The List Point Arboricity of Some Complete Multi-partite Graphs

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Abstract

Let G be a graph. The point arboricity of G, denoted by $\rho(G)$, is the minimum number of colors that can be used to color the vertices of G so that each color class induces an acyclic subgraph of G. The list point arboricity $\rho_l(G)$ is the minimum k so that there is an acyclic L−coloring for any list assignment L of G which $|L(v)| \geq k$. So $\rho(G) \leq \rho_l(G)$. Zhen and Wu conjectured that if $|V(G)| \leq 3\rho(G)$, then $\rho_l(G) = \rho(G)$. Motivated by this, we investigate the list point arboricity of some complete multi-partite graphs of order slightly larger than $3\rho(G)$, and obtain $\rho(K_{m(1),2(n-1)}) = \rho_l(K_{m(1),2(n-1)})$ $(m = 2, 3, 4)$.

Keywords: List point arboricity , Complete multi-partite graphs.

1 Introduction

All graphs considered here are finite, undirected and simple. We refer to [9] for unexplained terminology and notations. We say n graphs $G_1, G_2, ..., G_n$ are vertex disjoint if they have no vertex in common and denote their join by $G_1 + G_2 + ... + G_n$, which is obtained from their union by joining each vertex of G_i to the each vertex of G_j $(i \neq j)$.

The point arboricity of G, denoted by $\rho(G)$, is the minimum number of colors that can be used to color the vertices of G so that each color class induces an acyclic subgraph of G. The notion of list coloring of graphs was introduced by Vizing [8] and independently by Erdös et. al [4]. Borodin et. al [1] defined a similar concept for the point arboricity of a graph. A list assignment of a graph G is a function L defined on $V(G)$ such that $L(v) \subseteq N$ is the list of colors available for the vertex $v \in V(G)$. For a given

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positive integer k, if $|L(v)| = k$ for every vertex $v \in V(G)$, we say L is a k-list assignment of G. For a list assignment L of G, we say c is an L-coloring if $c(v) \in L(v)$ for every vertex $v \in V(G)$. Set $c(L) = \{c(v) | v \in V(G)\}$, that is the set of colors chosen for the vertices of G under c. An L-coloring c is called acyclic if for each color $i \in c(L)$, $G[V_i]$ is acyclic, or is a forest, where V_i is the set of vertices v of G with $c(v) = i$. In this case, we say G is acyclic L-colorable. If G is acyclic L-colorable for any possible k-list assignment L, G is called acyclic k-list colorable. The list point arboricity of a graph G, denoted by $\rho_l(G)$, is the minimum number k for which G is acyclic k-list colorable.

It is trivial that $\rho(G) \leq \rho_l(G)$. Ohba [5] conjectured that if a graph G has the chromatic number k and at most $2k+1$ vertices, then the chromatic number of G coincides with its list chromatic number. Enomoto et.al [3], Shen [7] and Cranston [2] have verified that the conjecture is true for some complete multi-partite graphs. Therefore it is significant to investigate the condition or find some graph classes, in which each graph satisfies $\rho_l(G) = \rho(G)$. Seymour [6] proved that $\rho_l(G) = \rho(G)$ hold if G is the line graph of any graph. Zhen and Wu [10] have the following results.

Theorem 1.1. *([10]) For any graph G, there exists a non-negative integer $n_0 = n_0(G)$ such that $\rho(G + K_n) = \rho_l(G + K_n)$, for any integer n with $n \geq n_0$.*

Theorem 1.2. *([10]) If $|V(G)| \leq 2\rho(G) + \sqrt{2\rho(G)} - 1$, then $\rho(G) = \rho_l(G)$.*

Conjecture 1.3. *([10]) If $|V(G)| \leq 3\rho(G)$, then $\rho(G) = \rho_l(G)$.*

We construct a graph $G = T_1 + T_2 + ... + T_i + ... + T_n$, where $T_i (1 \leq i \leq n)$ is a nontrivial tree. $G = T_1 + T_2 + ... + T_i + ... + T_n$ is an edge maximal graph such that $\rho(G) = n$. If we can prove equation $\rho(G) = \rho_l(G)$ holding for G, then the equation $\rho(G') = \rho_l(G')$ can also hold for the subgraph G' of G with $\rho(G') = n$.

2 Some Lemmas

For a graph $G = (V, E)$ and a subset $X \subseteq V$, let $G[X]$ denote the subgraph of G induced by X. For a list assignment L of G, let $L|_X$ denote L restricted to X, and $L(X)$ denote the union $\cup_{u \in X} L(u)$. If A is a set of colors, let $L \setminus A$ denote the list assignment from L by deleting the colors in A from each $L(u)$ with $u \in V(G)$. When A consists a single color a, we write $L \setminus a$.

Lemma 2.1. *([10]) If a graph G is not acyclic L-colorable, then there exists a set $X \subseteq V(G)$ such that $2|L(X)| < |X|$.*

Clearly, the following lemma can also hold.

Lemma 2.2. *Let L be a list assignment for a graph G. If $2|L(X)| \geq |X|$ for every subset $X \subseteq V(G)$, then G is acyclic $L-$colorable.*

Lemma 2.3. *Let L be a list assignment for a graph $G = (V, E)$ and let $X \subseteq V(G)$ be a maximal non-empty subset such that $2|L(X)| < |X|$. If $G[X]$ is acyclic $L|_X-$ colorable, then G is acyclic $L-$colorable.*

Proof. Let X be a maximal subset of V such that $X \neq \emptyset$ and $2|L(X)| < |X|$. Let $C = L(X)$. By the maximality of X, every subset $Y \subseteq V \setminus X$ satisfies $2|L(Y) \setminus C| \geq |Y|$. Let $L'(v) = L(v) \setminus C$ for every $v \in V \setminus X$. Note that $G[V \setminus X]$ and L' satisfy Lemma 2.2. Hence $G[V \setminus X]$ is acyclic L'-colorable. By hypothesis, $G[X]$ is acyclic $L|_X-$colorable. Since none of the colors used on X are used on $V \setminus X$, we can combine the two colorings to give an acyclic $L-$colloring of G. □

Lemma 2.4. *A graph $G = (V, E)$ is acyclic $k-$list colorable if G is acyclic $L-$colorable for every $k-$list assignment L such that $2|L(V)| < |V|$.*

Proof. We show the hypothesis of Lemma 2.3 holds. Let L be a $k-$list assignment for V and let X be a maximal non-empty subset of V such that $2|L(X)| < |X|$. We construct a new list assignment $L'(V)$ such that $L'(x) = L(x)$ for each $x \in X$. Choose an arbitrary vertex $u \in X$, for each vertex $v \notin X$, let $L'(v) = L(u)$. Note that $2|L'(V)| = 2|L(X)| < |X| \leq |V|$, by hypothesis G is acyclic $L'-$colorable. So $G[X]$ is acyclic $L'|_X-$colorable, and hence $G[X]$ is acyclic $L|_X-$colorable. The lemma follows from Lemma 2.3. □

3 Main results

Theorem 3.1. *Let $G = (V, E)$ be a graph with $G = T_1 + T_2 + ... + T_i + ... + T_n$. If $|T_i| = 3$ $(i = 1, 2, ...n)$, then $\rho(G) = \rho_l(G) = n$.*

Proof. We induct on n. The case $n = 1$ is easy. By Lemma 2.4 we may assume that $2|L(V)| < |V| = 3n$, then each three vertices of T_i has a color in common. If this is not true, each color in $L(T_i)$ appears in the lists of at most 2 vertices of T_i, and hence $2|L(V)| \geq 2|L(T_i)| \geq 3n$, a contradiction. Then we can use this common color on three vertices and proceed by induction. □

We use the notation $K_{2(n)}$ to denote a complete $n-$partite graph in which each part has 2 vertices. Analogously, the notation $K_{m(1),2(n-1)}$ to denote a complete $n-$partite graph, in which one part has m vertices and $n - 1$ parts have 2 vertices.

Corollary 3.2. *[10] $\rho_l(K_{2(n)}) = \rho(K_{2(n)})$.*

Proof. Since $\lceil \frac{2n}{3} \rceil = \rho(K_{2(n)}) \leq \rho_l(K_{2(n)})$, it suffices to show that $\rho_l(K_{2(n)}) \leq \lceil \frac{2n}{3} \rceil$. The graph $K_{2(n)}$ is the subgraph of $G = T_1 + T_2 + \ldots + T_{\lceil \frac{2n}{3} \rceil}$, where $|T_1| = |T_2| = \ldots = |T_{\lceil \frac{2n}{3} \rceil - 1}| = 3$, $|T_{\lceil \frac{2n}{3} \rceil}| \leq 3$, so $\rho_l(K_{2(n)}) \leq \rho_l(G) = \lceil \frac{2n}{3} \rceil$. □

Lemma 3.3. *Let H be a graph with $H = T_1 + T_2 + \ldots + T_i + \ldots + T_n$ and $T_1 = \{x\}$, $T_i = \{u_i, v_i, w_i\}(2 \leq i \leq n)$. Suppose that L is a list assignment of H satisfying that*
(1)$|L(x)| \geq n$;
(2)$|L(u_i)| = |L(v_i)| \geq n-1, |L(w_i)| \geq n$ for $i = 2, 3, \ldots, n$;
(3)$L(u_i) \cap L(v_i) \cap L(w_i) = \emptyset$ for $i = 2, 3, \ldots, n$;
Then H is acyclic $L-$colorable.

Proof. We shall prove that L satisfy Lemma 2.2. Assume to the contrary that there exists a subset $S \subseteq V(H)$ such that $2|L(S)| < |S|$.

Case 1. There exists some $T_i \subseteq S$. Since $L(u_i) \cap L(v_i) \cap L(w_i) = \emptyset$, then $2|L(T_i)| \geq (n-1) + (n-1) + n = 3n - 2$. From the hypothesis $2|L(S)| < |S|$, hence $3n - 2 \leq 2|L(T_i)| \leq 2|L(S)| < |S| \leq 1 + 3(n-1) = 3n - 2$, a contradiction.

Case 2. Each $T_i \nsubseteq S$. Since each $T_i \nsubseteq S$, $|S| \leq 2(n-1) + 1 = 2n - 1$. On the other hand, $|L(v)| \geq n - 1$ for any $v \in V(H)$, combining the hypothesis, $2(n-1) \leq 2|L(v)| \leq 2|L(S)| < |S|$, that is $|S| \geq 2(n-1) + 1$. Then S must contain vertex x, hence $|S| > 2|L(S)| \geq 2|L(x)| = 2n$, a contradiction. □

Theorem 3.4. *Let $G = (V, E)$ be a graph with $G = T_1 + T_2 + \ldots + T_i + \ldots + T_n$, $T_1 = \{u_1, v_1, w_1, x\}$, $T_i = \{u_i, v_i, w_i\}(2 \leq i \leq n)$, then $\rho(G) = \rho_l(G) = n$.*

Proof. If any part of size 3 or 4 has a color that appears on all its vertices, we use that color on these vertices and proceed by induction. So we may assume that no part of size 3 or 4 has a common color.

By Lemma 2.4 we may assume that $2|L(V)| < |V| = 3n + 1$, then there exist three vertices of T_1 having a common color. If this is not true, each color in $L(T_1)$ appears in the lists of at most 2 vertices of T_1, and hence $2|L(V)| \geq 2|L(T_1)| \geq 4n > 3n + 1 (n > 1)$, a contradiction. So there are three vertices in T_1, call them u_1, v_1 and w_1, that share a common color c. Use color c on u_1, v_1 and w_1. Now for each $v \in V \setminus \{u_1, v_1, w_1\}$, let $L'(v) = L(v) \setminus c$. Because no part of size 3 or 4 has a common color, then $|L'(x)| \geq n$, $|L'(u_i)| = |L'(v_i)| \geq n - 1, |L'(w_i)| \geq n$, by Lemma 3.3, we conclude the result. □

It is easy to see that $\rho(K_{3(1),2(n-1)}) = \lceil \frac{2n}{3} \rceil$. Moreover, $K_{3(1),2(n-1)}$ is subgraph of $T_1 + T_2 + \ldots + T_{\lceil \frac{2n}{3} \rceil}$, where $|T_1| = 4$, $|T_2| = \ldots = |T_{\lceil \frac{2n}{3} \rceil - 1}| = 3$, $|T_{\lceil \frac{2n}{3} \rceil}| \leq 3$. Therefore, the following corollary holds.

Corollary 3.5. $\rho_l(K_{3(1),2(n-1)}) = \rho(K_{3(1),2(n-1)}) = \lceil \frac{2n}{3} \rceil$.

Let $G = (V, E)$ be a graph with $G = T_1 + T_2 + ... + T_i + ... + T_n$, $T_1 = \{u_1, v_1, w_1, x, y\}$, $T_i = \{u_i, v_i, w_i\}(2 \leq i \leq n)$. We can conclude $\rho(G) = \rho_l(G)$ from a similar argument with Theorem 3.4 for $n \geq 4$. It is also trivial to be verified for $n \leq 3$. Therefore, we obtain the following corollary.

Corollary 3.6. $\rho_l(K_{4(1),2(n-1)}) = \rho(K_{4(1),2(n-1)}) = \lceil \frac{2n}{3} \rceil$.

Acknowledgement

The authors are much indebted to the referee for his/her valuable suggestions and corrections that improved the initial version of this paper.

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