Toughness and $[a, b]$-factors in graphs*

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Abstract

In this paper, we consider the relationship between toughness and the existence of $[a, b]$-factors. We obtain that a graph G has an $[a, b]$-factor if $t(G) \geq a - 1 + \frac{a-1}{b}$ with $b > a > 1$. Furthermore, it is showed that the result is best possible in some sense.

Keywords: $[a, b]$-factor, toughness, fractional factor

1 Introduction

All graphs considered are finite simple graphs. Let G be a graph with vertex set $V(G)$ and edge set $E(G)$. For $x \in V(G)$, the degree of x in G is denoted by $d_G(x)$. For any $S \subseteq V(G)$, we denote by $N_G(S)$ the neighborhood set of S in G. We use $G[S]$ and $G - S$ to denote the subgraph of G induced by S and $V(G) - S$, respectively. A subset S of $V(G)$ is called an independent set (a covering set) of G if every edge of G is incident with at most (at least) one vertex of S. We refer the readers to [1] for the terminologies not defined here. Let g and f be two integer-valued functions defined on $V(G)$ with $g(x) \leq f(x)$ for any $x \in V(G)$. A spanning subgraph F of G is called a (g, f)-factor if $g(x) \leq d_F(x) \leq f(x)$ holds for any vertex $x \in V(G)$. Moreover, a (g, f)-factor is called an $[a, b]$-factor if $g(x) \equiv a$ and $f(x) \equiv b$, an $[a, b]$-factor is called a k-factor if $a = b = k$. Let $h : E(G) \longrightarrow [0, 1]$ be

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a function and $k \geq 1$ be an integer. If $\sum_{e \ni x} h(e) = k$ holds for each vertex $x \in V(G)$, we call $G[F_h]$ a *fractional k-factor* of G with indicator function h where $F_h = \{e \in E(G) | \ h(e) > 0\}$. A fractional 1-factor is also called a fractional perfect matching [11].

Chvátal [4] first introduced the concept of toughness, denoted by $t(G) = \min\{\frac{|S|}{\omega(G-S)} : S \subseteq V(G),\ \omega(G-S) \geq 2\}$, where $\omega(G-S)$ denotes the number of components of $G-S$ and G is not a complete graph. If G is complete, then $t(G) = \infty$. A graph G is k-tough if $t(G) \geq k$. Chvátal mainly studied the relations between toughness and the existence of *Hamilton* cycles and k-factors. He conjectured that every k-tough graph G has a k-factor if $k|V(G)|$ is even. Enomoto et al. in [6] confirmed Chvátal's conjecture and showed that the result is sharp.

Theorem 1 (Enomoto et al. [6]) *Let G be a graph and $k \geq 2$. If $t(G) \geq k$, $|V(G)| \geq k+1$ and $k|V(G)|$ even, then G has a k-factor.*

Katerinis in [8] generalized Theorem 1 to $[a, b]$-factors. And Liu obtained a lower bound in [10] for fractional k-factors.

Theorem 2 (Liu [10]) *Let $k \geq 2$ be an integer. A graph G with $|V(G)| \geq k+1$ has a fractional k-factor if $t(G) \geq k - \frac{1}{k}$.*

Theorem 3 (Katerinis [8]) *Let G be a graph and a, b be two positive integers such that $b \geq a$. If $t(G) \geq (a-1) + \frac{a}{b}$ and $a|V(G)|$ is even when $a = b$, then G has an $[a, b]$-factor.*

Much work has been contributed to the relations between toughness and the existence of factors of a graph [2, 3, 5, 9]. In this paper, we get a better result about $[a, b]$-factors when $b > a$.

Theorem 4 *Let G be a graph and a, b be two positive integers such that $b > a > 1$. If $t(G) \geq (a-1) + \frac{a-1}{b}$, then G has an $[a, b]$-factor.*

2 Preliminary lemmas

We first give the characterization of (g, f)-factors due to Heinrich [7].

Lemma 5 (Heinrich [7]) *Let G be a graph and g, f be integer-valued functions defined on $V(G)$. If $g(x) < f(x)$ for every $x \in V(G)$, then G has a (g, f)-factor if and only if $g(T) - d_{G-S}(T) \leq f(S)$ for any set S of $V(G)$, where $T = \{x | \ x \in V(G) - S,\ d_{G-S}(x) \leq g(x)\}$.*

Lemma 6 (Chvátal [4]) *If a graph G is not complete, then $t(G) \leq \frac{1}{2}\delta(G)$.*

The following Lemma improves the result in Lemma 2.2 of [10].

Lemma 7 *Let G be a graph and $H = G[T]$ such that $d_G(x) = k-1$ for every $x \in V(H)$ and no component of H is isomorphic to K_k where $T \subseteq V(G)$, $k \geq 2$. Then there exists an independent set I and the covering set $C = V(H) - I$ of H satisfying $|V(H)| \leq (k - \frac{1}{k+1})|I'| + \sum_{j=0}^{k-2}(j+1)i''_j$, $|C| \leq (k-1-\frac{1}{k+1})|I'| + \sum_{j=0}^{k-2} ji''_j$, where $I' = \{x|x \in I,\ d_H(x) = k-1\}$, $i''_j = |\{x|x \in I'' = I - I',\ d_H(x) = j\}|$.*

Proof. Suppose that H has m components. For each component H_n, let I_n be a maximal independent set of H_n. It is obvious that some vertices of I_n have degree $k-1$ in H_n and some have degree less than $k-1$. Let $I_n = I'_n \cup I''_n$ where $I'_n = \{x|x \in I_n,\ d_{H_n}(x) = d_G(x) = k-1\}$ and $I''_n = I_n - I'_n = \{x|x \in I_n,\ d_{H_n}(x) < d_G(x) = k-1\}$. Thus for each vertex $x \in I''_n$, $d_{H_n}(x) \leq k-2$. Set $d_{H_n}(x) = j$ $(0 \leq j \leq k-2)$. Then $|I''_n| = \sum_{j=0}^{k-2} i''_{j_n}$ and $|N_{H_n}(I''_n)| \leq \sum_{j=0}^{k-2} ji''_{j_n}$, where $i''_{j_n} = |\{x|x \in I''_n,\ d_{H_n}(x) = j\}|$.

Claim. For each vertex $x \in I'_n$, there exists $y \in I_n - \{x\}$ such that $N_{H_n}(x) \cap N_{H_n}(y) \neq \emptyset$.

For this, we show that $H_n[N_{H_n}(x)]$ is not complete. Otherwise, $H'_n = H_n[\{x\} \cup N_{H_n}(x)] = K_k$. Since H_n is connected and for every vertex $x \in V(H_n)$, $d_{H_n}(x) \leq k-1$, it follows that $H_n = H'_n = K_k$, a contradiction. Therefore there exist two vertices x' and y' in $H_n[N_{H_n}(x)]$ that are not adjacent. Now if for any $y \in I_n - \{x\}$, $N_{H_n}(x) \cap N_{H_n}(y) = \emptyset$, then y is not adjacent to x' and y'. We can construct a new independent set $(I_n - \{x\}) \cup \{x', y'\}$ of H_n that is larger than I_n, a contradiction.

Combined by the previous discussion about I''_n we have $|V(H_n)| \leq k|I'_n| - \lceil \frac{|I'_n|}{2} \rceil + |I''_n| + |N_{H_n}(I''_n)| \leq (k - \frac{1}{k+1})|I'_n| + \sum_{j=0}^{k-2}(j+1)i''_{j_n}$, $(1 \leq n \leq m)$. Let $I' = \sum_{n=1}^{m} I'_n$, $I'' = \sum_{n=1}^{m} I''_n$ and $I = \sum_{n=1}^{m} I_n = I' \cup I''$. Then I is a maximum independent set of H. Thus $|V(H)| = \sum_{n=1}^{m} |V(H_n)| \leq \sum_{n=1}^{m}(k - \frac{1}{k+1})|I'_n| + \sum_{n=1}^{m}\sum_{j=0}^{k-2}(j+1)i''_{j_n} = (k - \frac{1}{k+1})|I'| + \sum_{j=0}^{k-2}(j+1)i''_j$, where $i''_j = \sum_{n=1}^{m} i''_{j_n} = |\{x|x \in I'',\ d_H(x) = j\}|$. Let $C = V(H) - I$. Then $|C| = |V(H)| - |I| = |V(H)| - |I'| - |I''| \leq (k-1-\frac{1}{k+1})|I'| + \sum_{j=0}^{k-2} ji''_j$.

Lemma 8 (Liu [10]) *Let G be a graph and let $H = G[T]$ such that $\delta(H) \geq 1$ and $1 \leq d_G(x) \leq k-1$ for $x \in V(H)$ where $T \subseteq V(G)$ and $k \geq 2$. Let $T_1,\cdots,T_{k-1}$ be a partition of the vertices of H satisfying $d_G(x) = j$ for $x \in T_j$ where we allow some T_j to be empty. If each component of H has a vertex of degree at most $k-2$ in G, then H has a maximal independent set I and a covering set $C = V(H)-I$ such that $\sum_{j=1}^{k-1}(k-j)c_j \leq \sum_{j=1}^{k-1}(k-2)(k-j)i_j$, where $c_j = |C \cap T_j|$ and $i_j = |I \cap T_j|$ for $1 \leq j \leq k-1$.*

3 Proof of the main result

Proof of Theorem 4. Suppose that G satisfies the conditions in Theorem 4, but G has no $[a,b]$-factors. Then, by Lemma 5 there exists $S \subseteq V(G)$ satisfying $a|T| - d_{G-S}(T) > b|S|$, (1) where $T = \{x|\ x \in V(G)-S,\ d_{G-S}(x) \leq a\}$ since $g(x) = a$ and $f(x) = b$ for every $x \in V(G)$. In addition, suppose that T is minimal with respect to (1). If there exists $x_0 \in T$ with $d_{G-S}(x_0) = a$, obviously S, $T - \{x_0\}$ satisfy (1), contradicting the minimality of T. So $T = \{x|x \in V(G)-S,\ d_{G-S}(x) \leq a-1\}$. And $\delta(G) \geq 2t(G) > a-1$ by Lemma 6. Therefore $S \neq \emptyset$ according to (1). Let $H' = G[T]$. If there exists components of H' that are isomorphic to K_a, let m be the number of these components and $T_0 = \{x \in V(H')|d_{G-S}(x) = 0\}$. Set $H = H' - mK_a - T_0$, $|T_0| = t_0$. If $|V(H)| = 0$, by (1) we get $at_0 + ma > b|S|$, that is, $1 \leq |S| < \frac{a}{b}(t_0 + m)$. Hence $\omega(G-S) \geq t_0 + m > 1$, we have $t(G) \leq \frac{|S|}{\omega(G-S)} \leq \frac{|S|}{t_0+m} < 1$. This contradicts that $t(G) \geq a - 1 + \frac{a-1}{b} > 1$.

Now we consider that $|V(H)| > 0$. Let $H = H_1 \cup H_2$ where H_1 is the union of components of H which satisfies that $d_{G-S}(x) = a-1$ for any vertex $x \in V(H_1)$ and $H_2 = H - H_1$. By Lemma 7, H_1 has a maximal independent set I_1 and the covering set $C_1 = V(H_1) - I_1$ such that $|V(H_1)| \leq (a - \frac{1}{a+1})|I_1'| + \sum_{j=0}^{a-2}(j+1)i_j''$, $|C_1| \leq (a - 1 - \frac{1}{a+1})|I_1'| + \sum_{j=0}^{a-2} j i_j''$, where $I_1' = \{x|x \in I_1,\ d_{H_1}(x) = d_{G-S}(x) = a-1\}$ and $i_j'' = |\{x|x \in I_1'' = I_1 - I_1',\ d_{H_1}(x) = j < d_{G-S}(x) = a-1\}|$.

On the other hand, we may assume that $\delta(H_2) \geq 1$. Since $\Delta(H_2) \leq a-1$, let $T_j = \{x \in V(H_2)|d_{G-S}(x) = j\}$ for $1 \leq j \leq a-1$. By the definition of H_2 we know that there exists one vertex with degree at most $a-2$ in $G-S$ from each component of H_2. According to Lemma 8, H_2 has a maximal independent set I_2 and the covering set $C_2 = V(H_2) - I_2$ such that $\sum_{j=1}^{a-1}(a-j)c_j \leq \sum_{j=1}^{a-1}(a-2)(a-j)i_j$, (2) where $c_j = |C_2 \cap T_j|$ and $i_j = |I_2 \cap T_j|$ for every $j = 1,\cdots,a-1$. Set $W = V(G) - S - T$ and $U = S \cup C_1 \cup (N_G(I_1'') \cap W) \cup C_2 \cup (N_G(I_2) \cap W)$. It follows that

$|U| \leq |S| + |C_1| + \sum_{j=0}^{a-2}(a-1-j)i''_j + \sum_{j=1}^{a-1} ji_j$, $\omega(G-U) \geq (t_0+m) + |I'_1| + \sum_{j=0}^{a-2} i''_j + \sum_{j=1}^{a-1} i_j$. Now we claim that $|U| \geq t(G)\omega(G-U)$. It holds obviously when $\omega(G-U) > 1$. When $\omega(G-U) = 1$, by the previous discussion we obtain that $t_0 = m = 0$, then for every vertex $x \in T$, $|U| \geq d_{G-S}(x) + |S| \geq \delta(G) \geq 2t(G)$. So $|S| + |C_1| + \sum_{j=0}^{a-2}(a-1-j)i''_j + \sum_{j=1}^{a-1} ji_j \geq t(G)(t_0+m+|I'_1| + \sum_{j=0}^{a-2} i''_j + \sum_{j=1}^{a-1} i_j)$. (3)

From (1), $a(t_0+m) + |V(H_1)| + \sum_{j=1}^{a-1}(a-j)i_j + \sum_{j=1}^{a-1}(a-j)c_j > b|S|$. Combined with (3), $a(t_0+m)+|V(H_1)|+b|C_1|+b\sum_{j=0}^{a-2}(a-1-j)i''_j+\sum_{j=1}^{a-1}(a-j)c_j > bt(G)(t_0+m+|I'_1|+\sum_{j=0}^{a-2} i''_j)+\sum_{j=1}^{a-1}(bt(G)-bj-a+j)i_j$. And according to the notation of $t(G)$, we have $|V(H_1)|+b|C_1|+\sum_{j=0}^{a-2} b(a-1-j)i''_j+\sum_{j=1}^{a-1}(a-j)c_j > bt(G)(|I'_1|+\sum_{j=0}^{a-2} i''_j)+\sum_{j=1}^{a-1}(bt(G)-bj-a+j)i_j+(bt(G)-a)(t_0+m) \geq bt(G)(|I'_1| + \sum_{j=0}^{a-2} i''_j) + \sum_{j=1}^{a-1}(bt(G)-bj-a+j)i_j$.

By Lemma 7, $|V(H_1)|+b|C_1|+\sum_{j=0}^{a-2} b(a-1-j)i''_j \leq (a-\frac{1}{a+1}+b(a-1-\frac{1}{a+1}))|I'_1|+\sum_{j=0}^{a-2}(j+1+bj)i''_j+\sum_{j=0}^{a-2} b(a-1-j)i''_j = ((1+b)(a-1)+1-\frac{1+b}{a+1})|I'_1|+\sum_{j=0}^{a-2}(ba-b+j+1)i''_j$. Combined with (2), $\sum_{j=1}^{a-1}(a-2)(a-j)i_j + (b(a-1) + a - \frac{1+b}{a+1})|I'_1| + \sum_{j=0}^{a-2}(ba-b+j+1)i''_j > bt(G)|I'_1| + bt(G)\sum_{j=0}^{a-2} i''_j + \sum_{j=1}^{a-1}(bt(G) - bj-a+j)i_j \geq b(a-1+\frac{a-1}{b})|I'_1| + bt(G)\sum_{j=0}^{a-2} i''_j + \sum_{j=1}^{a-1}(bt(G)-bj-a+j)i_j$. That is $\sum_{j=1}^{a-1}(a-2)(a-j)i_j + \sum_{j=0}^{a-2}(ba-b+j+1)i''_j > \sum_{j=1}^{a-1}(bt(G)-bj-a+j)i_j + bt(G)\sum_{j=0}^{a-2} i''_j$. Thus at least one of the following two cases must hold.

Case 1. At least one j satisfying $(a-2)(a-j) > bt(G)-bj-a+j$. Then $t(G) < \frac{a^2-a+(b-a+1)j}{b} \leq a-1+\frac{a-1}{b}$ $(j \leq a-1)$, a contradiction.

Case 2. $ba - b + j + 1 > bt(G)$ for some $j \in \{0, 1, 2, \cdots, a-2\}$.

In this case we have $t(G) < a - 1 + \frac{a-1}{b}$, contradicting to the toughness condition of Theorem 4, completing the proof of the theorem.

Remark. The bound of toughness in Theorem 4 is sharp. To see this, consider the graph G: $V(G) = V(A) \cup V(B) \cup V(C)$ where A, B and C are disjoint with $A = K_{(nb+1)(a-1)}$, $B = (nb+1)K_{a-1}$ and $C = K_{n(a-1)}$. Since $| V(A) |=| V(B) |$, we set the edges between A and B are a perfect matching between A and B. And we set each vertex of B is adjacent to all the vertices of C. This follows that $t(G) = \frac{(nb+1)(a-1)+n(a-1)}{nb+1} < a-1+\frac{a-1}{b}$, $t(G) \to a-1+\frac{a-1}{b}$ when $n \to \infty$. But we can get that G has no $[a,b]$-factor since (1) holds if we set $S = C$.

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