On the matching polynomial of theta graphs

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Abstract

A theta graph is denoted by $\theta(a,b,c), a \leq b \leq c$. It is obtained by subdividing the edges of the multigraph consisting of 3 parallel edges a times, b times and c times each. In this paper, we show that the theta graph is *matching unique* when $a \geq 2$ or $a = 0$, and all theta graphs are *matching equivalent* when only one of the edges is subdivided one time. We also completely characterize the relation between the largest matching root λ and the length of path a, b, c of a theta graph, and determine the extremal theta graphs.

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1 Introduction

Let G be a *simple graph*, $V(G)$ and $E(G)$ be its vertex set and edge set, respectively. Let $m(G,k)$ denote the number of $k-$*element matching* of G.

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For convenience, we set $m(G,0) = 1$. In addition, $m(G,1) = |E(G)|$ is the number of edges. In [3], E. J. Farrell defines the *matching polynomial* as

$$M_G(x) = \sum_{k\geq 0}^{n/2} (-1)^k m(G,k) x^{n-2k}. \quad (1)$$

The matching polynomial was defined formally in the framework of the theory of monomer-dimer systems (see in [2]). If $M_G(x) = M_H(x)$, then we say G and H are *comatching graphs*. A graph G is said to be *matching unique* if it has no comatching graphs. So far, only a few classes of matching unique graph have been shown to be matching unique. In [9], Beezer and Farrell have shown some classes of *matching unique graphs*.

Let P_n, C_n be the path and cycle of length n. $D(s,t)$ be the graph is obtained by joining a one degree vertex of P_t to a two degree vertex of C_s. $T_{i,j,k}$ be the graph that is obtained by coinciding three one degree vertices of P_i, P_j, P_k. *A theta graph* denoted by $\theta(a,b,c), a \leq b \leq c$ is obtained by subdividing each edge of the multigraph consisting of 3 parallel edges a times, b times and c times. $D(a,b,c)$ be the graph that is obtained by connecting two cycles C_a and C_c by a path P_b. In this paper, we show that the theta graph is matching unique when $a \geq 2$ or $a = 0$, an all theta graphs are matching equivalent when only one of edges is subdivided one time. We also study the largest root of matching polynomial of theta graphs, and determine the extremal graphs with respect to the largest matching root.

2 Basic lemmas

In general, calculate the matching polynomial of an outerplanar graph can be determined in polynomial time, but for general graphs it is NP-complete. The following Lemmas 2.1, 2.2 and 2.3 are quite helpful to calculate the matching polynomial of a graph.

Lemma 2.1 *[3][8] If $G_1, G_2, \ldots, G_k$ are the components of G, then*

$$M_G(x) = \prod_{i=1}^{i=k} M_{G_i}(x).$$

Lemma 2.2 *[3][8] Suppose $n, r \in z^+$, then*

1. $M_{P_n}(x) = \sum\limits_{r\geq 0}(-1)^r\binom{n-r}{r}x^{n-2r}$;

2. $M_{C_n}(x) = \sum\limits_{r\geq 0}(-1)^r\frac{n}{n-r}\binom{n-r}{r}x^{n-2r}$.

Lemma 2.3 *[4] Let G be a graph and suppose $uv \in E(G)$. Then*

1. $M_G(x) = M_{G-uv}(x) - M_{G-\{u,v\}}(x)$;

2. $M_{G-u}(x) = xM_{G-u}(x) - \sum\limits_{u_i\in N_G(u)} M_{G-\{uu_i\}}(x)$.

The following Lemmas 2.4, 2.5 and 2.6 give some properties of the coefficients of matching polynomial, and it is very important for characterizing the comatching graphs.

Lemma 2.4 *[10] Let $(d_1, d_2, \ldots, d_n)$ be the degree sequence of G, and $(d_1+t_1, d_2+t_2, \ldots, d_n+t_n)$ be the degree sequence of H. If $M_G(x) = M_H(x)$, then the following equations hold.*

1. *t_i are integral numbers;*

2. $\sum\limits_{i}^{n} t_i = 0$;

3. $\sum\limits_{i=1}^{n}(t_i^2 + 2d_it_i) = 0$.

Lemma 2.5 *[11] Let a_0, a_1, a_2, a_3 be the first four coefficients of $M_G(x)$. Then*

1. $a_0 = 1$;

2. *$|a_1| = m$, the number of edges;*

3. $a_2 = \binom{m}{2} - \sum\limits_{i=1}^{n}\binom{d_i}{2}$;

4. $a_3 = \binom{m}{3} - (m-2)\sum\limits_{i=1}^{n}\binom{d_i}{2} + 2\sum\limits_{i=1}^{n}\binom{d_i}{3} + \sum\limits_{i,j\in 1,2,\ldots,n}(d_i-1)(d_j-1) - t(\triangle)$, *where $t(\triangle)$ is the number of triangle of G.*

Lemma 2.6 *Let $M_{P_n}(x)$ be the matching polynomial of P_n. Then*

$$M_{P_n}(2) = n + 1.$$

Proof. We apply induction on the number of vertices. For a path on small number of vertices, we can easily check it. We only prove a path on large number of vertices. Suppose that result is correct for P_s and P_t, that is $M_{P_s}(2) = s + 1$ and $M_{P_t}(2) = t + 1$. Let us calculate $P_{s+t}(2)$. Form the basic Lemma 2.4, we have

$$M_{P_{s+t}}(x) = M_{P_s}(x)M_{P_t}(x) - M_{P_{s-1}}(x)M_{P_{t-1}}(x).$$

By our assumption

$$M_{P_{s+t}}(2) = M_{P_s}(2)M_{P_t}(2) - M_{P_{s-1}}(2)M_{P_{t-1}}(2)$$

$$= (s + 1)(t + 1) - st = s + t + 1.$$

Hence Lemma 2.6 is proved.

Similarly, we have an interesting result for cycles.

Lemma 2.7 *Let $M_{C_n}(x)$ be the matching polynomial of C_n. Then*

$$M_{C_n}(2) = 2.$$

Lemma 2.8 *Let G and H are simple graphs. $M_G(x)$ and $M_H(x)$ are the matching polynomial of G and H. If $M_G(x) < M_H(x)$ for any $x \geq \lambda(G)$, then $\lambda(G) > \lambda(H)$.*

Proof. Since $M_H(x) - M_G(x) > 0$, when $x \geq \lambda(G)$, so

$$M_H(\lambda(G)) - M_G(\lambda(G)) > 0,$$

that is

$$M_H(\lambda(G)) > 0.$$

Hence

$$\lambda(G) > \lambda(H).$$

3 Main results

In this section, we show the matching uniqueness and matching equivalence of theta graphs, and study the largest root of the matching polynomial of theta graphs.

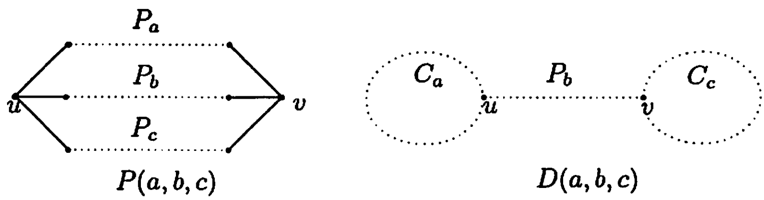

Fig. 1. Two type of graphs

3.1 The matching polynomial of $\theta(a,b,c)$ and $D(a,b,c)$

For the sake of simplicity and symbol convenience, in this section, we denote the matching polynomial $M_{P_r}(\lambda)$ **by** P_r. For convenience, let $P_{-2} = -1, P_{-1} = 0, P_0 = 1, P_1 = x$. By Lemma 2.1 and Lemma 2.3, the matching polynomial of $\theta(a,b,c)$ is

$$\begin{aligned}M_{\theta(a,b,c)}(\lambda) &= \lambda^2 P_a P_b P_c - 2\lambda(P_{a-1}P_bP_c + P_aP_{b-1}P_c + P_aP_bP_{c-1})\\ &+(P_{a-2}P_bP_c + P_{a-1}P_{b-1}P_c + P_{a-1}P_bP_{c-1})\\ &+(P_{a-1}P_{b-1}P_c + P_aP_{b-2}P_c + P_aP_{b-1}P_{c-1})\\ &+(P_{a-1}P_bP_{c-1} + P_aP_{b-1}P_{c-1} + P_aP_bP_{c-2})\end{aligned}$$

and by Lemma 2.6, the value on $\lambda = 2$ is

$$\begin{aligned}M_{\theta(a,b,c)}(2) &= 4(a+1)(b+1)(c+1)\\ &-4[a(b+1)(c+1) + (a+1)b(c+1) + (a+1)(b+1)c]\\ &+[(a-1)(b+1)(c+1) + ab(c+1) + a(b+1)c]\\ &+[ab(c+1) + (a+1)(b-1)(c+1) + (a+1)bc]\\ &+[a(b+1)c+(a+1)bc+(a+1)(b+1)(c-1)] = 1-a-b-c-ab-bc-ac-abc.\end{aligned}$$

As in [13], when G is a forest the matching polynomial and the characteristic polynomial are same, so by Lemma 2.4, we have a recurrence expression of the matching polynomial of a P_r,

$$P_r = \lambda P_{r-1} - P_{r-2}.$$

By solving this recurrence equation, we have:

$$P_r = \frac{x^{2r+2}-1}{x^{r+2}-x^r}, r \geq 2, \tag{2}$$

where x satisfies the equation $x^2 - \lambda x + 1 = 0$. Hence

$$M_{\theta(a,b,c)}(\lambda)x^{a+b+c}(x^6 - 3x^4 + 3x^2 - 1) = x^{2a+2b+2c+8} - 6x^{2a+2b+2c+6}$$
$$+9x^{2a+2b+2c+4} - 4x^{2a+2c+2} - 4x^{2b+2c+2} - 4x^{2a+2b+2} + x^{2a+4} + x^{2b+4}$$
$$+x^{2c+4} + 2x^{2a+2} + 2x^{2b+2} + 2x^{2c+2} + x^{2a} + x^{2b} + x^{2c} - 10x^2. \tag{3}$$

Similarly, we have matching polynomial of $D(a,b,c)$. From Lemma 2.1 and Lemma 2.3,

$$M_{D(a,b,c)}(\lambda) = (\lambda P_{a-1} - 2P_{a-2})(\lambda P_b P_{c-1} - P_{b-1}P_{c-1} - 2P_b P_{c-2})$$
$$-P_{a-1}(\lambda P_{b-1}P_{c-1} - 2P_{b-1}P_{c-2} - P_{b-2}P_{c-1}).$$

From Lemma 2.6,

$$D(a,b,c)(2) = 2(b+1)c - bc - 2(b+1)(c-1) - a(2bc - 2b(c-1) - (b-1)c)$$
$$= 2 + 2b - 2ab - bc - ac + abc.$$

$$D(a,b,c)(\lambda)x^{a+b+c}(x^6 - 3x^4 + 3x^2 - 1) = x^{2a+2b+2c+6}$$
$$-6x^{2a+2b+2c+4} + 9x^{2a+2b+2c+2} - x^{2b+2c+8} - x^{2a+2b+8} + 2x^{2a+2b+6}$$
$$+x^{2b+6} - 3x^{2a+2b+2} - 3x^{2b+2c+2} - 4x^{2a+2c}$$
$$+4x^{2a+2} + 4x^{2c+2} + 2x^{2b+4} + x^{2b+2} - 2x^4, \tag{4}$$

where x satisfies the equation $x^2 - \lambda x + 1 = 0$.

3.2 The possible matching equivalent graphs

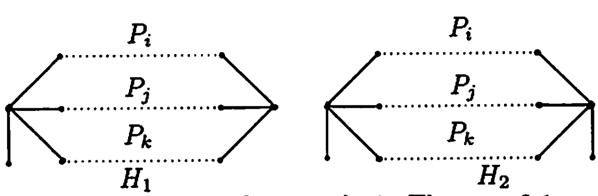

Fig. 2. Two graphs in Theorem 3.1

Theorem 3.1 *If graph H is matching equivalent to a theta graph $\theta(a,b,c)$, then H must be a theta graph or the graph $D(a,b,c)$ (see in Fig.1).*

Proof. For convenience, let $n = a+b+c+2$, and H be matching equivalent to G, with a degree sequence $D(H) = (d_1 + t_1, d_2 + t_2, \ldots, d_n + t_n)$. Then by Lemma 2.4, we have

$$\sum_{i=1}^{n+2} t_i = 0, \tag{5}$$

and

$$\sum_{i=1}^{n+2} (t_i^2 - 2d_i t_i) = 0. \tag{6}$$

From Equation (4), we have

$$\sum_{i=1}^{n+2} t_i^2 - 4\sum_{i=1}^{n} t_i - 6t_{n+1} - 6t_{n+2} = 0. \tag{7}$$

From the Equation (3), we have $\sum_{i=1}^{n} t_i = -t_{n+1} - t_{n+2}$ and substitute it into (7), we have

$$\sum_{i=1}^{n+2} t_i^2 - 2t_{n+1} - 2t_{n+2} = \sum_{i=1}^{n} t_i^2 + t_{n+1}^2 + t_{n+2}^2 - 2t_{n+1} - 2t_{n+2} = 0, \tag{8}$$

set $r = \sum_{i=1}^{n} t_i^2 + (t_{n+1} - 1)^2 \geq 0$ and substitute it into (5), we have

$$t_{n+2}^2 - 2t_{n+2} + r - 1 = 0. \tag{9}$$

By solving this equation, we have $t_{n+2} = 1 \pm \sqrt{2 - r}$, for all t_i are integers, so $r = 1, 2$ and $t_{n+2} = 0, 1, 2$. We discuss in different cases.

case 1: If $t_{n+2} = 0$, then from (6), we have

$$\sum_{i=1}^{n} t_i^2 + t_{n+1}^2 - 2t_{n+1} = 0. \tag{10}$$

case 2: If $t_{n+2} = 1$, form (6), we have

$$\sum_{i=1}^{n} t_i^2 + t_{n+1}^2 - 2t_{n+1} - 1 = 0. \tag{11}$$

case 3: If $t_{n+2} = 2$, then from (6), we have

$$\sum_{i=1}^{n} t_i^2 + t_{n+1}^2 - 2t_{n+1} = 0. \tag{12}$$

By applying the same discussion on the equations (8)-(10), finally we have the possible degree sequences of H are:

$$D_1 = (d_1, \ldots, d_n, 3, 3), d_i = 2, 0 \le i \le n;$$

$$D_2 = (d_1, \ldots, d_n, 5, 3), d_i = 2, 0 \le i \le n;$$

$$D_3 = (d_1, \ldots, d_{i-1}, d_i \pm 1, d_{i+1}, \ldots, d_n, 4, 3), d_i = 2, 0 \le i \le n;$$

$$D_4 = (d_1, \ldots, d_{i-1}, d_i \pm 1, d_{i+1}, \ldots, d_n, 3, 4), d_i = 2, 0 \le i \le n;$$

$$D_5 = (d_1, \ldots, d_{i-1}, d_i \pm 1, d_{i+1}, \ldots, d_n, 5, 4), d_i = 2, 0 \le i \le n;$$

$$D_6 = (d_1, \ldots, d_{i-1}, d_i \pm 1, d_{i+1}, \ldots, d_j \pm 1, \ldots, d_n, 4, 4), d_i = 2, 0 \le i \le n;$$

$$D_7 = (d_1, \ldots, d_{i-1}, d_i \pm 1, d_{i+1}, \ldots, d_n, 4, 5), d_i = 2, 0 \le i \le n;$$

$$D_8 = (d_1, \ldots, d_n, 5, 5), d_i = 2, 0 \le i \le n;$$

$$D_9 = (d_1, \ldots, d_n, 3, 5), d_i = 2, 0 \le i \le n,$$

in above cases we need $\sum\limits_{i=1}^{n+2} t_i = 0$, and the degree sequence of $\theta(a, b, c)$ is $D = (2, 2, \ldots, 2, 3, 3)$, so D_2, D_5, D_7, D_8, D_9 are impossible to be a degree sequence of a comatching graph of $\theta(a, b, c)$. The possible cases are

$$D1 = (2, 2, \ldots, 2, 3, 3); D2 = (2, \ldots, 2, 1, 2, \ldots, 2, 4, 3);$$

$$D3 = (2, 2, \ldots, 2, 1, 2, \ldots, 2, 1, 2, \ldots, 2, 4, 4),$$

these graphs are H_1, H_2 in Fig.2 and $D(a, b, c)$ in Fig.1.

By the Lemma 2.5 the number of edges of G is $a + b + c + 3$, and the number of edges of H is $a' + b' + c' + 3$. If H is a comatching graph of G, then $a + b + c + 3 = a' + b' + c' + 3 = n + 2$.
For graph H_1, we have

$$\binom{a+b+c+3}{2} + \sum_{i=1}^{n+2} \binom{d_i}{2} = \binom{a'+b'+c'+3}{2} + \sum_{i=1}^{n+2} \binom{d'_i}{2},$$

$$n + 6 = n - 1 + 3 + 6.$$

It is a contradiction.
For graph H_2, we have

$$\binom{a+b+c+3}{2} + \sum_{i=1}^{n+2} \binom{d_i}{2} = \binom{a'+b'+c'+3}{2} + \sum_{i=1}^{n+2} \binom{d'_i}{2},$$

$$n + 6 = n - 2 + 6 + 6.$$

It is a contradiction. So the only comatching graphs of $\theta(a, b, c)$ are theta graphs $\theta(a', b', c')$ and $D(a, b, c)$.

3.3 No $\theta(a,b,c)$ are matching equivalent when $a \geq 2$

In this section, we characterize the matching equivalent and matching unique theta graphs.

Theorem 3.2 *If $2 \leq a \leq b \leq c$, then the graph $\theta(a,b,c)$ is matching unique.*

Proof. It is obvious that $\theta(a,b,c)$ and $D(a',b',c')$ are not matching equivalent. Since from each matching polynomial (3) and (4), we find the coefficients of the lowest exponent of x are different.

Now, we prove that two non-isomorphic theta graphs are not matching equivalent when $a \geq 2$.

Suppose that $G = \theta(a,b,c)$ and $H = \theta(a',b',c')$ are matching equivalent. For convenience, let $a \leq b \leq c$ and $a' \leq b' \leq c'$. Since G and H have the same number of vertices, we have

$$a+b+c = a'+b'+c',$$

and from Lemma 2.6, we have:

$$1-a-b-c-ab-ac-bc-abc = 1-a'-b'-c'-a'b'-a'c'-b'c'-a'b'c'.$$

That is:

$$ab+ac+bc+abc = a'b'+a'c'+b'c'+a'b'c'. \tag{13}$$

From (3), we have

$$M_{\theta(a,b,c)}(\lambda)x^{a+b+c}(x^6-3x^4+3x^2-1) = x^{2a+2b+2c+8} - 6x^{2a+2b+2c+6} +9x^{2a+2b+2c+4} - 4x^{2a+2c+2} - 4x^{2b+2c+2} - 4x^{2a+2b+2} + x^{2a+4} + x^{2b+4} +x^{2c+4} + 2x^{2a+2} + 2x^{2b+2} + 2x^{2c+2} + x^{2a} + x^{2b} + x^{2c} - 10x^2, \tag{14}$$

and

$$M_{\theta(a',b',c')}(\lambda)x^{a'+b'+c'}(x^6-3x^4+3x^2-1) = x^{2a'+2b'+2c'+8} -6x^{2a'+2b'+2c'+6} + 9x^{2a'+2b'+2c'+4} - 4x^{2a'+2c'+2} - 4x^{2b'+2c'+2} -4x^{2a'+2b'+2} + x^{2a'+4} + x^{2b'+4} + x^{2c'+4} + 2x^{2a'+2} +2x^{2b'+2} + 2x^{2c'+2} + x^{2a'} + x^{2b'} + x^{2c'} - 10x^2. \tag{15}$$

(14)-(15), we have

$$(M_{\theta(a,b,c)}(\lambda) - M_{\theta(a',b',c')}(\lambda))x^{a+b+c}(x^6-3x^4+3x^2-1) =$$

$$4(x^{2a'+2c'+2}-x^{2a+2c+2})+4(x^{2b'+2c'+2}-x^{2b+2c+2})+4(x^{2a'+2b'+2}-x^{2a+2b+2})$$
$$+(x^{2a+4}-x^{2a'+4})+(x^{2b+4}-x^{ab'+4})+(x^{2c+4}-x^{2c'+4})+2(x^{2a+2}-x^{2a'+2})$$
$$+2(x^{2b+2}-x^{2b'+2})+2(x^{2c+2}-x^{2c'+2})+(x^{2a}-x^{2a'})+(x^{2b}-x^{2b'})+(x^{2c}-x^{2c'}).$$

If $\theta(a,b,c)$ and $\theta(a',b'c')$ are matching equivalent, then the above equation is equal to 0. Hence in the above equation we only need to show $a=a', b=b'$.

The smallest exponent of x in above equation is $2a$ or $2a+2$. We have the following cases:(1)$2a=2a'$, (2)$2a=2a'+2$, (3)$2a+2=2a'$.
case 1: If $2\le a=a'$, then $b+c=b'+c'$. From (13), we have

$$a[(b+c)-(b'+c')]+(a+1)(bc-b'c')=(a+1)(bc-b'c')=0.$$

Hence $bc=b'c'$, and with $b=b'+c'-c$, we have $(b'+c'-c)c=b'c'$, $c(c'-c)=b'(c'-c)$. We have $b'=c$.
If $a'=1$, then the coefficient of lowest exponent of x in (14) is -10 and in the (15) is -9, so $\theta(a,b,c)$ can not comatching with $\theta(a',b',c')$.
case 2: If $2\le a=a'+1$, then $b+c=b'+c'-1$. The second highest exponent term in (15) is $-4x^{2b'+2c'+2}=-4x^{2(b+c)+4}$ while the second highest exponent in (14) is $-4x^{2b+2c+2}$. It is impossible since two matching polynomial should be equal. The last case is the same as Case 2.

3.4 The matching equivalent of $\theta(1,b,c)$

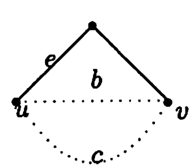

Fig. 3. $\theta(1,b,c)$

Theorem 3.3 *For theta graphs if $b+c=r+s$, then $\theta(1,b,c)$ comatching with $\theta(1,r,s)$, where a or b can be 0 but not both.*

Proof. For $\theta(1,b,c)$ we take an edge e as Fig. 3 show, and by Lemma 2.3

$$M_{\theta(1,b,c)}(x)=M_{D(b+c+2,1)}(x)-M_{P_{b+c+1}}(x),$$

for any $\theta(1,r,s)$ the matching polynomial is

$$M_{\theta(1,r,s)}(x)=M_{D(r+s+2,1)}(x)-M_{P_{r+s+1}}(x).$$

If $b+c=r+s$, then $D(b+c+2,1) \cong D(r+s+2,1)$ and $P_{b+c+1} \cong P_{r+s+1}$. Hence theorem holds.

3.5 The largest matching root of matching polynomial of theta graphs

In this section, we study the largest matching root of theta graphs.

Theorem 3.4 *For a theta graph $\theta(a,b,c), 0 \leq a < b \leq c, n = a+b+c+2$.*

1. *If $a = 0$, then λ is a decreasing function on $|b-c|$;*

2. *If $a = 1$, then the largest of matching root of all theta graphs are equal;*

3. *If $a \geq 2$ Then λ is an increasing function in $|b-c|$.*

Proof. (1) If $a = 0$, then

$$(M_{\theta(0,b-1,c+1)}(\lambda) - M_{\theta(0,b,c)}(\lambda))x^{b+c}(x^2-1) = (x^{2c+2}+x^{2b-2}) - (x^{2c}+x^{2b})$$
$$= x^{2c}(x^2-1) - x^{2b-2}(x^2-1) = (x^2-1)(x^{2c} - x^{2b-2}).$$

Since $x > 2, b \leq c$, then

$$(M_{\theta(0,b-1,c+1)}(\lambda) - M_{\theta(0,b,c)}(\lambda))x^{b+c} = (x^{2c} - x^{2b-2}) > 0.$$

By Lemma 2.8, we have $\lambda(\theta(0,b,c)) > \lambda(\theta(0,b-1,c+1))$.
(2) It is obvious by Theorem 3.3.
(3) If $a \geq 2$, then

$$\begin{aligned}(M_{\theta(a,b,c)}(\lambda) - M_{\theta(a,b-1,c+1)}(\lambda))x^{a+b+c}(x^2-1)^2 &= -4x^{2a+2c+2} \\ -4x^{2a+2b+2} + x^{2b+4} + x^{2c+4} + 2x^{2b+2} + 2x^{2c+2} + x^{2b} + x^{2c} \\ +4x^{2a+2c+4} + 4x^{2a+2b} - x^{2b+2} - x^{2c+6} - 2x^{2b} - 2x^{2c+4} - x^{2b-2} \\ -x^{2c+2} \\ = 4x^{2a+2}(x^{2c} - x^{2b-2}) - (x^{2c} - x^{2b-2})(x^2-1)^2 \\ = (x^{2c} - x^{2b-2})[4x^{2a+2} - (x^2-1)^2]. \qquad (16)\end{aligned}$$

When $x > 2, b \geq 2$ the above Equation (16)

$$(M_{\theta(a,b,c)}(\lambda) - M_{\theta(a,b-1,c+1)}(\lambda)) > 0.$$

Hence by Lemma 2.8

$$\lambda_{max}(\theta(a,b,c)) < \lambda_{max}(\theta(a,b-1,c+1)).$$

Corollary 3.5 *For $\theta(a,b,c)$, when $2 \le a \le b \le c$ and a,b,c are almost equal $\theta(a,b,c)$ has the minimal matching root; when $a = 0$ and b,c are almost equal $\theta(0,b,c)$ has the largest matching root.*

Proof.

$$\begin{aligned}
&(M_{\theta(2,2,n-6)}(\lambda) - M_{\theta(2,1,n-5)}(\lambda))x^{n-2}(x^2-1)^3 = \\
&-4x^{2n-6} - 4x^{2n-6} - 4x^{10} + x^8 + x^8 + x^{2n-8} + 2x^6 + 2x^6 + 2x^{2n-10} \\
&+x^4 + x^4 + x^{2n-12} + 4x^{2n-4} + 4x^{2n-6} + 4x^8 - x^8 - x^6 - x^{2n-6} \\
&-2x^6 - 2x^4 - 2x^{2n-8} - x^4 - x^2 - x^{2n-10} \\
&= 4x^{2n-6}(x^2-1) - 4x^8(x^2-1) + x^6(x^2-1) - x^{2n-8}(x^2-1) \\
&+2x^4(x^2-1) - 2x^{2n-10}(x^2-1) + x^2(x^2-1) - x^{2n-12}(x^2-1) \\
&= (4x^{2n-6} - 4x^8 + x^6 - x^{2n-8} + 2x^4 - 2x^{2n-10} + x^2 - x^{2n-12}) \\
&(x^2-1) \\
&= (x^{2n-12} - x^2)(4x^4 + 3x^2 + 1)(x^2-1)^2. \qquad (17)
\end{aligned}$$

When $n > 6$,

$$\begin{aligned}
&(M_{\theta(2,2,n-6)}(\lambda) - M_{\theta(2,1,n-5)}(\lambda))x^{n-2}(x^2-1) \\
&= (x^{2n-12} - x^2)(4x^4 + 3x^2 + 1) \ge 0.
\end{aligned}$$

Hence from Lemma 2.8, we have:

$$\begin{aligned}
&\lambda(\theta(2,i,n-4-i)) \le \ldots \le \lambda(\theta(2,2,n-6)) \le \lambda(\theta(1,2,n-5)) \\
&= \lambda(\theta(1,i,n-i-3))(i \ge 3).
\end{aligned}$$

$$\begin{aligned}
&(M_{\theta(2,0,n-4)}(\lambda) - M_{\theta(2,1,n-5)}(\lambda))x^{n-2}(x^2-1)^3 = -4x^{2n-2} - 4x^{2n-6} \\
&-4x^6 + x^8 + x^4 + x^{2n-4} + 2x^6 + 2x^2 + 2x^{2n-6} + x^4 + x^0 + x^{2n-8} \\
&+4x^{2n-4} + 4x^{2n-6} + 4x^8 - x^8 - x^6 - x^{2n-6} - 2x^6 - 2x^4 - 2x^{2n-8} \\
&-x^4 - x^2 - x^{2n-10} \\
&= -4x^{2n-4}(x^2-1) + 4x^6(x^2-1) - x^4(x^2-1) + x^{2n-6}(x^2-1) \\
&-2x^2(x^2-1) + 2x^{2n-8}(x^2-1) - (x^2-1) + x^{2n-10}(x^2-1) \\
&= (-4x^{2n-4} + 4x^6 - x^4 + x^{2n-6} - 2x^2 + 2x^{2n-8} - 1 + x^{2n-10}) \\
&(x^2-1) \\
&= (-x^{2n-10} + 1)(4x^4 + 3x^2 + 1)(x^2-1)^2 \qquad (18)
\end{aligned}$$

When $n \geq 5$, we have:

$$(M_{\theta(2,0,n-4)}(\lambda) - M_{\theta(2,1,n-5)}(\lambda))x^{n-2}(x^2-1)$$
$$= (-x^{2n-10}+1)(4x^4+3x^2+1) \leq 0.$$

Hence from Lemma 2.8, we have:

$$\lambda(\theta(1,2,n-5)) \leq \lambda(\theta(0,2,n-4)),$$

so

$$(\lambda(\theta(2,i,n-i-4))(i \geq 3) < \lambda(\theta(2,2,n-4)) < \lambda(\theta(1,2,n-5)) =$$
$$\lambda(\theta(1,i,n-3-i))(i \geq 3) = \theta(0,1,n-3) < \lambda(\theta(0,i,n-2-i))(i \geq 2)$$
$$< \lambda(\theta(0,i,j))(i+j=n-2, |i-j| \leq 1).$$

From above argument, when $2 \leq a \leq b \leq c$ and a, b, c are almost equal $\theta(a,b,c)$ has the minimal matching root; when $a = 0$ and b, c are almost equal, $\theta(0,b,c)$ has the largest matching root.

Corollary 3.6 *The theta graph $\theta(0,b,c), 0 < b \leq c$ are matching unique.*

Proof. By the argument in the subsection 3.1, the matching polynomials of two graphs $M_{\theta(0,b,c)}(x) \neq M_{D(a',b',c')}(x)$. Then the only possible comatching graphs of $\theta(0,b,c)$ are $\theta(a',b',c')$, where $b+c = a'+b'+c'$. By the Theorem 3.4 and Corollary 3.5, any change of the length of the paths P_a, P_b and P_c will change the largest matching root of theta graphs so $\theta(0,b,c)$ are matching unique graphs.

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