A FAMILY OF CHROMATICALLY UNIQUE K_4-HOMEOMORPHS

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Abstract: We discuss the chromaticity of one family of K_4-homeomorphs which has girth 7 and has exactly 1 path of length 1, and give a sufficient and necessary condition for the graphs in the family to be chromatically unique.

keywords: Chromatic polynomial, K_4-homeomorph, Chromatic Uniqueness

1. Introduction

In this paper, we consider graphs which are simple. For such a graph G, let $P(G;\lambda)$ denote the chromatic polynomial of G. Two graphs G and H are chromatically equivalent, denoted by $G \sim H$, if $P(G;\lambda) = P(H;\lambda)$. A graph G is chromatically unique if for any graph H such that $H \sim G$, we have $H \cong G$, i.e., H is isomorphic to G.

A K_4-homeomorph is a subdivision of the complete graph K_4. Such a homeomorph is denoted by $K_4(\alpha,\beta,\gamma,\delta,\varepsilon,\eta)$ if the six edges of K_4 are replaced by the six paths of length $\alpha,\beta,\gamma,\delta,\varepsilon,\eta$, respectively, as shown in Fig.1.

So far, the study of the chromaticity of K_4-homeomorphs with at least 2 paths of length 1 has been fulfiled(see[2],[4],[5],[12]). And the study of the chromaticity of K_4-homeomorphs which have girth 3,4,5 or 6 has been fulfiled. When referring to the chromaticity of K_4-homeomorphs which has girth 7, we showed before that only three types of K_4-homeomorphs, $K_4(1,2,4,\delta,\varepsilon,\eta)$,$K_4(3,2,2,\delta,\varepsilon,\eta)$,$K_4(1,3,3,\delta,\varepsilon,\eta)$ need to be solved. The type of $K_4(1,3,3,\delta,\varepsilon,\eta)$ was already solved([8]). In order to complete the study of the chromaticity of K_4-homeomorphs with girth 7,in this paper, we study another type $K_4(3,2,2,\delta,\varepsilon,\eta)$ (as Fig.2).

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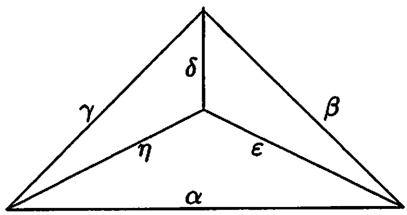

Fig.1 $K_4(\alpha,\beta,\gamma,\delta,\varepsilon,\eta)$

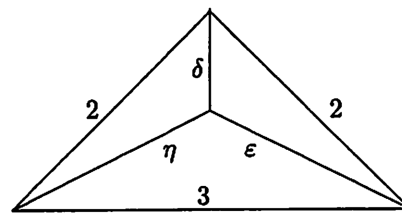

Fig.2 $K_4(3,2,2,\delta,\varepsilon,\eta)$

2. Auxiliary results

In this section we cite some known results used in the sequel.

Proposition 1. Let G and H are chromatically equivalent. Then

(1) $|V(G)| = |V(H)|$, $|E(G)| = |E(H)|$ (see [3]);
(2) G and H have the same girth and same number of cycles with the length equal to their girth(see [11]);
(3) If G is a K_4-homeomorph, then H is a K_4-homeomorph as well(see [1]);
(4) If G and H are homeomorphic to K_4, then both the minimum values of parameters and the number of parameters equal to this minimum value of the graphs G and H coincide (see [10]).

Proposition 2 (Ren[9]). Let $G = K_4(\alpha,\beta,\gamma,\delta,\varepsilon,\eta)$(see Fig.1) when exactly three of $\alpha,\beta,\gamma,\delta,\varepsilon,\eta$ are the same. Then G is not chromatically unique if and only if G is isomorphic to $K_4(s,s,s-2,1,2,s)$ or $K_4(s,s-2,s,2s-2,1,s)$ or $K_4(t,t,1,2t,t+2,t)$ or $K_4(t,t,1,2t,t-1,t)$ or $K_4(t,t+1,t,2t+1,1,t)$ or $K_4(1,t,1,t+1,3,1)$ or $K_4(1,1,t,2,t+2,1)$, where $s \geq 3$, $t \geq 2$.

Proposition 3 (Peng[7]). Let K_4-homeomorphs $K_4(1,3,3,\delta,\varepsilon,\eta)$ and $K_4(3,2,2,\delta',\varepsilon',\eta')$ be chromatically equivalent, then $K_4(1,3,3,\delta,\varepsilon,\eta)$ is isomorphic to $K_4(3,2,2,\delta',\varepsilon',\eta')$.

Proposition 4 (Peng[6]). Let K_4-homeomorphs $K_4(3,2,2,\delta,\varepsilon,\eta)$ and $K_4(3,2,2,\delta',\varepsilon',\eta')$ be chromatically equivalent, then $K_4(3,2,2,\delta,\varepsilon,\eta)$ is isomorphic to $K_4(3,2,2,\delta',\varepsilon',\eta')$.

3. Main results

Lemma. If $G \cong K_4(3,2,2,\delta,\varepsilon,\eta)$ and $H \cong K_4(1,2,4,\delta',\varepsilon',\eta')$, then we have

(1) $P(G) = (-1)^{n+1}[r/(r-1)^2][-r^2 - r^3 - r^4 - r^{n+1} + 2 + Q(G)]$, where

$$\begin{aligned} Q(G) \;=\;& -2r^3 - r^\delta - r^\varepsilon - r^\eta - r^{\delta+1} - r^{\varepsilon+1} - r^{\eta+1} + 3r + r^{\varepsilon+2} + r^{\eta+2} \\ & + r^{\delta+3} + r^{\delta+4} + r^{\varepsilon+5} + r^{\eta+5} + r^{\delta+\varepsilon+\eta} \end{aligned}$$

where $r = 1 - \lambda$, n is the number of vertices of G.
(2) $P(H) = (-1)^{m+1}[r/(r-1)^2][-r^2 - r^3 - r^4 - r^{m+1} + 2 + Q(H)]$, where

$$\begin{aligned} Q(H) \;=\;& -r^5 - r^{\delta'} - r^{\varepsilon'} - r^{\eta'} - r^{\varepsilon'+1} - r^{\eta'+1} + 2r + r^{\eta'+2} + r^{\varepsilon'+3} \\ & + r^{\varepsilon'+4} + r^{\eta'+5} + r^{\delta'+6} + r^{\delta'+\varepsilon'+\eta'} \end{aligned}$$

where $r = 1 - \lambda$, m is the number of vertices of H.
(3) If $P(G) = P(H)$, then $Q(G) = Q(H)$.

Proof(1). Let $r = 1 - \lambda$. From [10], we have the chromatic polynomial of K_4-homeomorph $K_4(\alpha,\beta,\gamma,\delta,\varepsilon,\eta)$ as follows

$$\begin{aligned} P(K_4(\alpha,\beta,\gamma,\delta,\varepsilon,\eta)) \;=\;& (-1)^{n+1}[r/(r-1)^2][(r^2+3r+2) - (r+1) \\ & (r^\alpha + r^\beta + r^\gamma + r^\delta + r^\varepsilon + r^\eta) + (r^{\alpha+\delta} + r^{\beta+\eta} \\ & + r^{\gamma+\varepsilon} + r^{\alpha+\beta+\varepsilon} + r^{\beta+\delta+\gamma} + r^{\alpha+\gamma+\eta} + r^{\delta+\varepsilon+\eta} \\ & - r^{n+1})] \end{aligned}$$

Then

$$\begin{aligned} P(G) \;=\;& P(K_4(3,2,2,\delta,\varepsilon,\eta)) \\ =\;& (-1)^{n+1}[r/(r-1)^2][(r^2+3r+2) - (r+1)(r^3 + r^2 + r^2 + r^\delta \\ & + r^\varepsilon + r^\eta) + (r^{\delta+3} + r^{\eta+2} + r^{\varepsilon+2} + r^{\varepsilon+5} + r^{\delta+4} + r^{\eta+5} \\ & + r^{\delta+\varepsilon+\eta} - r^{n+1})] \\ =\;& (-1)^{n+1}[r/(r-1)^2][-r^{n+1} + 3r + 2 - r^2 - 3r^3 - r^4 - r^\delta - r^\varepsilon \\ & - r^\eta - r^{\delta+1} - r^{\varepsilon+1} - r^{\eta+1} + r^{\varepsilon+2} + r^{\eta+2} + r^{\delta+3} + r^{\delta+4} + r^{\varepsilon+5} \\ & + r^{\eta+5} + r^{\delta+\varepsilon+\eta}] \\ =\;& (-1)^{n+1}[r/(r-1)^2][-r^2 - r^3 - r^4 - r^{n+1} + 2 + Q(G)] \end{aligned}$$

where

$$Q(G) = -2r^3 - r^\delta - r^\varepsilon - r^\eta - r^{\delta+1} - r^{\varepsilon+1} - r^{\eta+1} + 3r + r^{\varepsilon+2} + r^{\eta+2} + r^{\delta+3} + r^{\delta+4} + r^{\varepsilon+5} + r^{\eta+5} + r^{\delta+\varepsilon+\eta}$$

Proof(2). We can handle this case in the same fashion as case(1), and get the result(2).

Proof(3). If $P(G) = P(H)$, then it is easy to see that $Q(G) = Q(H)$.

Theorem. K_4-homeomorphs $K_4(3,2,2,\delta,\varepsilon,\eta)$(see Fig.2) which has exactly 1 path of length 1 and has girth 7 is not chromatically unique if and only if it is $K_4(3,2,2,a,1,a+3)$, $K_4(3,2,2,b,1,5)$, where $a > 3$, $b > 3$ and

$$K_4(3,2,2,a,1,a+3) \sim K_4(1,2,4,a+2,2,a)$$

$$K_4(3,2,2,b,1,5) \sim K_4(1,2,4,4,b,2)$$

Proof. Let $G \cong K_4(3,2,2,\delta,\varepsilon,\eta)$. If there is a graph H such that $P(H) = P(G)$, then from Proposition 1, we know that H is a K_4- homeomorph $K_4(\alpha',\beta',\gamma',\delta',\varepsilon',\eta')$ which has exactly 1 path of length 1, and the girth of H is 7. So H must be one of the following four types:
Type 1:

$$K_4(1,2,\gamma',2,\varepsilon',2)(\varepsilon' \geq 4, \gamma' \geq 4)$$

Type 2:

$$K_4(3,2,2,\delta',\varepsilon',\eta')(\delta'+\varepsilon' \geq 5, \varepsilon'+\eta' \geq 4, \delta'+\eta' \geq 5)$$

Type 3:

$$K_4(1,3,3,\delta',\varepsilon',\eta')(\delta'+\varepsilon' \geq 4, \varepsilon'+\eta' \geq 6, \delta'+\eta' \geq 4)$$

Type 4:

$$K_4(1,2,4,\delta',\varepsilon',\eta')(\delta'+\varepsilon' \geq 5, \varepsilon'+\eta' \geq 6, \delta'+\eta' \geq 3)$$

We now solve the equation $P(G) = P(H)$ to get all solutions.

If H has Type 1 , then from Proposition 2, we know that H is chromatically unique. Since $G \sim H$, we have $G \cong H$. But it is obvious that G is not isomorphic to H. This is a contradiction.

If H has Type 2, then from Proposition 4, we know G is isomorphic to H.

If H has Type 3, then from Proposition 3, we know G is isomorphic to H.

Suppose that H has Type 4, We solve the equation $Q(G) = Q(H)$. From Lemma, we have

$$\begin{aligned} Q(G) &= -2r^3 - r^{\delta} - r^{\varepsilon} - r^{\eta} - r^{\delta+1} - r^{\varepsilon+1} - r^{\eta+1} + 3r + r^{\varepsilon+2} + r^{\eta+2} \\ &\quad + r^{\delta+3} + r^{\delta+4} + r^{\varepsilon+5} + r^{\eta+5} + r^{\delta+\varepsilon+\eta} \\ Q(H) &= -r^5 - r^{\delta'} - r^{\varepsilon'} - r^{\eta'} - r^{\varepsilon'+1} - r^{\eta'+1} + 2r + r^{\eta'+2} + r^{\varepsilon'+3} \\ &\quad + r^{\varepsilon'+4} + r^{\eta'+5} + r^{\delta'+6} + r^{\delta'+\varepsilon'+\eta'} \end{aligned}$$

We denote the lowest remaining power by l.r.p. and the highest remaining power by h.r.p.. We can assume $\varepsilon \le \eta$ and $min\{\delta', \varepsilon', \eta'\} \ge 2$. From Proposition 1, we have

$$\delta + \varepsilon + \eta = \delta' + \varepsilon' + \eta' \qquad (1)$$

Since $K_4(3, 2, 2, \delta, \varepsilon, \eta)$ has exactly 1 path of length 1 and $\varepsilon \le \eta$, we have $\delta = 1$ or $\varepsilon = 1$. There are two cases to be considered.

Case 1 If $\delta = 1$, then $\varepsilon > 1$ and $\eta > 1$. We obtain the following after simplification:

$$Q(G): -r^2 - 2r^3 - r^{\varepsilon} - r^{\eta} - r^{\varepsilon+1} - r^{\eta+1} + r^4 + r^5 + r^{\varepsilon+2} + r^{\eta+2} + r^{\varepsilon+5} + r^{\eta+5}$$

$$Q(H): -r^5 - r^{\delta'} - r^{\varepsilon'} - r^{\eta'} - r^{\varepsilon'+1} - r^{\eta'+1} + r^{\eta'+2} + r^{\varepsilon'+3} + r^{\varepsilon'+4} + r^{\eta'+5} + r^{\delta'+6}$$

Comparing the l.r.p. in Q(G) and the l.r.p. in Q(H), we have $\delta' = 2$ or $\varepsilon' = 2$ or $\eta' = 2$.

Case 1.1 If $\delta' = 2$,then we obtain the following after simplification:

$$Q(G): -2r^3 - r^{\varepsilon} - r^{\eta} - r^{\varepsilon+1} - r^{\eta+1} + r^4 + r^5 + r^{\varepsilon+2} + r^{\eta+2} + r^{\varepsilon+5} + r^{\eta+5}$$

$$Q(H): -r^5 - r^{\varepsilon'} - r^{\eta'} - r^{\varepsilon'+1} - r^{\eta'+1} + r^{\eta'+2} + r^{\varepsilon'+3} + r^{\varepsilon'+4} + r^{\eta'+5} + r^8$$

Since $-2r^3$ can't be cancelled by the terms in $Q(G)$, there are two terms in $Q(H)$ which are equal to $-r^3$. So we have $\varepsilon' = \eta' + 1 = 3$ or $\varepsilon' = \eta' = 3$ or $\varepsilon' + 1 = \eta' = 3$ or $\varepsilon' + 1 = \eta' + 1 = 3$.

If $\varepsilon' = \eta' + 1 = 3$, then we obtain the following after simplification:

$$\begin{aligned} Q(G) &: -2r^3 - r^{\varepsilon} - r^{\eta} - r^{\varepsilon+1} - r^{\eta+1} + r^4 + r^5 + r^{\varepsilon+2} + r^{\eta+2} + r^{\varepsilon+5} \\ &\quad + r^{\eta+5} \\ Q(H) &: -r^2 - 2r^3 - r^5 + r^6 + 2r^7 + r^8 \end{aligned}$$

Comparing the l.r.p. in Q(G) and the l.r.p. in Q(H), we have $\varepsilon = 2$. From $\delta + \varepsilon + \eta = \delta' + \varepsilon' + \eta'$(from equation(1)), and $\delta = 1$ and $\delta' = 2$, we have $\eta = 4$. So $Q(G) \ne Q(H)$, a contradiction.

If $\varepsilon' = \eta' = 3$ or $\varepsilon' + 1 = \eta' = 3$ or $\varepsilon' + 1 = \eta' + 1 = 3$, we can handle these cases in the same fashion as above and get $Q(G) \ne Q(H)$, a contradiction.

Case 1.2 If $\varepsilon' = 2$, From $\delta + \varepsilon + \eta = \delta' + \varepsilon' + \eta'$, and $\delta = 1$, we have

$$\varepsilon + \eta = \delta' + \eta' + 1 \tag{2}$$

After simplifying, we have

$$\begin{aligned} Q(G) &: -r^3 - r^\varepsilon - r^\eta - r^{\varepsilon+1} - r^{\eta+1} + r^4 + r^{\varepsilon+2} + r^{\eta+2} + r^{\varepsilon+5} + r^{\eta+5} \\ Q(H) &: -r^5 - r^{\delta'} - r^{\eta'} - r^{\eta'+1} + r^6 + r^{\eta'+2} + r^{\eta'+5} + r^{\delta'+6} \end{aligned}$$

comparing the h.r.p. in Q(G) and the h.r.p. in Q(H), we have $\eta + 5 = \eta' + 5$ or $\eta + 5 = \delta' + 6$.

If $\eta + 5 = \eta' + 5$, from equation(2), we have $\varepsilon = \delta' + 1$. We obtain the following after simplification:

$$\begin{aligned} Q(G) &: -r^3 - r^\varepsilon - r^{\varepsilon+1} + r^4 + r^{\varepsilon+2} \\ Q(H) &: -r^5 - r^{\delta'} + r^6 \end{aligned}$$

It is easy to see that $\delta' = 3$ and $\varepsilon = 4$. So G is isomorphic to H.

If $\eta + 5 = \delta' + 6$, from equation(2), we have $\varepsilon = \eta'$. We obtain the following after simplification:

$$\begin{aligned} Q(G) &: -r^3 - r^\eta - r^{\eta+1} + r^4 + r^{\eta+2} \\ Q(H) &: -r^5 - r^{\delta'} + r^6 \end{aligned}$$

It is easy to see that $\delta' = 3$ and $\eta = 4$. We get G is isomorphic to H.

Case 1.3 If $\eta' = 2$, From $\delta + \varepsilon + \eta = \delta' + \varepsilon' + \eta'$, and $\delta = 1$, we have

$$\varepsilon + \eta = \delta' + \varepsilon' + 1 \tag{3}$$

After simplifying, we have

$$\begin{aligned} Q(G) &: -r^3 - r^\varepsilon - r^\eta - r^{\varepsilon+1} - r^{\eta+1} + r^5 + r^{\varepsilon+2} + r^{\eta+2} + r^{\varepsilon+5} + r^{\eta+5} \\ Q(H) &: -r^5 - r^{\delta'} - r^{\varepsilon'} - r^{\varepsilon'+1} + r^7 + r^{\varepsilon'+3} + r^{\varepsilon'+4} + r^{\delta'+6} \end{aligned}$$

comparing the h.r.p. in Q(G) and the h.r.p. in Q(H), we have $\eta + 5 = \varepsilon' + 4$ or $\eta + 5 = \delta' + 6$, where $\delta' \geq 2$, $\varepsilon' \geq 2$.

If $\eta + 5 = \varepsilon' + 4$, then $\varepsilon' = \eta + 1$. From equation(3), we have $\varepsilon = \delta' + 2$. We obtain the following after simplification:

$$\begin{aligned} Q(G) &: -r^3 - r^\varepsilon - r^\eta - r^{\varepsilon+1} + r^5 + r^{\varepsilon+2} + r^{\eta+2} + r^{\varepsilon+5} \\ Q(H) &: -r^5 - r^{\delta'} - r^{\varepsilon'+1} + r^{\varepsilon'+3} + r^7 + r^{\delta'+6} \end{aligned}$$

Consider $r^{\eta+2}$ in $Q(G)$ and $-r^{\varepsilon'+1}$ in $Q(H)$. It is due to $\varepsilon \leq \eta$ and $\eta > 1$ that $r^{\eta+2}$ can cancel none of the negative terms in $Q(G)$. Thus, no term in $Q(G)$ is equal to $-r^{\varepsilon'+1}$(noting $\varepsilon' = \eta + 1$)in Q(H). Therefore, $-r^{\varepsilon'+1}$ must be cancelled by the positive term in Q(H) and $r^{\eta+2}$ must equal the positive term in Q(H). So, $\eta+2=\varepsilon'+1=7=\delta'+6$. Thus $\delta'=1$ which contradicts $\delta'\geq 2$.

If $\eta+5=\delta'+6$, then $\eta=\delta'+1$. From equation(3), we have $\varepsilon=\varepsilon'$. We obtain the following after simplification:

$$\begin{aligned} Q(G) &: -r^3-r^\eta-r^{\eta+1}+r^{\varepsilon+2}+r^{\eta+2}+r^5+r^{\varepsilon+5} \\ Q(H) &: -r^5-r^{\delta'}+r^7+r^{\varepsilon'+3}+r^{\varepsilon'+4} \end{aligned}$$

It is easy to see that $\delta'=3$ and $\eta=4$. So, $\eta=4=\varepsilon+2$ and $\varepsilon=\varepsilon'=2$. We get G is isomorphic to H.

Case 2 If $\varepsilon=1$, then we obtain the following after simplification:

$$\begin{aligned} Q(G) &= -r^2-r^3-r^\delta-r^\eta-r^{\delta+1}-r^{\eta+1}+r^6+r^{\eta+2}+r^{\delta+3}+r^{\delta+4} \\ &\quad +r^{\eta+5} \\ Q(H) &= -r^5-r^{\delta'}-r^{\varepsilon'}-r^{\eta'}-r^{\varepsilon'+1}-r^{\eta'+1}+r^{\eta'+2}+r^{\varepsilon'+3}+r^{\varepsilon'+4} \\ &\quad +r^{\eta'+5}+r^{\delta'+6} \end{aligned}$$

Comparing the l.r.p. in Q(G) and the l.r.p. in Q(H), we have $\varepsilon'=2$ or $\delta'=2$ or $\eta'=2$. There are three cases to be considered.

Case 2.1 If $\varepsilon'=2$, From $\delta+\varepsilon+\eta=\delta'+\varepsilon'+\eta'$, and $\varepsilon=1$, we have

$$\delta+\eta=\delta'+\eta'+1 \tag{4}$$

After simplifying, we have

$$\begin{aligned} Q(G) &= -r^\delta-r^\eta-r^{\delta+1}-r^{\eta+1}+r^{\eta+2}+r^{\delta+3}+r^{\delta+4}+r^{\eta+5} \\ Q(H) &= -r^{\delta'}-r^{\eta'}-r^{\eta'+1}+r^{\eta'+2}+r^{\eta'+5}+r^{\delta'+6} \end{aligned}$$

Comparing the l.r.p. in Q(G) and the l.r.p. in Q(H), we have $min\{\delta,\eta\}=min\{\delta',\eta'\}$.

If $\delta=\delta'$, then from equation(4), we have $\eta=\eta'+1$. After simplifying, we have

$$\begin{aligned} Q(G) &= -r^{\delta+1}-r^{\eta+1}+r^{\eta+2}+r^{\delta+3}+r^{\delta+4}+r^{\eta+5} \\ Q(H) &= -r^{\eta'}+r^{\eta'+2}+r^{\eta'+5}+r^{\delta'+6} \end{aligned}$$

Consider $-r^{\eta+1}$ in $Q(G)$ and $r^{\eta'+2}$ in $Q(H)$. It is due to $\eta=\eta'+1$ that no term in $Q(H)$ is equal to $-r^{\eta+1}$. So, $-r^{\eta+1}$ must be cancelled by the term in Q(G) and $r^{\eta'+2}$ must equal one of the terms in Q(G). Therefore, there are two terms in Q(G) which are equal $r^{\eta'+2}$(noting $\eta=\eta'+1$), a contradiction.

If $\delta=\eta'$, then from equation(4), we have $\eta=\delta'+1$. After simplifying, we have

$$\begin{aligned} Q(G) &= -r^{\eta}-r^{\eta+1}+r^{\eta+2}+r^{\delta+3}+r^{\delta+4} \\ Q(H) &= -r^{\delta'}+r^{\eta'+2}+r^{\eta'+5} \end{aligned}$$

It is easy to see that $\eta=\delta+3$. Since $\delta=\eta'$, we have $\eta=\eta'+3$. So, $\delta'=\delta+2$ and $\delta'=\eta'+2$. After simplifying, we have $Q(G)=Q(H)$. Let $\delta=a(a>3)$. We obtain the solution where G is isomorphic to $K_4(3,2,2,a,1,a+3)$ and H is isomorphic to $K_4(1,2,4,a+2,2,a)$. That is

$$K_4(3,2,2,a,1,a+3)\sim K_4(1,2,4,a+2,2,a)$$

If $\eta=\delta'$, then from equation(4), we have $\delta=\eta'+1$. After simplifying, we have

$$\begin{aligned} Q(G) &= -r^{\delta+1}-r^{\eta+1}+r^{\eta+2}+r^{\delta+3}+r^{\eta+5} \\ Q(H) &= -r^{\eta'}+r^{\eta'+2}+r^{\delta'+6} \end{aligned}$$

Consider $-r^{\delta+1}$ in $Q(G)$ and $r^{\eta'+2}$ in $Q(H)$. It is due to $\delta=\eta'+1$ that no term in $Q(H)$ is equal to $-r^{\delta+1}$. So, $-r^{\delta+1}$ must be cancelled by the term in Q(G) and $r^{\eta'+2}$ must equal one of the terms in Q(G). Therefore, there are two terms in Q(G) which are equal $r^{\eta'+2}$(noting $\delta=\eta'+1$), a contradiction.

If $-r^{\eta}=-r^{\eta'}$, then from equation(4), we have $\delta=\delta'+1$. After simplifying, we have

$$\begin{aligned} Q(G) &= -r^{\delta}-r^{\delta+1}+r^{\delta+3}+r^{\delta+4} \\ Q(H) &= -r^{\delta'}+r^{\delta'+6} \end{aligned}$$

It is easy to say that $Q(G)\neq Q(H)$, this is a contradiction.

Case 2.2 If $\eta'=2$, From $\delta+\varepsilon+\eta=\delta'+\varepsilon'+\eta'$, and $\varepsilon=1$, we have

$$\delta+\eta=\delta'+\varepsilon'+1 \tag{5}$$

After simplifying, we have

$$\begin{aligned} Q(G) &: -r^{\delta}-r^{\eta}-r^{\delta+1}-r^{\eta+1}+r^{6}+r^{\eta+2}+r^{\delta+3}+r^{\delta+4}+r^{\eta+5} \\ Q(H) &: -r^{5}-r^{\delta'}-r^{\varepsilon'}-r^{\varepsilon'+1}+r^{4}+r^{7}+r^{\varepsilon'+3}+r^{\varepsilon'+4}+r^{\delta'+6} \end{aligned}$$

comparing the h.r.p. in Q(G) and the h.r.p. in Q(H), we have $\delta+4=\varepsilon'+4$ or $\eta+5=\delta'+6$ or $\delta+4=\delta'+6$ or $\eta+5=\varepsilon'+4$. There are four cases to be considered.

Case 2.2.1 If $\delta+4=\varepsilon'+4$, from equation(5), we have $\eta=\delta'+1$. We obtain the following after simplification:

$$\begin{aligned} Q(G) &: -r^{\eta}-r^{\eta+1}+r^{6}+r^{\eta+2} \\ Q(H) &: -r^{5}-r^{\delta'}+r^{4}+r^{7} \end{aligned}$$

It is easy to see that $\eta=5$. So, $\delta'=4$. After simplifying, we have $Q(G)=Q(H)$. Let $\delta=b(b>3)$. We obtain the solution where G is isomorphic to $K_4(3,2,2,b,1,5)$ and H is isomorphic to $K_4(1,2,4,4,b,2)$. That is

$$K_4(3,2,2,b,1,5)\sim K_4(1,2,4,4,b,2)$$

Case 2.2.2 If $\eta+5=\delta'+6$, then $\eta=\delta'+1$. From equation(5), we have $\delta=\varepsilon'$. Thus, this case can be handled in the same fashion as case 2.2.1 and we get the same result as above.

Case 2.2.3 If $\delta+4=\delta'+6$, then $\delta=\delta'+2$. From equation(5), we have $\varepsilon'=\eta+1$. We obtain the following after simplification:

$$\begin{aligned} Q(G) &: -r^{\eta}-r^{\delta}-r^{\delta+1}+r^{6}+r^{\eta+2}+r^{\delta+3} \\ Q(H) &: -r^{5}-r^{\delta'}-r^{\varepsilon'+1}+r^{4}+r^{7}+r^{\varepsilon'+3} \end{aligned}$$

Consider $r^{\eta+2}$ in $Q(G)$ and $-r^{\varepsilon'+1}$ in $Q(H)$. It is due to $\varepsilon'=\eta+1$ that $-r^{\varepsilon'+1}$ must be equal one of $-r^{\delta}$ and $-r^{\delta+1}$. So, $r^{\eta+2}$ must be equal one of r^4 and r^7. Thus, we have $\delta=\varepsilon'+1=\eta+2=4$ or $\delta=\varepsilon'+1=\eta+2=7$ or $\delta+1=\varepsilon'+1=\eta+2=4$ or $\delta+1=\varepsilon'+1=\eta+2=7$.

If $\delta=\varepsilon'+1=\eta+2=4$, then G is isomorphic to H.

If $\delta=\varepsilon'+1=\eta+2=7$, then $Q(G)\neq Q(H)$. This a contradiction.

If $\delta+1=\varepsilon'+1=\eta+2=4$, then $\delta=3$. From $\delta=\delta'+2$, we have $\delta'=1$ which contradict $\delta'\geq 2$.

If $\delta+1=\varepsilon'+1=\eta+2=7$, after simplifying, we have $Q(G)=Q(H)$. So, we obtain the solution where G is isomorphic to $K_4(3,2,2,6,1,5)$ and H is isomorphic to $K_4(1,2,4,4,6,2)$. That is

$$K_4(3,2,2,6,1,5)\sim K_4(1,2,4,4,6,2)$$

Case 2.2.4 If $\eta+5=\varepsilon'+4$, then $\eta+1=\varepsilon'$. From equation(5), we have $\delta=\delta'+2$. Thus, this case can be handled in the same fashion as case 2.2.3 and we get the same result as above.

Case 2.3 If $\delta'=2$, From $\delta+\varepsilon+\eta=\delta'+\varepsilon'+\eta'$, and $\varepsilon=1$, we have

$$\delta+\eta=\varepsilon'+\eta'+1 \tag{6}$$

After simplifying, we have

$$\begin{aligned} Q(G) &= -r^3 - r^\delta - r^\eta - r^{\delta+1} - r^{\eta+1} + r^6 + r^{\eta+2} + r^{\delta+3} + r^{\delta+4} + r^{\eta+5} \\ Q(H) &= -r^5 - r^{\varepsilon'} - r^{\eta'} - r^{\varepsilon'+1} - r^{\eta'+1} + r^{\eta'+2} + r^{\varepsilon'+3} + r^{\varepsilon'+4} + r^{\eta'+5} \\ &\quad + r^8 \end{aligned}$$

comparing the h.r.p. in Q(G) and the h.r.p. in Q(H), we have $\delta+4 = \varepsilon'+4$ or $\delta+4 = \eta'+5$ or $\delta+4 = 8$ or $\eta+5 = \varepsilon'+4$ or $\eta+5 = \eta'+5$ or $\eta+5 = 8$. There are six cases to be considered.

Case 2.3.1 If $\delta + 4 = \varepsilon' + 4$, from equation(6), we have $\eta = \eta' + 1$. We obtain the following after simplification:

$$\begin{aligned} Q(G) &= -r^3 - r^{\eta+1} + r^6 + r^{\eta+2} + r^{\eta+5} \\ Q(H) &= -r^5 - r^{\eta'} + r^{\eta'+2} + r^{\eta'+5} + r^8 \end{aligned}$$

It is due to $\eta > 1$ that no term in Q(G) can cancel $-r^3$ in Q(G). So, $r^3 = -r^{\eta'}$. We get $\eta' = 3$ and $\eta = 4$. Thus, $Q(G) \neq Q(H)$. This a contradiction.

Case 2.3.2 If $\delta + 4 = \eta' + 5$, then $\delta = \eta' + 1$. From equation(6), we have $\eta = \varepsilon'$. We obtain the following after simplification:

$$\begin{aligned} Q(G) &= -r^3 - r^{\delta+1} + r^6 + r^{\eta+2} + r^{\delta+3} + r^{\eta+5} \\ Q(H) &= -r^5 - r^{\eta'} + r^{\eta'+2} + r^{\varepsilon'+3} + r^{\varepsilon'+4} + r^8 \end{aligned}$$

It is due to $\eta > 1$ and $\delta > 1$ that no term in Q(G) can cancel $-r^3$ in Q(G). So, $r^3 = -r^{\eta'}$. We get $\eta' = 3$ and $\delta = 4$. After simplifying Q(G)=Q(H) and comparing the degrees of both the sides of Q(G)=Q(H), we have $\eta = 3$ and $\varepsilon' = 3$. Thus, we get G is isomorphic to H.

Case 2.3.3 If $\delta + 4 = 8$, then $\delta = 4$. From equation(6), we have $\eta + 3 = \varepsilon' + \eta'$. We obtain the following after simplification:

$$\begin{aligned} Q(G) &= -r^3 - r^4 - r^\eta - r^{\eta+1} + r^6 + r^7 + r^{\eta+2} + r^{\eta+5} \\ Q(H) &= -r^{\varepsilon'} - r^{\eta'} - r^{\varepsilon'+1} - r^{\eta'+1} + r^{\eta'+2} + r^{\varepsilon'+3} + r^{\varepsilon'+4} + r^{\eta'+5} \end{aligned}$$

By considering the l.r.p. in Q(G) and the l.r.p. in Q(H), we have $\varepsilon' = 3$ or $\eta' = 3$ or $\eta = \varepsilon'$ or $\eta = \eta'$. Noting $\eta + 3 = \varepsilon' + \eta'$ and comparing the degrees of both the sides of $Q(G) = Q(H)$, we get $Q(G) = Q(H)$, and G is isomorphic to H.

Case 2.3.4 If $\eta + 5 = \varepsilon' + 4$, then $\varepsilon' = \eta + 1$. From equation(6), we have $\delta = \eta' + 2$. We obtain the following after simplification:

$$\begin{aligned} Q(G) &= -r^3 - r^\delta - r^\eta - r^{\delta+1} + r^6 + r^{\eta+2} + r^{\delta+4} \\ Q(H) &= -r^5 - r^{\eta'} - r^{\varepsilon'+1} - r^{\eta'+1} + r^{\eta'+2} + r^{\varepsilon'+3} + r^8 \end{aligned}$$

By considering $-r^3$ in Q(G), we have $-r^3 = -r^{\eta'}$ or $-r^3 = -r^{\varepsilon'+1}$ or $-r^3 = -r^{\eta'+1}$.

If $-r^3 = -r^{\eta'}$, then $\eta' = 3$ and $\delta = 5$. After simplifying Q(G)=Q(H) and comparing the degrees of both the sides of Q(G)=Q(H), we have $\eta = 4$ and $\varepsilon' = 5$. Thus, $Q(G) \neq Q(H)$, a contradiction.

If $-r^3 = -r^{\varepsilon'+1}$, then $\varepsilon' = 2$. By $\varepsilon' = \eta + 1$, we have $\eta = 1$ which contradicts $\eta > 1$.

If $-r^3 = -r^{\eta'+1}$, then $\eta' = 2$ and $\delta = 4$. After simplifying Q(G)=Q(H) and comparing the degrees of both the sides of Q(G)=Q(H), we have $\eta = 2$ and $\varepsilon' = 3$. Thus, we get $Q(G) = Q(H)$, and G is isomorphic to H.

Case 2.3.5 If $\eta + 5 = \eta' + 5$, then $\eta = \eta'$. From equation(6), we have $\delta = \varepsilon' + 1$. We obtain the following after simplification:

$$\begin{aligned} Q(G) &= -r^3 - r^{\delta+1} + r^6 + r^{\delta+4} \\ Q(H) &= -r^5 - r^{\varepsilon'} + r^{\varepsilon'+3} + r^8 \end{aligned}$$

it is easy to see that $\varepsilon' = 3$ and $\delta = 4$. Thus, we get G is isomorphic to H.

Case 2.3.6 If $\eta + 5 = 8$, then $\eta = 3$. From equation(6), we have $\varepsilon' + \eta' = \delta + 2$. We obtain the following after simplification:

$$\begin{aligned} Q(G) &= -2r^3 - r^4 - r^{\delta} - r^{\delta+1} + r^5 + r^6 + r^{\delta+3} + r^{\delta+4} \\ Q(H) &= -r^5 - r^{\varepsilon'} - r^{\eta'} - r^{\varepsilon'+1} - r^{\eta'+1} + r^{\eta'+2} + r^{\varepsilon'+3} + r^{\varepsilon'+4} + r^{\eta'+5} \end{aligned}$$

Consider $-2r^3$ and $-r^4$ in $Q(G)$. We have $\varepsilon' = \eta' = 3$ or $\varepsilon' = \eta' + 1 = 3$ or $\varepsilon' + 1 = \eta' = 3$.

If $\varepsilon' = \eta' = 3$, from $\varepsilon' + \eta' = \delta + 2$, we have $\delta = 4$. Thus, G is isomorphic to H.

If $\varepsilon' = \eta' + 1 = 3$, from $\varepsilon' + \eta' = \delta + 2$, we have $\delta = 3$. Thus, $Q(G) \neq Q(H)$, this is a contradiction.

If $\varepsilon' + 1 = \eta' = 3$, from $\varepsilon' + \eta' = \delta + 2$, we have $\delta = 3$. Thus, $Q(G) \neq Q(H)$, this is a contradiction.

So far, we have solved the equation $P(G) = P(H)$ and got the solution as follows:

$$K_4(3,2,2,a,1,a+3) \sim K_4(1,2,4,a+2,2,a)$$

$$K_4(3,2,2,b,1,5) \sim K_4(1,2,4,4,b,2)$$

where $a > 3$, $b > 3$.
The proof is completed. □

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