Prime Cordial labeling of Flower Snark and related graphs *

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Abstract

A graph with vertex set V is said to have a prime cordial labeling if there is a bijection f from V to $\{1, 2, \ldots, |V|\}$ such that if each edge uv is assigned the label 1 for the greatest common divisor $gcd(f(u), f(v)) = 1$ and 0 for $gcd(f(u), f(v)) > 1$ then the number of edges labeled with 0 and the number of edges labeled with 1 differ by at most 1. In this paper, we show that the Flower Snark and its related graphs are prime cordial for all $n \geq 3$.

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1 Introduction

We consider only finite undirected graphs without loops or multiple edges. Let $G = (V, E)$ be a graph with vertex set V and edge set E.

A graph with vertex set V is said to have a prime labeling if its vertices are labelled with distinct integers from $\{1, 2, \ldots, |V|\}$ such that for every edge uv, the labels assigned to u and v are relatively prime or coprime [2–5].

A graph with vertex set V is said to have a prime cordial labeling if there is a bijection f from V to $\{1, 2, \ldots, |V|\}$ such that if each edge uv is assigned the label 1 for $gcd(f(u), f(v)) = 1$ and 0 for $gcd(f(u), f(v)) > 1$ then the number of edges labeled with 0 and the number of edges labeled with 1 differ by at most 1. A graph is called prime cordial if it has a prime cordial labeling. This concept was introduced in [1] by M. Sumndarm, R. Ponraj, and S. Somasundram. They proved that the following graphs are prime cordial: C_n if and only if $n \geq 6$; P_n if and only if $n \neq 3$ or 5; $K_{1,n}$ (n odd); bistars; dragons; crowns; triangular snakes T_n if and only if $n \geq 3$; ladders. We refer the readers to the dynamic survey by Gallian [6].

Let G_n be a simple nontrival connected cubic graph with vertex set $V(G_n) = \{a_i, b_i, c_i, d_i : 0 \leq i \leq n-1\}$ and edge set $E(G_n) = \{a_i a_{i+1}, b_i b_{i+1}, c_i c_{i+1}, d_i a_i, d_i b_i, d_i c_i : 0 \leq i \leq n-1\}$, where the vertex labels are read modulo n. Let H_n be a graph obtained from G_n by replacing the edges $b_{n-1}b_0$ and $c_{n-1}c_0$ with $b_{n-1}c_0$ and $c_{n-1}b_0$

respectively. For odd $n \geq 5$, H_n is called a Snark, namely Flower Snark. While the other graphs, i.e. all G_n, H_3 and all H_n with even $n \geq 4$, are called the related graphs of Flower Snark. In this paper we denote the vertices a_i, b_i, c_i and d_i as v_{4i}, v_{4i+1}, v_{4i+2} and v_{4i+3} for $0 \leq i \leq n-1$(the vertex labels are read modulo n). (see the Figure 1.1).

In this paper, we show that Flower Snark and related graph are prime cordial for all $n \geq 3$.

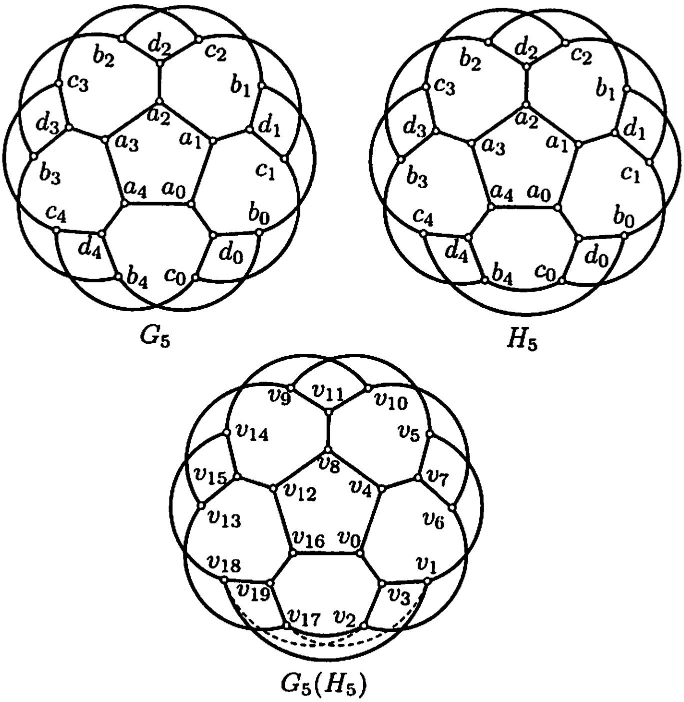

Figure 1.1. The Flower Snark and its related graphs.

2 Prime cordial labeling of Flower Snark graph

Observation 2.1. Let x, y be two positive integers, and let $|x-y| = p_1^{\alpha_1} p_2^{\alpha_2} \dots p_t^{\alpha_t}$, where p_i are distinct prime factors and α_i are their orders. Then they are coprime if $x \not\equiv 0 \bmod p_i$ for $1 \le i \le t$.

Theorem 2.1. $G_n(H_n)$ is prime cordial for all $n \ge 3$.

Proof. For $0 \le i \le 4n-1$, we define the function f as follows:

$$f(v_i) = \begin{cases} i+1, & i \equiv 0,1,8 \pmod{12}; \\ i+2, & i \equiv 2,3,10 \pmod{12}; \\ i+3, & i \equiv 4,5 \pmod{12}; \\ i+4, & i \equiv 6,7 \pmod{12}; \\ i-3, & i \equiv 9 \pmod{12}; \\ i-8, & i \equiv 11 \pmod{12}. \end{cases}$$

Case 1. $n \equiv 0 \pmod 3$. Let $f_0(v_i) = f(v_i)$ for $0 \le i \le 4n-1$. One can easily verify that f_0 is a bijection from $V(G_n(H_n))$ to $\{1, 2, \dots, 4n\}$. Table 2.1 shows $f_0(G_n(H_n))$ for $n \equiv 0 \pmod 3$.

Table 2.1. $f_0(G_n(H_n))$ for $n \equiv 0 \mod 3$

i	0	1	2	3	4	5	6	7	8	9	10	11
$f(v_i)$	1	2	4	5	7	8	10	11	9	6	12	3
i	12	13	14	15	16	17	18	19	20	21	22	23
$f(v_i)$	13	14	16	17	19	20	22	23	21	18	24	15
						...						
i	$4n-12$	$4n-11$	$4n-10$	$4n-9$	$4n-8$	$4n-7$	$4n-6$	$4n-5$	$4n-4$	$4n-3$	$4n-2$	$4n-1$
$f(v_i)$	$4n-11$	$4n-10$	$4n-8$	$4n-7$	$4n-5$	$4n-4$	$4n-2$	$4n-1$	$4n-3$	$4n-6$	$4n$	$4n-9$

Now, we verify that f is a prime cordial labeling of $G_n(H_n)$ for $n \equiv 0 \mod 3$. We leave for the reader to verify the prime cordiality for other cases of $G_n(H_n)$.

The edges of $G_n(H_n)$ can be divided into six subsets: $E_1 = \{v_i v_{i+4} : 0 \le i \le 4n-4 \text{ and } i \equiv 0 \bmod 4\}$, $E_2 = \{v_i v_{i+3} : 0 \le$

$i \leq 4n-4$ and $i \equiv 0 \bmod 4\}$, $E_3 = \{v_{i+1}v_{i+3} : 0 \leq i \leq 4n-4$ and $i \equiv 0 \bmod 4\}$, $E_4 = \{v_{i+2}v_{i+3} : 0 \leq i \leq 4n-4$ and $i \equiv 0 \bmod 4\}$, $E_5 = \{v_{i+1}v_{i+5} : 0 \leq i \leq 4n-4$ and $i \equiv 0 \bmod 4\}$ and $E_6 = \{v_{i+2}v_{i+6} : 0 \leq i \leq 4n-4$ and $i \equiv 0 \bmod 4\}$.

For $i = 0 \pmod{12}$, since $|f_0(v_i) - f_0(v_{i+4})| = 2 \times 3$, $f_0(v_i) \neq 0 \pmod 3$ and $f_0(v_i)$ is odd, by Observation 2.1, we have $f_0(v_iv_{i+4}) = 1$. For $i = 4 \pmod{12}$, since $|f_0(v_i) - f_0(v_{i+4})| = 2$ and $f_0(v_i)$ is odd, we have $f_0(v_iv_{i+4}) = 1$. For $i = 8 \pmod{12}$ and $i \neq 4n-4$, since $|f_0(v_i) - f_0(v_{i+4})| = 2^2$ and $f_0(v_i)$ is odd, we have $f_0(v_iv_{i+4}) = 1$. For $i = 4n-4$, since $f_0(v_0) = 1$, we have $f_0(v_{4n-4}v_0) = 1$. Hence, there are n edges labeled with 1 in E_1.

For $i = 0, 4 \pmod{12}$, since $|f_0(v_i) - f_0(v_{i+3})| = 2^2$ and $f_0(v_i)$ is odd, we have $f(v_iv_{i+3}) = 1$. For $i = 8 \pmod{12}$, since $gcd(f_0(v_i), f_0(v_{i+3})) \geq 3$, we have $f_0(v_iv_{i+3}) = 0$. Hence, there are $2n/3$ edges labeled with 1 and $n/3$ edges labeled with 0 in E_2.

For $i = 0, 4 \pmod{12}$, since $|f_0(v_{i+1}) - f_0(v_{i+3})| = 3$ and $f_0(v_i) \neq 0 \pmod 3$, by Observation 2.1, we have $f_0(v_{i+1}v_{i+3}) = 1$. For $i = 8 \pmod{12}$, since $gcd(f_0(v_{i+1}), f_0(v_{i+3})) \geq 3$, we have $f_0(v_{i+1}v_{i+3}) = 0$. Hence, there are $2n/3$ edges labeled with 1 and $n/3$ edges labeled with 0 in E_3.

For $i = 0, 4 \pmod{12}$, since $|f_0(v_{i+2}) - f_0(v_{i+3})| = 1$ we have $f_0(v_{i+2}v_{i+3}) = 1$. For $i = 8 \pmod{12}$, since $gcd(f_0(v_{i+2}), f(v_{i+3})) \geq 3$, we have $f_0(v_{i+2}v_{i+3}) = 0$. Hence, there are $2n/3$ edges labeled with 1 and $n/3$ edges labeled with 0 in E_4.

For $i = 0, 4, 8 \pmod{12}$ and $i \neq 4n-4$, since $gcd(f_0(v_{i+1}), f_0(v_{i+5})) \geq 2$, we have $f_0(v_{i+1}v_{i+5}) = 0$. For $i = 4n-4$, since $gcd(f_0(v_{4n-3}), f_0(v_2)) \geq 2$ for H_n (or $gcd(f_0(v_{4n-3}), f_0(v_1)) \geq 2$ for G_n), we have

$f_0(v_{4n-3}v_2) = 0$ for H_n (or $f_0(v_{4n-3}v_1) = 0$ for G_n). Hence, there are n edges labeled with 0 in E_5.

For $i = 0, 4, 8 \pmod{12}$ and $i \neq 4n-4$, since $gcd(f_0(v_{i+2}), f_0(v_{i+6})) \geq 2$, we have $f_0(v_{i+2}v_{i+6}) = 0$. For $i = 4n-4$, since $gcd(f_0(v_{4n-2}), f_0(v_1)) \geq 2$ for H_n (or $gcd(f_0(v_{4n-2}), f_0(v_2)) \geq 2$ for G_n), we have $f_0(v_{4n-2}v_1) = 0$ for H_n (or $f_0(v_{4n-2}v_2) = 0$ for G_n). Hence, there are n edges labeled with 0 in E_6.

Therefore, there are $3n$ edges labeled with 0 and $3n$ edges labeled with 1 in total under f_0 defined in Table 2.1, i.e, f_0 is a prime cordial labeling of $G_n(H_n)$ for $n \equiv 0 \mod 3$.

Case 2. $n \equiv 1 \pmod 3$. We define the function f_1 as follows:

$$f_1(v_i) = \begin{cases} f(v_i), & 0 \leq i \leq 4n-5; \\ 4n-1, & i = 4n-4; \\ 4n-2, & i = 4n-3; \\ 4n, & i = 4n-2; \\ 4n-3, & i = 4n-1. \end{cases}$$

Case 3. $n \equiv 2 \pmod 3$. We define the function f_2 as follows:

$$f_2(v_i) = \begin{cases} f(v_i), & 0 \leq i \leq 4n-7; \\ 4n-5, & i = 4n-6; \\ 4n-3, & i = 4n-5; \\ 4n-1, & i = 4n-4; \\ 4n-4, & i = 4n-3; \\ 4n-2, & i = 4n-2; \\ 4n, & i = 4n-1. \end{cases}$$

In Figure 2.1, we show the prime cordial labeling of $G_n(H_n)$ for $n = 9, 10, 11$, where the edges with label 1 are in dark.

From the Theorem 2.1 the Corollary 2.2 holds.

Corollary 2.2. Flower Snark and related graph are prime cordial for all $n \geq 3$.

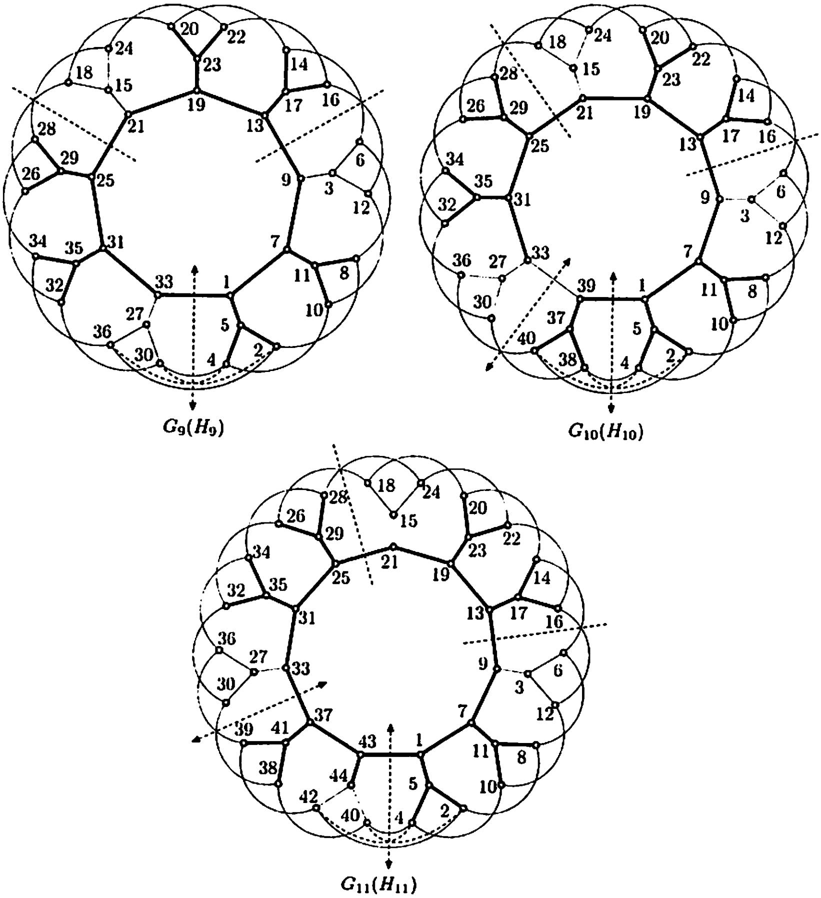

Figure 2.1. Prime cordial labeling of $G_n(H_n)$ for $n = 9, 10, 11$.

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