On Self-Clique Graphs all of whose Cliques have Equal Size

G.L. Chia[a] and Poh-Hwa Ong[b]
[a]Institute of Mathematical Sciences, University Malaya, 50603 Kuala Lumpur, Malaysia
[b]Department of Mathematical and Actuarial Sciences, Universiti Tunku Abdul Rahman, 46200 Petaling Jaya, Selangor, Malaysia

Abstract

The *clique graph* of a graph G is the graph whose vertex set is the set of cliques of G and two vertices are adjacent if and only if the corresponding cliques have non-empty intersection. A graph is *self-clique* if it is isomorphic to its clique graph. In this paper, we present several results on connected self-clique graphs in which each clique has the same size k for $k = 2$ and $k = 3$.

1 Introduction

Let G be a graph. By a *clique* in G, we mean a maximal complete subgraph of G. Let $\mathcal{K}(G)$ denote the set of all cliques in G. The *clique graph* of G, denoted $K(G)$, is the graph whose vertex set is $\mathcal{K}(G)$ and two vertices are adjacent if and only if the corresponding cliques have non-empty intersection. A graph is *self-clique* if it is isomorphic to its clique graph. Self-clique graphs have been the subject of much discussion lately (see [2], [3], [4], [5], [9] and [10] for instance). This paper follows in the similar vein of thought by confining the attention on those self-clique graphs whose clique sizes are uniform.

Let $\mathcal{G}(k)$ denote the set of all connected self-clique graphs where each clique is of size k. In the present section, we record some known results concerning $\mathcal{G}(2)$ (Theorem 1). In the next section, while unable to determine all graphs in $\mathcal{G}(3)$, we turn to determine all those in $\mathcal{G}(3)$ which are 4-regular (Corollary 3) and all those in which the degree of any vertex is at most 4 (Theorem 3). In the subsequent sections, we show the existence of 5-regular graphs and 6-regular graphs in $\mathcal{G}(3)$ by constructions (Propositions 2 and 3). In the final section, we examine the existence of a graph in $\mathcal{G}(3)$ whose set of vertices admits two degrees r and s where $2 \leq r < s \leq 6$. It is shown that, with the exceptions of $s = 6$ and $r \in \{2, 5\}$, such graphs do not exist in $\mathcal{G}(3)$ unless $r = 4$ and $s = 5$ (Propositions 4 to 7).

Let K_n, C_n and P_n denote a complete graph, a cycle and a path on n vertices respectively. If G is a graph and x is a vertex in G, let $d_G(x)$, or just $d(x)$ denote the degree of x in G.

Suppose $G \in \mathcal{G}(2)$. In [6], Escalante showed that, if G is finite, then G is the cycle C_n for some $n \geq 4$. It is easy to see that if the finiteness condition is dropped, then the two graphs of Figure 1 are the only other members of $\mathcal{G}(2)$. We omit the proof. However for completeness, we record this fact here.

Theorem 1 *A graph is in $\mathcal{G}(2)$ if and only if it is either the cycle C_n with $n \geq 4$ or else one of the infinite paths of Figure 1.*

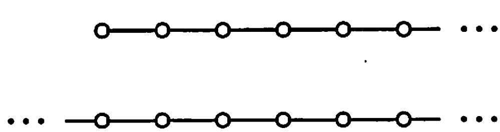

Figure 1: Two infinite self-clique graphs.

2 4-regular graphs in $\mathcal{G}(3)$ and beyond

Let G be a graph and let $x \in V(G)$. Let $N(x)$ denote the neighborhood of x. Further, if A is a subset of $V(G)$, let $G[A]$ denote the subgraph of G induced by the vertices in A.

Lemma 1 *Let $G \in \mathcal{G}(3)$. Then any edge in G is contained in at most two cliques of G.*

Proof: If there is an edge uv of G which is contained in s cliques of G, then these s cliques will give rise to a K_s in $K(G)$. Since $G \in \mathcal{G}(3)$, we must have $s \leq 3$.

Suppose $s = 3$. Then these three cliques give rise to an induced subgraph H in G consisting of three triangles overlapping at the common edge uv. Since G is self-clique, $K(G)$ must also contain an induced subgraph isomorphic to H.

Let U, V and Q_i, $i = 1, 2, 3$ be some cliques of G which form an induced subgraph H^* of $K(G)$ isomorphic to H. Assume further that $\{U, V, Q_i\}$, $i = 1, 2, 3$ are three cliques in $K(G)$ that have UV as the common edge in the subgraph H^*.

Since $U \cap Q_i \neq \emptyset$, and the Q_i's are pairwise disjoint in G, we may assume that $U = \{x_1, x_2, x_3\}$ and that $Q_i = \{x_i, w_i, z_i\}$, $i = 1, 2, 3$. Now assume that $V = \{y_1, y_2, y_3\}$. There are two cases to consider.

Case (i): $|U \cap V| = 2$

In this case, assume that $x_1 = y_1$ and $x_2 = y_2$. Since $Q_3 \cap V \neq \emptyset$, we see that either $y_3 = w_3$ or $y_3 = z_3$ (because $x_3 \notin V$ and $x_1, x_2 \notin Q_3$). In either case, we have y_3 adjacent to x_3 which means that $\{x_1, x_2, x_3, y_3\}$ is a K_4 in G, a contradiction.

Case (ii): $|U \cap V| = 1$

In this case, assume that $x_1 = y_1$. Since $Q_i \cap V \neq \emptyset$, for $i = 2, 3$, we may assume that $y_i = w_i$. But this means that w_2 and w_3 are both adjacent to x_1 in G so that $R_i = \{x_1, x_i, w_i\}$, $i = 2, 3$, U and V are four cliques in G all with the common vertex x_1. This yields a K_4 in $K(G)$, a contradiction because $K(G) \cong G$.

This completes the proof. □

Proposition 1 *Suppose $G \in \mathcal{G}(k)$ where $k \geq 2$. Then for any vertex $x \in V(G)$, we have $k - 1 \leq d(x) \leq k(k-1)$.*

Proof: It is clear that $d(x) \geq k - 1$ since each clique in G is of size k.

Since G is self-clique, at each vertex x, there are at most k cliques containing x. Hence the degree of x is at most $k(k-1)$. □

Theorem 2 *Suppose $G \in \mathcal{G}(3)$ and let x be a vertex of degree r in G. Then $G[N(x)]$ is*

(i) P_r if $2 \leq r \leq 4$,
(ii) $P_2 \cup P_3$ if $r = 5$ and
(iii) $3P_2$ if $r = 6$.

Proof: Let $Q_1, \ldots, Q_t$ denote the set of cliques in G containing the vertex x. Then clearly, $1 \leq t \leq 3$ because these t cliques form a complete subgraph K_t in $K(G)$. Consequently, we have

(O1) $G[N(x)]$ contains at most 3 edges because each edge in $G[N(x)]$, together with the vertex x, induce a clique of size 3 in G.

Also, since G contains neither cliques of size 2 nor cliques of size 4, we have

(O2) $G[N(x)]$ contains neither isolated vertices nor triangles.

These two observations immediately imply that $G[N(x)]$ is P_r if $2 \leq r \leq 3$.

Suppose $r = 4$. If $G[N(x)]$ is disconnected, then $G[N(x)] \cong 2P_2$ by (O2). In this case, $t = 2$ and Q_1 and Q_2 are such that Q_1Q_2 forms a clique of size 2 in $K(G)$ which is impossible because $K(G) \cong G$. Hence $G[N(x)]$ is connected.

By (O1) and (O2), $G[N(x)]$ is a tree on 4 vertices. If $G[N(x)]$ contains a vertex v of degree 3, then the edge xv is contained in the three cliques Q_1, Q_2 and Q_3, a contradiction to Lemma 1. Hence $G[N(x)] \cong P_4$. This proves (i).

Applying observations (O1) and (O2) to the cases $r = 5$ and $r = 6$ lead to the conclusions (ii) and (iii). □

A consequence to the above theorem is the following.

Corollary 1 *If there is an r-regular graph in $\mathcal{G}(3)$, then $r \geq 4$.*

Proof: Let G be an r-regular graph in $\mathcal{G}(3)$. Clearly, $r \geq 3$.

Suppose $r = 3$. Let x be a vertex of degree 3 in G. By Theorem 2(i), we may assume that $x_1x_2x_3$ is the path on 3 vertices in $G[N(x)]$. By Theorem 2(i), we may assume that $G[N(x_1)]$ is the path xx_2y for some vertex $y \in V(G)$ where $y \neq \{x, x_1, x_2, x_3\}$. But then this means that $d(x_2) \geq 4$, a contradiction. Hence $r \geq 4$. □

Let G be a graph. The k-th *power* of G, denoted G^k, is the graph having the same vertex set as G and two vertices u and v are adjacent in G^k if and only if the distance from u to v is no more than k. Let $\mathbb{Z}_n = \{0, 1, \ldots, n-1\}$ and let $\mathbb{Z}$ denote the set of all integers.

We shall invoke the following result of Hall ([7], 4.9). Note that, under the notation adopted in [7] (page 421), the infinite graph C^2_∞ is a special case of C^2_n.

Theorem 3 *([7]) Let G be a connected graph. Then $G[N(x)] \cong P_4$ for any vertex x in G if and only if either $G \cong C^2_n$ for some $n \geq 7$ or else $G \cong C^2_\infty$.*

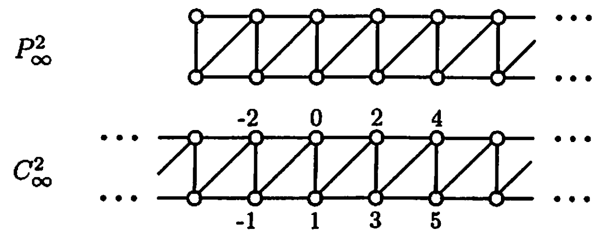

Figure 2: Two more infinite self-clique graphs.

Theorem 4 *Let G be a graph with no vertices of degree 5 or 6. Then $G \in \mathcal{G}(3)$ if and only if G is either the graph C_n^2 for some $n \geq 7$ or else one of the infinite graphs C^2_∞ or P^2_∞ of Figure 2.*

Proof: The sufficiency is by direct verification. We now prove the necessity part.

First, we consider the case where G is 4-regular. By Theorem 2(i), $G[N(x)]$ is a path on 4 vertices for any vertex x in G. By Theorem 3, either $G \cong C_n^2$ for some $n \geq 7$ or else $G \cong C^2_\infty$.

Now, consider the case where G is not 4-regular. In this case, G contains vertices of degree $2, 3$ or 4.

Suppose G contains a vertex x_1 of degree 2 and let $N(x_1) = \{x_2, x_3\}$. By Theorem 2(i), $Q_1 = \{x_1, x_2, x_3\}$ is a clique of G, and further, $d(x_2) \geq 3$ and $d(x_3) \geq 3$.

If $d(x_2) = 3 = d(x_3)$, then, by Theorem 2(i) again, G is isomorphic to the graph obtained from K_4 by deleting an edge. But this means that $K(G) \not\cong G$, a contradiction.

Hence assume that $d(x_3) = 4$. Further, let $N(x_3) = \{x_1, x_2, x_4, x_5\}$ so that $x_1x_2x_4x_5$ is a path on 4 vertices and that $Q_i = \{x_i, x_{i+1}, x_{i+2}\}$ is a clique of G for each $i = 2, 3$, by Theorem 2(i).

If $d(x_2) = 4$, then by Theorem 2(i), $G[N(x_2)]$ is the path $x_1x_3x_4z$, for some vertex $z \in V(G)$, $z \neq x_i$, $i = 1, 2, \ldots, 5$ and $\{x_2, x_4, z\}$ is a clique of G. But, on taking the clique graph of G, we see that $K(G)$ contains a K_4 which is absurd since $K(G) \cong G$. Hence $d(x_2) = 3$.

Now, if $d(x_4) = 3$, then $d(x_5) = 2$ and we have a contradiction because $K(G) \cong K_3 \not\cong G$. Hence $d(x_4) = 4$.

By Theorem 2(i), $G[N(x_4)]$ is the path $x_2x_3x_5x_6$ for some vertex $x_6 \in V(G)$ and $x_6 \neq x_i$, $i = 1, 2, \ldots, 5$ so that $Q_4 = \{x_4, x_5, x_6\}$ is a clique of G.

Now, if $d(x_5) = 3$, then $d(x_6) = 2$ and we have a contradiction because $K(G) \not\cong G$. Hence $d(x_5) = 4$.

Repeat the similar argument as before to the vertex x_k successively, for each $k \geq 5$ where x_k is adjacent to x_{k-1} and x_{k+1} (and by noting that $G[N(x_k)]$ is a path on 4 vertices in G), we see that G is an infinite graph isomorphic to the graph P^2_∞ (shown in Fig. 2).

Hence we may assume that G contains no vertices of degree 2. In this case, G contains only vertices of degree 3 or 4. By Proposition 6, G is not in $\mathcal{G}(3)$, a contradiction.

This completes the proof. □

Corollary 2 *Suppose G is a 4-regular graph. Then $G \in \mathcal{G}(3)$ if and only if G is either the graph C^2_n for some $n \geq 7$ or else the infinite graph C^2_∞ of Figure 2.*

3 5-regular graphs in $\mathcal{G}(3)$

Despite that 4-regular graphs in $G \in \mathcal{G}(3)$ have been completely characterized, it seems to be the case that 5-regular graphs in $G \in \mathcal{G}(3)$ are much more difficult to characterize unless further restriction is imposed on them. In this section, we shall only show the existence of 5-regular graphs in $\mathcal{G}(3)$ by construction.

Definition 1 *Let m, $n \geq 2$ be two integers. Let $L(m,n)$ denote the graph whose vertex set is the set of ordered pairs (i,j) where $i \equiv j \pmod 2$, $i \in \mathbb{Z}_{4m}$ and $j \in \mathbb{Z}_{4n}$ and whose edge set is $E_1 \cup E_2$ where $E_1 = \{(i,j)(k,l) : i \in \mathbb{Z}_{4m},\ j \in \mathbb{Z}_{4n},\ |i-k| = 1 = |j-l|\}$ and $E_2 = \{(2i,2j)(2i+2,2j),\ (2i+1,2j+1)(2i+1,2j+3) : i \in \mathbb{Z}_{2m},\ j \in \mathbb{Z}_{2n},\ i+j \equiv 1 \pmod 2\}$. Here, the operations on the first (respectively second) index are reduced modulo $4m$ (respectively modulo $4n$).*

The above definition gives a graph whose set of vertices is finite. We may allow the second index to be any integer and obtain an infinite graph $L(m)$.

It might appear that these definitions look unnatural, but the general drawing of $L(m,n)$ on the torus can easily be extended in a natural way from the smallest such graph $L(2,2)$ which is depicted in Figure 3. Notice that those edges that are 'horizontal' or 'vertical' are in E_2 whereas those that are 'diagonal' are in E_1. It is routine to verify that $L(2,2)$ is a 5-regular self-clique graph all of whose cliques have size equal to 3. More generally, we have the following result.

Proposition 2 *For each $m, n \geq 2$, the graphs $L(m,n)$ and $L(m)$ are 5-regular and are both in $\mathcal{G}(3)$.*

Proof: Let G be the graph $L(m,n)$ or $L(m)$.

Let Q be a clique in G. Then Q is one of the following four types.

(i) $(a+1,b-1)(a,b)(a+1,b+1)$, a even,
(ii) $(a-1,b-1)(a,b)(a-1,b+1)$, a even,
(iii) $(a-1,b+1)(a,b)(a+1,b+1)$, a odd, and
(iv) $(a-1,b-1)(a,b)(a+1,b-1)$, a odd.

Let φ be a mapping from $V(K(G))$ to $V(G)$ defined by

$$\varphi(Q) = (a+2, b).$$

Then it is readily checked that φ is an isomorphism from $K(G)$ onto G. □

By allowing the first index of the vertex set of $L(m)$ to include any integers, we obtain another infinite 5-regular graph which is in $\mathcal{G}(3)$.

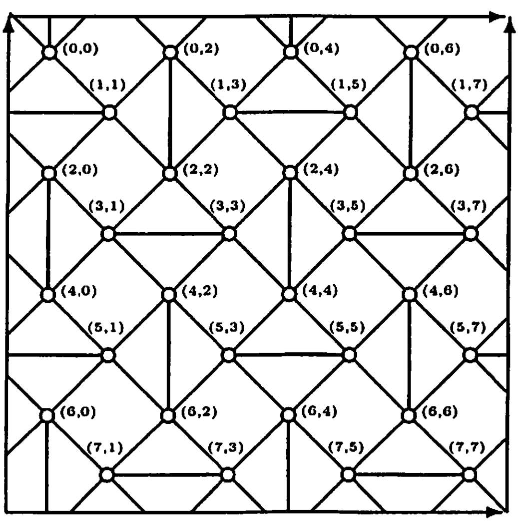

Figure 3: The graph $L(2,2)$ drawn on the torus.

4 6-regular graphs in $\mathcal{G}(3)$

Likewise, in this section, we shall only show the existence of 6-regular graphs in $\mathcal{G}(3)$ by construction. Let $A_n = \{i \in \mathbb{Z}_n : i \not\equiv 0 \pmod 4\}$.

Definition 2 *Let $n \geq 4$ be an integer. Let $M(n)$ denote the graph whose vertex set is $V_1 \cup V_2$ and whose edge set is $E_1 \cup E_2 \cup E_3 \cup E_4$ where*

(i) $V_1 = \{(i,j) : \ i \in A_{12}, \ j \in \mathbb{Z}_{2n}, \ i \equiv j \pmod 2\}$
$V_2 = \{x_{2i+1} : \ i \in \mathbb{Z}_n\}$.

(ii) $E_1 = \{(i,j)(k,l) : \ i,k \in A_{12}, \ j,l \in \mathbb{Z}_{2n}, \ |i-k| = 1 = |j-l|\}$
$E_2 = \{(i,j)(i,j+2), (i,j)x_j : \ i \in A_{12}, \ j \in \mathbb{Z}_{2n}, \ i,j \equiv 1 \pmod 2\}$
$E_3 = \{(i,j)(i+2,j) : \ i \in A_{12}, \ j \in \mathbb{Z}_{2n}, \ i \equiv 3 \pmod 4), \ j \equiv 1 \pmod 2\}$
$E_4 = \{(i,j)(i+4,j) : \ i \in A_{12}, \ j \in \mathbb{Z}_{2n}, \ i,j \equiv 0 \pmod 2\}$

Here, the operations on the first (respectively second) index are reduced modulo 12 (respectively modulo $2n$).

The above definition gives a graph whose set of vertices is finite. We may allow the second index to be any integer and obtain an infinite graph M.

Figure 4 shows a toroidal drawing of the graph $M(4)$. It is routine to verify that $M(4)$ is a 6-regular self-clique graph all of whose cliques have size equal to 3. More generally, we have the following result.

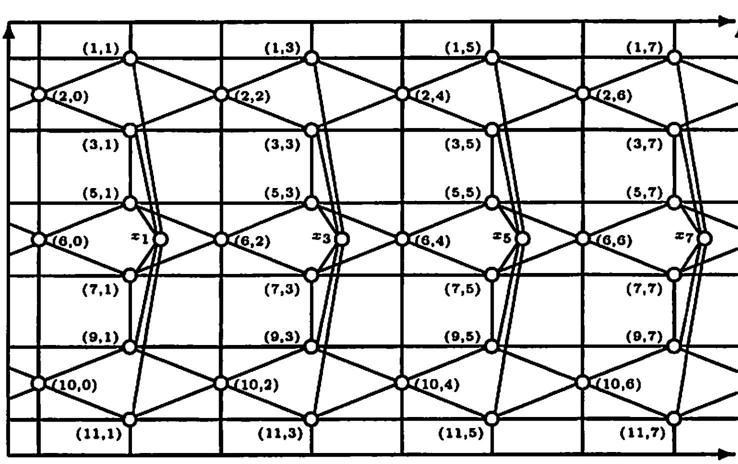

Figure 4: The graph $M(4)$ drawn on the torus.

Proposition 3 *For each $n \geq 4$, the graphs $M(n)$ and M are 6-regular and are both in $\mathcal{G}(3)$.*

Proof: Let G be the graph $M(n)$ or M.

Let Q be a clique in G. Then Q is one of the following four types.

(i) $(a-1,b-1)(a,b)(a-1,b+1)$,
(ii) $(a+1,b-1)(a,b)(a+1,b+1)$,
(iii) $(a-4,b)(a,b)(a+4,b)$,
(iv) $(a,b)(a+2,b)x_b$,

where $a \equiv 2 \pmod 4$, $b \equiv 0 \pmod 2$ for (i), (ii), (iii) and $a \equiv 3 \pmod 4$, $b \equiv 1 \pmod 2$ for (iv).

Let φ be a mapping from $V(K(G))$ to $V(G)$ defined by

$$\varphi(Q) = \begin{cases} (a+1,b+1) & \text{if } Q \text{ is type (i)} \\ (a+3,b+1) & \text{if } Q \text{ is type (ii)} \\ x_{b+1} & \text{if } Q \text{ is type (iii)} \\ (a+3,b+1) & \text{if } Q \text{ is type (iv)} \end{cases}$$

Then it is readily checked that φ is an isomorphism from $K(G)$ onto G. □

5 S-graphs in $\mathcal{G}(3)$

In this section, we shall investigate the existence of graphs in $\mathcal{G}(3)$ whose sets of vertices are of mixed degrees. By Proposition 1, if $G \in \mathcal{G}(3)$, then $2 \leq d(x) \leq 6$ for any vertex x in G. The more interesting case seems to be those graphs in $\mathcal{G}(3)$ which are almost regular in the sense that, for every vertex x in G, $d(x) = r$ or $d(x) = s$ for some $2 \leq r < s \leq 6$.

Definition 3 *If S is the set of degrees of G, then G is called an S-graph.*

We shall now confine our attention to S-graphs in $\mathcal{G}(3)$ where $|S| = 2$. Of course, one could also investigate S-graphs in $\mathcal{G}(3)$ where $3 \leq |S| \leq 6$ but we feel that this could be done elsewhere.

Lemma 2 *Let x be a vertex of degree 2 in a $\{2,s\}$-graph $G \in \mathcal{G}(3)$ where $3 \leq s \leq 6$. Then x is adjacent to another vertex of degree 2 in G.*

Proof: Let $N(x) = \{x_1, x_2\}$. By Theorem 2(i), $Q = \{x, x_1, x_2\}$ is a clique of G.

Suppose $d(x_1) = s = d(x_2)$ and let $N(x_1) = \{x, x_2, y_1, \ldots, y_{s-2}\}$ and let $N(x_2) = \{x, x_1, z_1, \ldots, z_{s-2}\}$.

Suppose $3 \leq s \leq 4$. Since $G[N(x_1)]$ and $G[N(x_2)]$ are both paths on s vertices, by Theorem 2(i), we may assume that $y_1 = z_1$ so that $\{x_1, x_2, y_1\}$ is a clique of G, and in addition, if $s = 4$, then so are $\{x_1, y_1, y_{s-2}\}$ and $\{x_2, y_1, z_{s-2}\}$. Now, by taking the clique graph of G, we see that, if $s = 3$, then Q is a vertex of degree 1 in $K(G)$, while if $s = 4$, then $K(G)$ contains a K_4. Either case is a contradiction since $K(G) \cong G$.

Suppose $5 \leq s \leq 6$. By Theorem 2, (ii) and (iii), each of $G[N(x_1)]$ and $G[N(x_2)]$ is a union of paths. Moreover, if $t = |\{y_1, \ldots, y_{s-2}\} \cap \{z_1, \ldots, z_{s-2}\}|$, then $t \leq 1$ by Lemma 1.

Note that, if $s = 6$, then $t = 0$ and that if $s = 5$, then either $t = 0$ or $t = 1$.

If $t = 0$, then $\{x_1, y_1, y_2\}$, $\{x_1, y_{s-3}, y_{s-2}\}$, $\{x_2, z_1, z_2\}$ and $\{x_2, z_{s-3}, z_{s-2}\}$ are cliques of G, each has a non-empty intersection with Q so that Q is a vertex of degree 4 in $K(G)$.

If $t = 1$, we may take $y_1 = z_1$ so that $\{x_1, y_1, x_2\}$, $\{x_1, y_2, y_3\}$ and $\{x_2, z_2, z_3\}$ are cliques of G so that Q is a vertex of degree 3 in $K(G)$.

In either case, we have a contradiction since $K(G) \cong G$.

This completes the proof. □

Proposition 4 *There exist no $\{2, s\}$-graphs in $\mathcal{G}(3)$ for any $3 \leq s \leq 5$.*

Proof: Suppose there is a $\{2, s\}$-graph $G \in \mathcal{G}(3)$. Let x be a vertex of degree 2 in G and let $N(x) = \{y, z\}$. Then $Q = \{x, y, z\}$ is a clique in G, by Theorem 2(i).

By Lemma 2, we may assume that $d(y) = 2$ in G. Then, clearly $d(z) = s$ for some $3 \leq s \leq 5$. Suppose $N(z) = \{x, y, y_1, \ldots, y_{s-2}\}$.

By Theorem 2(i), we have $s \notin \{3, 4\}$.

Therefore $s = 5$. Then, by Theorem 2(ii), we may assume that $y_1y_2y_3$ is a path on 3 vertices in $G[N(z)]$, so that $Q_1 = \{z, y_1, y_2\}$ and $Q_2 = \{z, y_2, y_3\}$ are cliques of G.

Clearly, $d(y_2) = 5$. Let $N(y_2) = \{z, y_1, y_3, z_1, z_2\}$ so that $Q_3 = \{y_2, z_1, z_2\}$ is a clique of G. By taking the clique graph of G, we see that the subgraph of $K(G)$ induced by Q, Q_1, Q_2 and Q_3 is such that Q is a vertex of degree 2 in $K(G)$ not adjacent to any vertex of degree 2 in $K(G)$. This, however, contradicts Lemma 2 because $K(G) \cong G$. □

We now show that if $G \in \mathcal{G}(3)$ is a $\{2, 6\}$-graph, then G is an infinite graph.

Proposition 5 *There exist no finite $\{2, 6\}$-graphs in $\mathcal{G}(3)$.*

Proof: Suppose G is a $\{2, 6\}$-graph in $\mathcal{G}(3)$. Assume that G has m vertices of degree 2 and n vertices of degree 6. We shall obtain a contradiction by showing that the number of triangles in G is less than $m + n$.

Let x be a vertex of degree 2 in G and let $N(x) = \{y, z\}$. Then $Q = \{x, y, z\}$ is a clique in G, by Theorem 2(i).

By Lemma 2, we may assume that $d(y) = 2$ in G. Then, clearly $d(z) = 6$. Suppose $N(z) = \{x, y, z_1, \ldots, z_4\}$.

By Theorem 2(iii), we may assume that z_1 is adjacent to z_2 and that z_3 is adjacent to z_4 so that $Q_1 = \{z, z_1, z_2\}$ and $Q_2 = \{z, z_3, z_4\}$ are cliques of G. As such, Q is a vertex of degree 2 which is contained in the clique $\{Q, Q_1, Q_2\}$ of $K(G)$. By Lemma 2, we may assume that the degrees of Q_1 and Q_2 in $K(G)$ are 2 and 6 respectively. But this implies that $d(z_3) = 6 = d(z_4)$.

Suppose $Q_3 = \{z_3, w_1, w_2\}$ and $Q_4 = \{z_3, w_3, w_4\}$ (respectively $Q_5 = \{z_4, w_5,$
$w_6\}$ and $Q_6 = \{z_4, w_7, w_8\}$) are the other two cliques containing the vertex z_3 (respectively z_4). Since $G \cong K(G)$, on taking the clique graph of G, it follows from Lemma 2 that we may assume that w_5, w_6, w_7, w_8 are vertices each of degree 6. We can then repeat the similar argument to the cliques that are incident to the vertices w_5, w_6, w_7, w_8 and continue in the like manner.

Since G is a finite graph, this argument must terminate in a finite number of steps. In that case, G has the following property. Each triangle in G contains either exactly two vertices of degree 2 or else exactly three vertices of degree 6. But this implies that the number of triangles in G is at most $\frac{m}{2} + n$ (which is less that $m + n$), a contradiction.

This completes the proof. □

On the other hand, the graph J defined below is an infinite $\{2, 6\}$-graph in $\mathcal{G}(3)$.

Let $\mathbb{N} = \{0, 1, 2, \ldots\}$ denote the set of all non-negative integers and let $2\mathbb{N} = \{2x \mid x \in \mathbb{N}\}$. Let $V(J) = \mathbb{N} \times \mathbb{N}$ and $E(J) = E_1 \cup E_2$ where $E_1 = \{(a, b)(a+1, b) \mid a \in 2\mathbb{N},\ b \in \mathbb{N}\}$ and $E_2 = \{(a, b)(4a+t, b-1) \mid a \in \mathbb{N},\ b \in \mathbb{N} - \{0\},\ t \in \{0, 1, 2, 3\}\}$.

Part of this graph is depicted in Figure 5.

Next, we observe that each triangle in J is given by $T_{2a,b} = \{(2a, b), (2a+1, b), (\lfloor \frac{a}{2} \rfloor, b+1)\}$ where $a, b \in \mathbb{N}$. Now, each $T_{2a,b}$ becomes a vertex in $K(J)$. Let $N_2 = \{T_{8a+2t,b-1} \mid t \in \{0, 1, 2, 3\}\}$ if $b \geq 1$ and let $N_2 = \emptyset$ otherwise. Then the neighborhood of $T_{2a,b}$ in $K(J)$ is given by

$$N(T_{2a,b}) = \begin{cases} \{T_{2a+2,b},\ T_{2\lfloor \frac{a}{4} \rfloor, b+1}\} \cup N_2 & \text{if } a \equiv 0 \pmod 2 \\ \{T_{2a-2,b},\ T_{2\lfloor \frac{a}{4} \rfloor, b+1}\} \cup N_2 & \text{if } a \equiv 1 \pmod 2. \end{cases}$$

Hence, the mapping that sends the vertex (x, b) to the triangle $T_{2x,b}$ is an isomorphism from J to $K(J)$.

To see that J is an infinite $\{2,6\}$-graph in $\mathcal{G}(3)$, first we observe the following. Let $x, b \in \mathbb{N}$ and let $N_1 = \{(4x+t, b-1) \mid t \in \{0,1,2,3\}\}$ if $b \geq 1$ and let $N_1 = \emptyset$ otherwise. Then the neighborhood of each vertex in J is given by

$$N((x,b)) = \begin{cases} \{(x+1,b),\ (\lfloor \frac{x}{4} \rfloor, b+1)\} \cup N_1 & \text{if } x \equiv 0 \pmod 2 \\ \{(x-1,b),\ (\lfloor \frac{x}{4} \rfloor, b+1)\} \cup N_1 & \text{if } x \equiv 1 \pmod 2. \end{cases}$$

Proposition 6 *There exist no* $\{3,s\}$*-graphs in* $\mathcal{G}(3)$ *for any* $4 \leq s \leq 6$.

Proof: Let x be a vertex of degree 3 in a $\{3,s\}$-graph $G \in \mathcal{G}(3)$. Let $N(x) = \{x_1, x_2, x_3\}$. By Theorem 2(i), we may assume that $x_1x_2x_3$ is the path on 3 vertices in $G[N(x)]$ so that $Q_1 = \{x, x_1, x_2\}$ and $Q_2 = \{x, x_2, x_3\}$ are cliques of G.

If $d(x_2) = 3$ in G, then this implies that Q_1Q_2 is a clique of size 2 in $K(G)$. But this is impossible because $K(G) \cong G$. Hence $d(x_2) = s$.

Let $N(x_2) = \{x, x_1, x_3, y_1, \ldots, y_{s-3}\}$.

Now, $s \neq 6$, by Theorem 2(iii), because $G[N(x_2)]$ contains a path on 3 vertices.

Suppose $s = 4$. Since $G[N(x_2)]$ is a path on 4 vertices by Theorem 2(i), y_1 must be adjacent to x_1, say. Then $d(x_3)$ must be either 3 or 4. Either case leads to absurdity because $G[N(x_3)]$ is then either P_3 or P_4 which is impossible because $d(x) = 3$ and $d(x_2) = 4$.

Suppose $s = 5$. Then $Q_3 = \{x_2, y_1, y_2\}$ is a clique of G. Moreover, $d(x_1) = 3$ or 5. Let $N(x_1) = \{x, x_2, z_1, \ldots, z_t\}$ where $t \in \{1,3\}$.

If $t = 1$, then $z_1 = y_i$ for some $i \in \{1,2\}$ because $G[N(x_1)] \cong P_3$ by Theorem 2(i). But then Q_1, Q_2, Q_3 and $\{x_1, x_2, y_i\}$ form a clique K_4 in $K(G)$ which is impossible because $K(G) \cong G$.

Hence $t = 3$. By Theorem 2(ii), we may take $z_1z_2z_3$ to be a path on 3 vertices in $G[N(x_1)]$ so that $\{x_1, z_1, z_2\}$ and $\{x_1, z_2, z_3\}$ are cliques in G. Taking the clique graph of G, we see that Q_1 is a vertex of degree 4 in $K(G)$. But this is impossible because $K(G) \cong G$.

This completes the proof. □

Figure 6 depicts two $\{4,5\}$-graphs all of whose cliques are of size 3. They are both drawn on the torus. It is routine to check that these two graphs are self-clique. These graphs can easily be extended to other $\{4,5\}$-graphs in $\mathcal{G}(3)$. Moreover there exist $\{4,5\}$-graphs in $\mathcal{G}(3)$ which do not resemble

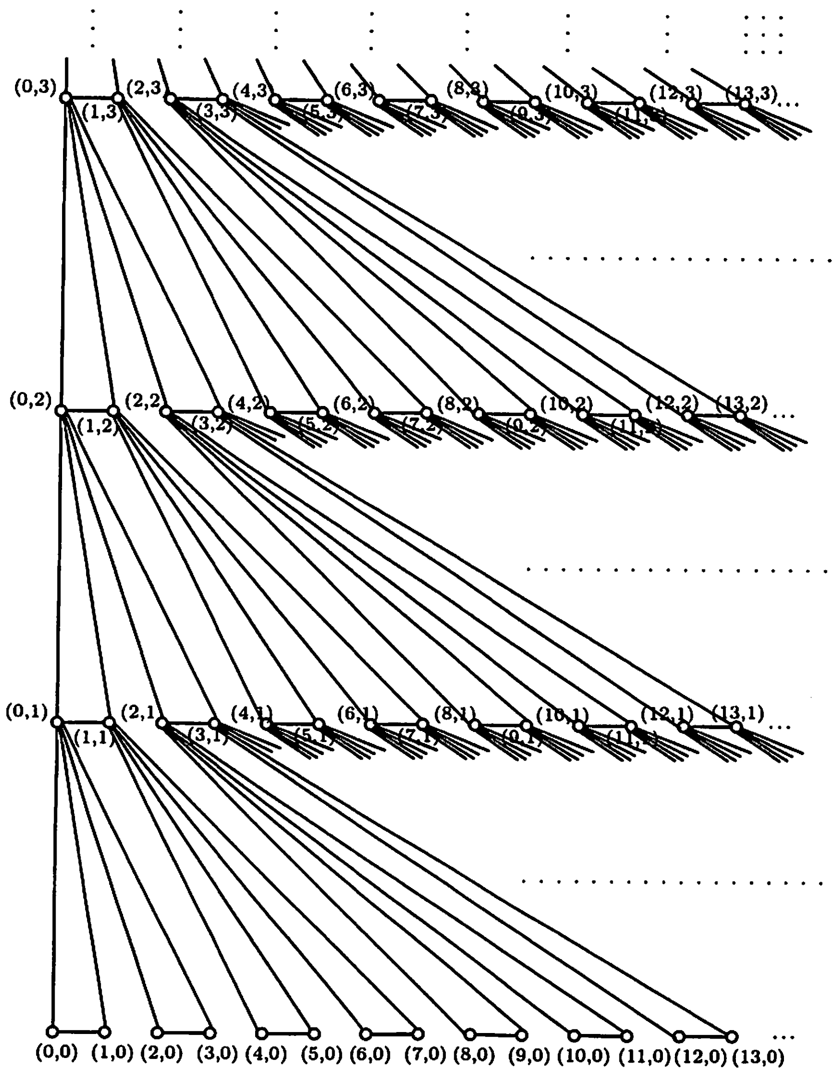

Figure 5: Infinite $\{2,6\}$-graph J in $\mathcal{G}(3)$

those shown in Figure 6. Further, one could easily modify these examples to yield an infinite number of finite graphs and also an infinite number of infinite ones.

Proposition 7 *There exist* $\{4,5\}$*-graphs in* $\mathcal{G}(3)$.

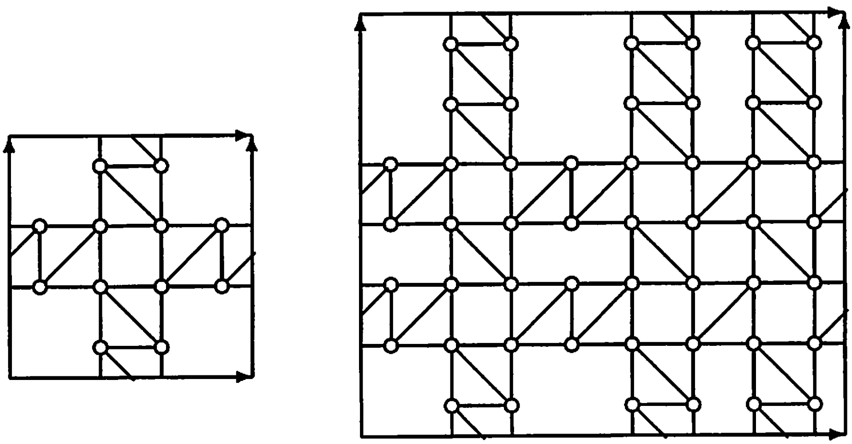

Figure 6: Some $\{4,5\}$-graphs in $\mathcal{G}(3)$ drawn on the torus.

Proposition 8 *There exist no* $\{4,6\}$*-graphs in* $\mathcal{G}(3)$.

Proof: Let x be a vertex of degree 4 in a $\{4,6\}$-graph $G \in \mathcal{G}(3)$. Let $N(x) = \{x_1, x_2, x_3, x_4\}$. By Theorem 2(i), we may assume that $x_1x_2x_3x_4$ is the path on 4 vertices in $G[N(x)]$ so that $Q_i = \{x, x_i, x_{i+1}\}$ is a clique of G for each $i = 1, 2, 3$.

Theorem 2(iii) implies that $d(x_2) = 4 = d(x_3)$. Let $N(x_2) = \{x, x_1, x_3, y_1\}$ and $N(x_3) = \{x, x_2, x_4, y_2\}$. Then $y_1 \neq y_2$, otherwise $\{x_2, x_3, y_1\}$ is a clique of G which, together with Q_1, Q_2 and Q_3, form a K_4 in $K(G)$ which is impossible because $K(G) \cong G$.

Hence y_1 is adjacent to x_1, and y_2 is adjacent to x_4. By Theorem 2, (i) and (iii), $d(x_1) = 4 = d(x_4)$. This implies that there exist vertices $z_1, z_2 \in V(G) - \{x\}$ such that $xx_2y_1z_1$ and $xx_3y_2z_2$ are paths on 4 vertices in G.

Applying Theorem 2(iii) to the vertices y_1 and y_2, and continue with similar argument as before, we see that G is a 4-regular graph, a contradiction.

This completes the proof. □

6 Remark

The results in preceding sections lead to the following questions. (i) Can 5-regular graphs or 6-regular graphs in $\mathcal{G}(3)$ be classified? (ii) Does there exist a $\{5,6\}$-graph in $\mathcal{G}(3)$?

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