N_0 + N_3 + 3N_4 = n - (4d + 4 - 6\lambda). \tag{4}$$

This proves the first two statements of the lemma. Note that $n = 4d+4-6\lambda$ implies $N_0 = N_3 = N_4 = 0$, allowing (3) to be solved for N_1 and N_2. We find $N_2 = 6(\lambda-2)$ and $N_1 = n+8-6\lambda = 4(d+3-3\lambda)$. The latter equation proves the last two statements of the lemma. ∎

Lemma 3 *Let $G \in ER(n,d,\lambda)$ for some n, d, and λ such that $\lambda > 0$ and $n = 3(d-\lambda)+1$. Suppose that G contains at least one K_4. Then $d = 3(\lambda-1)$, $n = 6\lambda - 8$, and the hypothesis of statement 4 of Lemma 2 is satisfied.*

Proof : Let K denote a K_4 in G, and let $N_0, \ldots, N_4$ be as in the proof of Lemma 2. As in that proof, we have $N_0 + N_3 + 3N_4 = n - (4d + 4 - 6\lambda)$. By the last statement in Lemma 1 we conclude that $N_3 = N_4 = 0$, so $N_0 = n - (4d + 4 - 6\lambda)$.

Let T be any one of the 4 triangles in K and let n_0, n_1, n_2, and n_3 be as in the proof of Lemma 1. As in that proof we have that $n_0 + n_3 = n - 3(d-\lambda) = 1$. But n_3 is at least 1 because T is in K. Therefore $0 = n_0 \geq N_0 \geq 0$, so $N_0 = n - (4d + 4 - 6\lambda) = 0$. Putting this together with $n = 3(d-\lambda)+1$ and solving for n and d in terms of λ, the conclusions of this lemma are easily obtained. ∎

Let $K(m:2)$ denote the Kneser graph whose vertices are the 2-subsets of an m-set, with two vertices adjacent if and only if they are disjoint. The complement of $K(m:2)$ is called the triangular graph on m points, and will be denoted $T(m)$.

Lemma 4 *Suppose that G satisfies the hypothesis of Lemma 3 and is strongly regular. Then $\lambda \in \{3,4,6\}$ and G is one of*

1. *$T(5)$, which is also known as the complement of the Petersen graph, and is the only graph in $SR(10,6,3,4)$;*
2. *the two graphs in $SR(16,9,4,6)$, which are the complements of the 4×4 grid and of the Shrikhande graph [9];*
3. *the four graphs in $SR(28,15,6,10)$, one of which is $K(8:2)$; the other 3 are the complements of the Chang graphs.*

Proof : By Lemma 3, $d = 3(\lambda-1)$ and $n = 6\lambda - 8$. From this and well known relations among n, d, λ, and μ when $SR(n,d,\lambda,\mu)$ is non-empty (see, for instance, [2]), it is easy to see that λ divides 12 and $\mu = 2(\lambda-1)$. In case $\lambda = 2$ we have $d = 3$ and $n = 4$, so G would have to be K_4, which is not strongly regular. When $\lambda = 3$, 4, 6, we get the graphs mentioned in 1, 2, and 3; see [2], Section VI.5.2. In the case $\lambda = 12$, $(n, d, \lambda, \mu) = (64, 33, 12, 22)$ do not satisfy the absolute bound (see [2] again), so $SR(64, 33, 12, 22)$ is empty. ∎

Theorem 1 *Let $G \in \mathrm{ER}(6\lambda-8, 3\lambda-3, \lambda)$ with $\lambda > 0$, and suppose that for some $t > 0$ every edge belongs to exactly t cliques of size 4. Then $\lambda = 2, 3, 4$ or 6 and G is isomorphic to one of the following 4 graphs:*

1. *The complete graph K_4 on 4 vertices. ($n = 4$, $d = 3$, $\lambda = 2$, $t = 1$.)*
2. *The graph $T(5)$, i.e., the complement of the Petersen graph. ($n = 10$, $d = 6$, $\lambda = 3$, $t = 1$.)*
3. *The complement of the 4×4 grid. ($n = 16$, $d = 9$, $\lambda = 4$, $t = 2$.)*
4. *$K(8:2)$. ($n = 28$, $d = 15$, $\lambda = 6$, $t = 3$.)*

In all cases but the first G is strongly regular.

Proof : Let u be a vertex of G and consider the "local geometry" $G(u)$ with points the neighbors of u and lines the triangles in $N(u)$ (which when joined to u are precisely all K_4's through u). By statement 4 of Lemma 1, any two points of $G(u)$ lie on at most one line (and hence $G(u)$ is a partial linear space). Moreover, the conditions of this theorem imply that every point of $G(u)$ lies on exactly t lines. The number of points of $G(u)$ is $d = 3(\lambda - 1)$ and the number of lines is $dt/3 = (\lambda - 1)t$.

Consider a line of $G(u)$ and a vertex v of G, $v \neq u$. By statement 4 of Lemma 2, we know that either v belongs to the given line, is adjacent to exactly 1 point of that line (when u and v are adjacent) or to exactly 2 points of that line (when u and v are not adjacent).

This property allows us to count the number μ of vertices x adjacent to both u and v, which we will do for the case that u and v are not adjacent. We count in two ways the number N of pairs (x, L) such that x is a point of $G(u)$ adjacent to v and L is a line of $G(u)$ containing x. For each of the μ points adjacent to both u and v there are exactly t lines of this type. Hence $N = \mu t$. On the other hand, for each of the $(\lambda - 1)t$ lines of $G(u)$ there are exactly two such points. Hence $N = 2(\lambda - 1)t$.

This yields $\mu = 2(\lambda - 1)$, independent of the choice of u and v, proving that G is strongly regular (unless G is a complete graph, in which case G must be isomorphic to K_4).

The conclusion of the theorem now follows from Lemma 4. (The 4 graphs mentioned in Lemma 4 other than $T(5)$, the complement of the 4×4 grid, and $K(8:2)$ do not satisfy the requirement that every edge is in exactly t K_4's, for some t.) ∎

It is surely worthy of note that of the 8 graphs mentioned in Lemma 4 and the Theorem, only for $T(5)$, the complement of the Petersen graph, is $p = n - 2d + \lambda$ odd. To sum up, Lemma 4 and the Theorem settle the question of which graphs are edge-regular with parameters n, d, and $\lambda > 0$ satisfying $n = 3(d - \lambda) + 1$ (an extremal condition when d or p is odd) under the strong additional requirements that either

1. the graph contains a K_4 and is strongly regular, or
2. there is a number $t > 0$ such that every edge of the graph is in exactly t K_4's.

There turn out to be 7 graphs in the first group and 4 in the second, with an overlap of 3, for a total of 8. Whether there are infinitely many, or any, such graphs that do not satisfy one or the other of these special requirements is an open question.

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