$\{P_r\}$-free colorings of Sierpiński-like graphs

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Abstract

Suppose $\{P_r\}$ is a nonempty family of paths for $r \geq 3$, P_r is a path on r vertices. An r-coloring of a graph G is said to be *$\{P_r\}$-free* if G contains no 2-colored subgraph isomorphic to any path P_r in $\{P_r\}$. The minimum k such that G has a $\{P_r\}$-free coloring using k colors is called the *$\{P_r\}$-free chromatic number* of G and is denoted by $\chi_{\{P_r\}}(G)$. If the family $\{P_r\}$ consists of a single graph P_r, then we use $\chi_{P_r}(G)$. In this paper, $\{P_r\}$-free colorings of Sierpiński-like graphs are considered. In particular, $\chi_{P_3}(S_n)$, $\chi_{P_4}(S_n)$, $\chi_{P_4}(S(n,k))$, $\chi_{P_3}(S^{++}(n,k))$, and $\chi_{P_4}(S^{++}(n,k))$ are determined.

Keywords: $\{P_r\}$-free coloring; $\{P_r\}$-free chromatic number; Sierpiński-like graphs

AMS Subject Classification (2010): 05C15

1 Introduction

Suppose $\mathcal{F}$ is a nonempty family of connected bipartite graphs, each with at least 3 vertices. An r-coloring of a graph G is said to be *$\mathcal{F}$-free* [3] if G contains no 2-colored subgraph isomorphic to any graph F in $\mathcal{F}$. We denote the minimum number of colors in an $\mathcal{F}$-free coloring of G by $\chi_{\mathcal{F}}(G)$. If the family $\mathcal{F}$ consists of a single graph F, then we use $\chi_F(G)$. If $\mathcal{F}$ is the family of all even cycles, then $\mathcal{F}$-free coloring is the acyclic coloring [1]. In this paper, we concentrate on the case when $\mathcal{F} = \{P_r\}$, each path with at least r vertices, $r \geq 3$. The *$\{P_r\}$-free chromatic number* of an undirected graph G, denoted by $\chi_{\{P_r\}}(G)$, is the smallest integer k for which G admits a $\{P_r\}$-free coloring with k colors.

$\{P_3\}$-free coloring is the *2-distance coloring* [4]. An r-coloring of G is called a 2-distance coloring if there are no 2-colored paths on 3 vertices. The minimum k such that G has a 2-distance coloring using k colors is called the *2-distance*

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chromatic number of G and is denoted by $\chi_{P_3}(G)$ or $\chi_{2d}(G)$. Alon [2] proved that $\chi_{P_3}(G) = (1 + o(1))\Delta^2$ if the girth $g(G) = 3, 4, 5$ or 6, and $\chi_{P_3}(G) = O(\Delta^2/\log\Delta)$ if the girth $g(G) \geq 7$. For studies of the 2-distance coloring of planar and outerplanar graphs, see [19, 21].

If $r = 4$, then $\{P_4\}$-free coloring is equivalent to the *star coloring* [5, 6]. A star coloring of G is a proper coloring of G such that no path of length 3 in G is bicolored. The *star chromatic number* of G, denoted by $\chi_{P_4}(G)$ or $\chi_s(G)$, is the smallest integer k for which G admits a star coloring with k colors. Albertson et al.[3] proved that $\chi_s(G) \leq \Delta(\Delta - 1) + 2$ and Fertin et al.[5] proved that $\chi_s(G) = O(\Delta^{\frac{3}{2}})$ for any graph G of maximum degree Δ. The $\{P_4\}$-free coloring has been quite extensively studied by now, see [3].

The Sierpiński-like graphs appeared naturally in many different areas of mathematics and were applied in several other scientific fields. One of the most important families of such graphs are *Sierpiński gasket graphs* that were introduced by Scorer et al.[23]. These graphs play an important role in psychology [17], dynamical systems [9] and probability [12]. For some recent results on Sierpiński gasket graphs, see [11, 13, 14]. In [15], the graphs $S(n, 3)$ were generalized to the *Sierpiński graphs* $S(n, k)$ for $k \geq 3$. The motivation for this generalization came from topological studies of Lipscomb's space [18, 20], where it is shown that this space is a generalization of the Sierpiński triangular curve (Sierpiński gasket). As it turned out, the $S(n, k)$ possess many appealing properties and were studied from different perspectives, as for instance existence of codes [8, 16] and several metric properties [22]. These graphs have been quite extensively studied by now, see [13, 15, 16, 22]. The graphs $S(n, k)$ are almost regular and there are at least two natural ways to extend them to regular graphs. In this spirit, $S^+(n, k)$ and $S^{++}(n, k)$ were proposed in [14].

Now the colorings of Sierpiński-like graphs have also been previously studied in [7, 10, 13]. In section 3, we determine the $\{P_r\}$-free chromatic number of S_n for $r = 3, 4$. In section 4 we give the $\{P_4\}$-free chromatic number of $S(n, k)$. Fu and Xie [7] proved that $\chi_{P_3}(S(n, k)) = k + 1$. In the last section, we consider the $\{P_r\}$-free chromatic number of $S^+(n, k)$ and $S^{++}(n, k)$ for $r = 3, 4$. The results obtained in this paper together with the previously known results are summarized in Table 1.

Table 1 Summary of the results

	S_n $n \geq 2$	$S(n,k)$ $n \geq 2, k \geq 2$	$S^+(n,k)$ $n \geq 2, k \geq 3$	$S^{++}(n,k)$ $n \geq 2, k \geq 2$
χ_{P_∞}	3	k	k	k
χ_{P_3}	6	$k+1$	$k+1$, *n is odd* \ , *n is even*	$k+1$
χ_{P_4}	4 $n = 2, 3$ 5 $n \geq 4$	$k+1$	$k+1$, *n is odd* $k+1$ *or* $k+2$, *n is even*	$k+1$
χ_{P_r}	3 or 4 or 5	k or $k+1$	$k+1$, *n is odd* $k+1$ *or* $k+2$, *n is even*	k or $k+1$

2 Preliminaries

In this section we define the families of Sierpiński-like graphs. We first introduce the Sierpiński graphs $S(n,k)$ that are defined on vertex set $V(S(n,k)) = \{1,\ldots,k\}^n$ ($|V(S(n,k))| = k^n$) for any $n \geq 1$ and $k \geq 1$ as follows. We will write $(u_1,\ldots,u_n)$, $u_r \in \{1,\ldots,k\}$, $r \in \{1,\ldots,n\}$, for the vertex of graphs $S(n,k)$. Two different vertices $u = (u_1,\ldots,u_n)$ and $v = (v_1,\ldots,v_n)$, $u_r, v_r \in \{1,\ldots,k\}$, $r \in \{1,\ldots,n\}$, are adjacent iff there exists an $h \in \{1,\ldots,n\}$ such that
(i) $u_t = v_t$, for $t = 1,\ldots,h-1$;
(ii) $u_h \neq v_h$; and
(iii) $u_t = v_h$ and $v_t = u_h$ for $t = h+1,\ldots,n$.
In the rest of the paper we will write $(u_1u_2\ldots u_n)$ for $(u_1,u_2,\ldots,u_n)$ or even shorter $u_1u_2\ldots u_n$. See Fig.1 for the Sierpiński graphs $S(2,5)$ and $S(3,4)$.

For $i = 1,\ldots,k$, let $S_i(n+1,k)$ be the subgraph of $S(n+1,k)$ induced by the vertex set $V_i = \{(ij_1\ldots j_n)|\ j_r \in \{1,\ldots,k\}, r \in \{1,\ldots,n\}\}$. Clearly $S_i(n+1,k)$ is isomorphic to $S(n,k)$. Consequently, for any $k \geq 2$, $S(n+1,k)$ contains k copies of the graph $S(n,k)$ and k^n copies of the complete graph $K_k = S(1,k)$. The $S(n+1,k)$ can be constructed inductively from $S(n,k)$ as follows (cf. Fig.1):

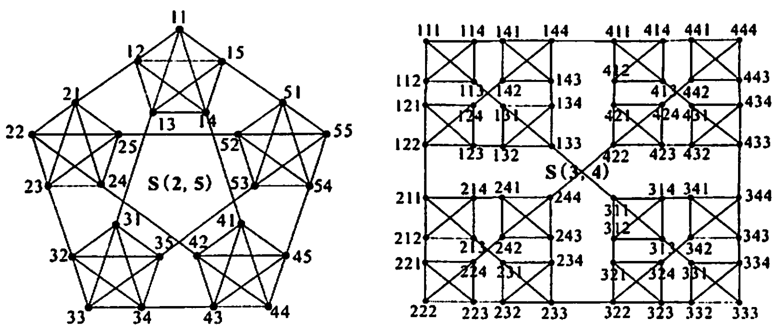

Fig. 1. Sierpiński graphs $S(2,5)$ and $S(3,4)$

• Take k copies $S_1(n+1,k),\ldots,S_k(n+1,k)$ of $S(n,k)$. Then we have,

$$V(S_i(n+1,k)) = \{(ij_1\ldots j_n)|\ (j_1\ldots j_n) \in V(S(n,k)), i, j_r = 1,\ldots,k, r = 1,\ldots,n\}.$$

• For any $i \neq j$, we add an edge between the vertex $(ijj\ldots j)$ of $S_i(n+1,k)$ and the vertex $(jii\ldots i)$ of $S_j(n+1,k)$.

Note that no edge incident with the vertex $(ii\ldots i)$ of $S_i(n,k)$ is added for any $i = 1,\ldots,k$.

Sierpiński gasket graph S_n, for any $n \geq 1$, is obtained from $S(n,3)$ by contracting all the edges of $S(n,3)$ that lie in no triangle, see Fig.2 for S_4.

We label the vertices of S_n by using a labeling technique proposed in [13]. Let $(u_1\ldots u_rij\ldots j)$ and $(u_1\ldots u_rji\ldots i)$ be end vertices of an edge of $S(n,3)$ that is contracted to a vertex x of S_n. Then the vertex x is labeled with $(u_1\ldots u_r)\{i,j\}$, where $0 \leq r \leq n-2$. There are three special vertices in S_n that are not merged with any other vertex, namely $(1,\ldots,1), (2,\ldots,2), (3,\ldots,3)$, called the extreme vertices of S_n. For $n = 4$, S_4 is shown on the left-hand side of Fig. 2.

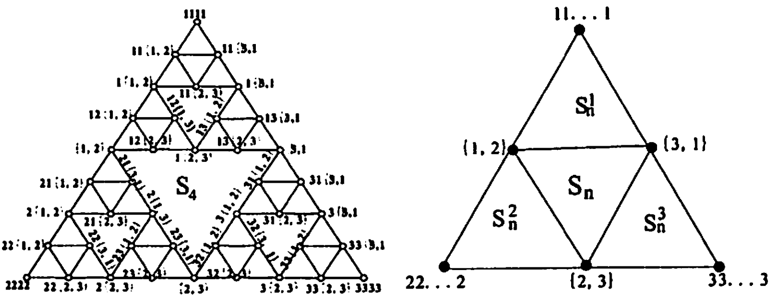

Fig. 2. Sierpiński gasket graphs S_4 and S_n

For any $n \geq 1$ and $i \in \{1,2,3\}$, let S_n^i be the subgraph of S_n induced by $(i\ldots i)$, $\{i,j\}$, $\{j,k\}$, $\{k,i\}$, $(iu_1\ldots u_r)\{i,j\}$, $(iu_1\ldots u_r)\{j,k\}$ and $(iu_1\ldots u_r)\{k,i\}$, where $u_r, j, k \in \{1,2,3\}$, $r \in \{0,\ldots,n-3\}$. Note that S_n^i is isomorphic to S_{n-1}. S_n, $n \geq 2$, is schematically shown on the right-hand side of Fig. 2.

The extended Sierpiński graphs $S^+(n,k)$ and $S^{++}(n,k)$ were introduced in the following way. $S^+(n,k)$, $n \geq 1$ and $k \geq 1$, is obtained from $S(n,k)$ by adding a new vertex w, called the *special vertex* of $S^+(n,k)$, and the edges joining w with all extreme vertices of $S(n,k)$. (As an example $S^+(3,4)$ is shown on the left-hand side of Fig. 3).

$S^{++}(n,k)$, $n \geq 2$ and $k \geq 1$, can be defined as the graph obtained from the disjoint union of a copy of $S(n,k)$ and a copy of $S(n-1,k)$ such that the extreme vertices of $S(n,k)$ and the extreme vertices of $S(n-1,k)$ are connected by a matching. (As an example $S^{++}(3,3)$ is shown on the right-hand side of Fig. 3).

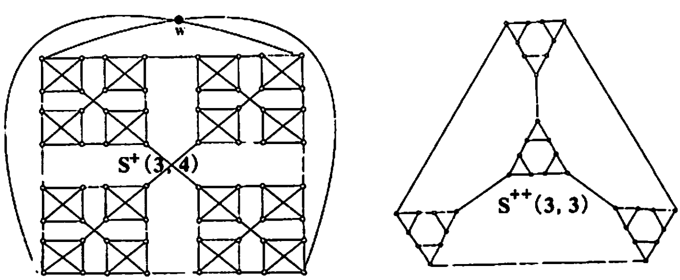

Fig. 3. Graphs $S^+(3,4)$ and $S^{++}(3,3)$

By the definition of $\mathcal{F}$-free coloring, if H is a subgraph of F then an H-free coloring of G is certainly an F-free coloring of G. Every member of the nonempty family of paths P_r ($r \geq 3$) has a 3 vertex path as a subgraph, combining Proposition 2.2 in [3] ($\chi_{P_r}(G) \leq \chi_{P_3}(G) = \chi(G^2) \leq min\{\Delta(G)^2+1, n\}$), we give the following lemma:

Lemma 2.1 *For* $5 \le r \le |V(G)|$, $\chi(G) = \chi_{P_\infty}(G) \le \chi_{P_r}(G) \le \chi_{P_4}(G) \le \chi_{P_3}(G) \le min\{\Delta(G)^2 + 1, n\}$.

Corollary 2.2 $\chi_{P_\infty}(G) = \chi(G) = \chi_{P_r}(G)$ *for* $r > |V(G)|$.

3 $\{P_r\}$-free coloring of S_n

Theorem 3.1. *For any* $n \ge 2$, $\chi_{P_3}(S_n) = 6$.

Proof. Since the diameter of S_2 is no more than 2, $\chi_{P_3}(S_2) = |V(S_2)| = 6$ by the definition of $\{P_3\}$-free coloring. Clearly, for any $n \ge 3$, S_2 is a subgraph of S_n, thus $\chi_{P_3}(S_n) \ge \chi_{P_3}(S_2) = 6$. We only need to show that $\chi_{P_3}(S_n) \le 6$ for any $n \ge 3$. Now, we will use the following notations. Let $f_n = \bigcup_{i=1}^{3} f_n^i$ be a coloring of S_n and f_n^i be a coloring of S_n^i for $i = 1, 2, 3$. Let $G^n_{\{i,j\}}$ be an induced subgraph of S_n by V_n, $V_n = \{\{i,j\}, j\overbrace{i\ldots i}^{n-3}\{i,j\}, j\overbrace{i\ldots i}^{n-3}\{k,i\}, i\overbrace{j\ldots j}^{n-3}\{j,k\}, i\overbrace{j\ldots j}^{n-3}\{i,j\}\}$, $i,j,k \in \{1,2,3\}$ and $i \ne j \ne k$. Note that $f_n(G^n_{\{i,j\}})$ is the set of colors appearing on the vertices of $G^n_{\{i,j\}}$. Now, we construct a $\{P_3\}$-free coloring f_n of S_n with six colors by induction on n.

Suppose that $n = 3$. A $\{P_3\}$-free coloring f_3 with 6 colors of S_3 is shown in Fig.4. $f_3(111) = f_3(2\{1,2\}) = f_3(3\{3,1\}) = 1$; $f_3(222) = f_3(1\{1,2\}) = f_3(3\{1,2\}) = 2$; $f_3(\{1,2\}) = f_3(3\{2,3\}) = 3$; $f_3(1\{2,3\}) = f_3(\{2,3\}) = 4$; $f_3(\{3,1\}) = f_3(2\{2,3\}) = 5$; $f_3(333) = f_3(1\{3,1\}) = f_3(2\{3,1\}) = 6$.

Suppose that the result holds for $n-1$, $n-1 \ge 2$, i.e., there exists a $\{P_3\}$-free coloring f_{n-1} of S_{n-1} that uses six colors. Now we form a coloring f_n of S_n that uses six colors as follows:

Suppose that $n-1$ is odd.

$f_n^1(u) = f'_{n-1}(u)$, if $u \in V(S_n^1)$, where $f'_{n-1} = f_{n-1}$.

$f_n^2(u) = f''_{n-1}(u)$, if $u \in V(S_n^2)$, f''_{n-1} is obtained from f_{n-1} by applying permutation (123)(645), where if $f_{n-1}(v) = 1$ then $f_n^2(u) = 2$; if $f_{n-1}(v) = 2$ then $f_n^2(u) = 3$; if $f_{n-1}(v) = 3$ then $f_n^2(u) = 1$; if $f_{n-1}(v) = 4$ then $f_n^2(u) = 5$; if $f_{n-1}(v) = 5$ then $f_n^2(u) = 6$; if $f_{n-1}(v) = 6$ then $f_n^2(u) = 4$ (Since S_n^i is isomorphic to S_{n-1}, there exists a mapping θ, $\theta(v) = u$ for $v \in V(S_{n-1})$ and $u \in V(S_n^i)$).

$f_n^3(u) = f'''_{n-1}(u)$, if $u \in V(S_n^3)$, f'''_{n-1} is obtained from f_{n-1} by applying permutation (165)(243), then exchange the colors of vertices $3\overbrace{2\ldots 2}^{n-3}\{2,3\}$ and $3\overbrace{2\ldots 2}^{n-3}\{3,1\}$.

Suppose that $n-1$ is even.

$f_n^1(u) = f'_{n-1}(u)$, if $u \in V(S_n^1)$, where $f'_{n-1} = f_{n-1}$.

$f_n^2(u) = f''_{n-1}(u)$, if $u \in V(S_n^2)$, f''_{n-1} is obtained from f_{n-1} by applying permutation (132)(654).

$f_n^3(u) = f'''_{n-1}(u)$, if $u \in V(S_n^3)$, f'''_{n-1} is obtained from f_{n-1} by applying permutation (156)(234), then exchange the colors of vertices $3\overbrace{2\ldots 2}^{n-3}\{2,3\}$ and

$3\overbrace{2\ldots2}^{n-3}\{3,1\}$.

By the way in which f_n has been constructed, the only possible bicolored P_3 must contain middle vertex $\{i,j\}$ where $i,j \in \{1,2,3\}$ and $i \neq j$. We only need to show that f_n is a $\{P_3\}$-free coloring of S_n. It is trivial for $n = 2,3$ (f_2 and f_3 are shown in Fig.4). Assume that it holds for $n-1$, i.e., f_{n-1} is a $\{P_3\}$-free coloring of S_{n-1}.

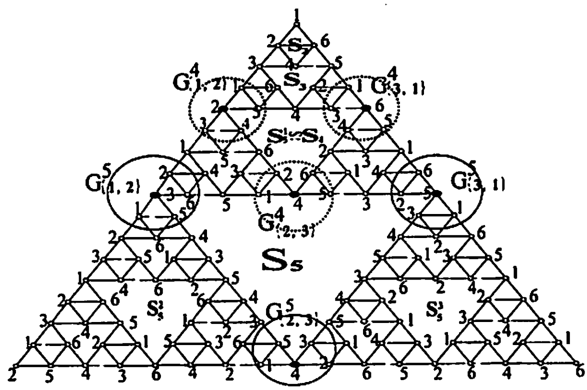

Fig.4. $\{P_3\}$-free colorings of $S_n, n = 2,3,4,5$.

Suppose that n is even.

Note that $f_n^1 = f_{n-1}$, and f_n^2 and f_n^3 are obtained from f_{n-1} by applying permutations (123)(645) and (165)(243), respectively. Suppose that $u \in V(S_n^i)$ and $v \in V(S_n^j)$, $i \neq j$. For $d(u,v) \leq 2$, by the structure of S_n, then $u,v \in V(G^n_{\{i,j\}})$. Since we exchanged the colors of the vertices $3\overbrace{2\ldots2}^{n-3}\{2,3\}$ and $3\overbrace{2\ldots2}^{n-3}\{1,3\}$, by the definition of f_n, $f_n(G^n_{\{1,2\}}) = \{2,5,1,3,4\}$, $f_n(G^n_{\{2,3\}}) = \{4,5,6,2,1\}$, $f_n(G^n_{\{3,1\}}) = \{6,1,3,4,5\}$. Thus, $f_n(u) \neq f_n(v)$.

Suppose that n is odd.

Note that $f_n^1 = f_{n-1}$, and f_n^2 and f_n^3 are obtained from f_{n-1} by applying permutations (132)(654) and (156)(234), respectively. Suppose that $u \in V(S_n^i)$ and $v \in V(S_n^j)$ and $i \neq j$. For $d(u,v) \leq 2$, by the structure of S_n, then $u,v \in V(G^n_{\{i,j\}})$. Since we exchanged the colors of the vertices $3\overbrace{2\ldots2}^{n-3}\{2,3\}$ and $3\overbrace{2\ldots2}^{n-3}\{1,3\}$, by the definition of f_n, $f_n(G_{\{1,2\}}) = \{3,6,2,1,5\}$, $f_n(G_{\{2,3\}}) = \{4,2,3,5,1\}$, $f_n(G_{\{3,1\}}) = \{5,6,2,3,1\}$. Thus, $f_n(u) \neq f_n(v)$.

By the principle of induction, f_n is a $\{P_3\}$-free coloring of S_n. The theorem is proved. □

Remark 3.2. *For $n = 1$, S_1 is isomorphic to K_3, $\chi_{P_3}(S_1) = \chi(K_3) = 3$.*

Lemma 3.3. *S_3 is uniquely 4-$\{P_4\}$-free-colorable (up to isomorphism).*

Proof. Clearly, $\chi_{P_4}(S_3) \geq 4$. Now we construct a $\{P_4\}$-free coloring f_3 of S_3 that uses 4 colors. Let $f_3(111) = 1$, $f_3(1\{1,2\}) = 2$ and $f_3(1\{1,3\}) = 3$. There are now two options: either $f_3(1\{2,3\}) = 1$ or $f_3(1\{2,3\}) = 4$. Let us detail each of those two cases.

Case 1 $f_3(1\{2,3\}) = 4$.

If $f_3(\{1,2\}) = f_3(\{1,3\}) = 1$, then it is impossible to assign colors to vertices $2\{1,2\}$ or $2\{1,3\}$; if $f_3(\{1,2\}) = 1$ and $f_3(\{1,3\}) = 2$ ($f_3(\{1,2\}) = 3$ and $f_3(\{1,3\}) = 1$; $f_3(\{1,2\}) = 3$ and $f_3(\{1,3\}) = 2$), then it is impossible to assign colors to vertices $3\{1,2\}$ or $3\{1,3\}$ ($2\{1,2\}$ or $2\{1,3\}$; $2\{1,2\}$ or $2\{1,3\}$). Hence this case cannot happen.

Case 2 $f_3(1\{2,3\}) = 1$.

Since $f_3(111) = 1$, $f_3(1\{1,2\}) = 2$, $f_3(1\{1,3\}) = 3$, $f_3(1\{2,3\}) = 1$, by the definition of $\{P_4\}$-free coloring, $f_3(\{1,2\}) = f_3(\{1,3\}) = 4$ and $f_3(2\{1,2\}) \neq 1,4$, $f_3(2\{1,3\}) \neq 1,4$, $f_3(3\{1,2\}) \neq 1,4$, $f_3(3\{1,3\}) \neq 1,4$. We distinguish four cases.

Case 2.1 $f_3(2\{1,2\}) = f_3(3\{1,3\}) = 2$ and $f_3(2\{1,3\}) = f_3(3\{1,2\}) = 3$.

By the definition of $\{P_4\}$-free coloring, $f_3(\{2,3\}) \neq 2,3,4$. Thus $f_3(\{2,3\}) = 1$, no color can be given to $3\{2,3\}$. Hence this case cannot happen.

Case 2.2 $f_3(2\{1,2\}) = f_3(3\{1,2\}) = 3$ and $f_3(2\{1,3\}) = f_3(3\{1,3\}) = 2$.

Then as in case 2.1, this case cannot happen.

Case 2.3 $f_3(2\{1,2\}) = f_3(3\{1,3\}) = 3$ and $f_3(2\{1,3\}) = f_3(3\{1,2\}) = 2$.

Then as in case 2.1, this case cannot happen.

Case 2.4 $f_3(2\{1,2\}) = f_3(3\{1,2\}) = 2$ and $f_3(2\{1,3\}) = f_3(3\{1,3\}) = 3$.

By the definition of $\{P_4\}$-free coloring, $f_3(\{2,3\}) \neq 2,3$, then $\{2,3\}$ can be assigned either color 1 or 4. If $f_3(\{2,3\}) = 1$, then no color can be given to $3\{2,3\}$. If $f_3(\{2,3\}) = 4$, then $f_3(2\{2,3\}) \neq 2,3,4$ and $f_3(3\{2,3\}) \neq 2,3,4$. If $f_3(2\{2,3\}) = f_3(3\{2,3\}) = 1$, then $f_3(222) \neq 1,2,4$ and $f_3(333) \neq 1,3,4$ ($\{P_4\}$-free coloring of S_2 and S_3 are shown on the left-hand side of Fig.5). □

Theorem 3.4. *For any* $n \geq 4$, $\chi_{P_4}(S_n) = 5$.

Proof. In order to prove the theorem, we first show that $\chi_{P_4}(S_n) = \chi_s(S_n) \geq 5$ for any $n \geq 4$. Suppose that we can assign four colors to S_4. Since S_3 is uniquely 4-$\{P_4\}$-free-colorable, no color can be given to $21\{1,2\}$ or $21\{1,3\}$. Hence, $\chi_{P_4}(S_4) \geq 5$. We only need to construct a $\{P_4\}$-free coloring of S_n with five colors by induction on n. Now, we will use the following notations. Let f_n^i be a coloring of S_n^i for $i \in \{1,2,3\}$ and $f_n = \bigcup_{i=1}^3 f_n^i$ be a coloring of S_n. Let $f_n(G_{\{i,j\}}^n)$ be the set of colors appearing on the vertices of $G_{\{i,j\}}^n$ (Graphs $G_{\{i,j\}}^n$ are shown on the right-hand side of Fig.5 for $i, j \in \{1,2,3\}$ and $i \neq j$).

Suppose that $n = 4$. We form a $\{P_4\}$-free coloring f_4 of S_4 that uses five colors (f_4 is shown on the left-hand side of Fig.5).

Suppose that the result holds for n (for any $n \geq 4$), i.e., there exists a $\{P_4\}$-free coloring of S_n that uses five colors. Now we form a coloring f_{n+1} of S_{n+1} that uses five colors as follows:

$$f_{n+1}^i(u) = f_n(u), \quad if\ u \in V(S_{n+1}^i)\ for\ \ i = 1,2,3.$$

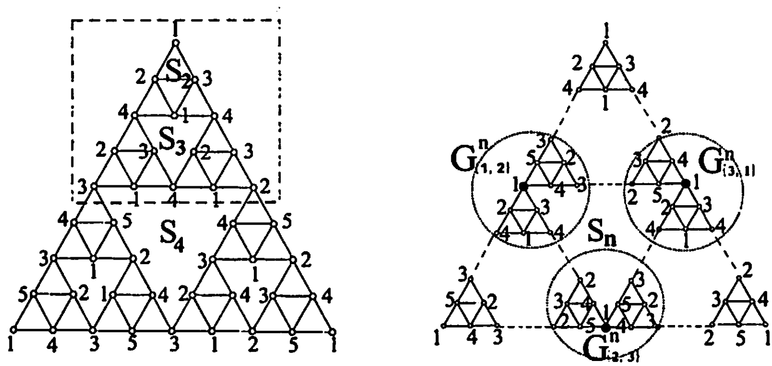

Fig. 5. $\{P_4\}$-free colorings of S_2, S_3, S_4, and S_n.

Now we only need to show that f_{n+1} is a $\{P_4\}$-free coloring of S_{n+1} for $n \geq 4$. We proceed by induction on n. It is trivial for $n = 5$ (Since f_4 is a $\{P_4\}$-free coloring of S_4 and $f_5^i = f_4$, f_5 is a $\{P_5\}$-free coloring of S_5^i. By the definition of $f_5(G^5_{\{i,j\}})$, $f_5 = \bigcup_{i=1}^{3} f_5^i$ is a $\{P_4\}$-free coloring of S_5). Assume that it holds for n, i.e., f_n is a $\{P_4\}$-free coloring of S_n. Note that $f_{n+1}^i = f_n$, by the induction hypothesis, f_{n+1}^i is a $\{P_4\}$-free coloring of S_{n+1}^i for $i = 1, 2, 3$. By the definition of $\{P_4\}$-free coloring, one may easily check from the right hand side of Fig.5 that $G^n_{\{i,j\}}$ has a $\{P_4\}$-free coloring for $i, j \in \{1, 2, 3\}$ and $i \neq j$. Therefore f_{n+1} is a $\{P_4\}$-free coloring of graph S_{n+1}. The theorem follows from the principle of induction. □

Remark 3.5. *For $n = 1$, S_1 is isomorphic to K_3, $\chi_{P_4}(S_1) = \chi(K_3) = 3$; for S_2 and S_3, $\chi_{P_4}(S_2) = 4$, $\chi_{P_4}(S_3) = 4$ and f_2 and f_3 are shown in Fig.* 5.

Corollary 3.6. *For any $n \geq 1$ and any $5 \leq r \leq |V(S_n)|$, $3 \leq \chi_{P_r}(S_n) \leq 5$.*

4 $\{P_r\}$-free coloring of $S(n, k)$

Let $\varphi_n = \bigcup_{i=1}^{k} \varphi_n^i$ be a coloring of $S(n,k)$ and φ_n^i be a coloring of $S_i(n,k)$ for $i \in \{1, \ldots, k\}$. By the structure of $S(n,k)$, let $u = (ij_1 \ldots j_{n-1}) \in V(S_i(n,k))$ and $u^{(i)} = (j_1 \ldots j_{n-1}) \in V(S(n-1,k))$, $i, j_r \in \{1, \ldots, k\}$, $r \in \{1, \ldots, n-1\}$. The following lemma is a direct consequence of the proof of Theorem 3.1 in [7].

Lemma 4.1. *Let φ_n be a coloring of $S(n,k)$.*

(i) For $n = 2$,

$\varphi_2(ij) = j$, *if* $i \neq j$, *for* $i, j \in \{1, \ldots, k\}$,

$\varphi_2(ii) = k + 1$, *for* $i \in \{1, \ldots, k\}$.

(ii) For $n \geq 3$

$\varphi_n(u) = \varphi^i_{n-1}(u)$, *if* $u = (ij_1 \ldots j_{n-1}) \in V(S_i(n,k))$, *for* $i, j_r \in \{1, \ldots, k\}$, $r \in \{1, \ldots, n-1\}$, *where* $\varphi^i_{n-1}(u)$ *is obtained from* $\varphi_{n-1}(u^{(i)})$ *using permuting of colors* $(1)\ldots(i-1)(i\ k+1)(i+1)\ldots(k), i \in \{1, \ldots, k\}$. *Then,* φ_n *is a* $\{P_3\}$*-free coloring of* $S(n,k)$ *and for any* $n \geq 2$ *and* $k \geq 2$, $\chi_{P_3}(S(n,k)) = k+1$.

As an example, for $k = 5$, φ_2 and φ_3 are shown in Fig.6.

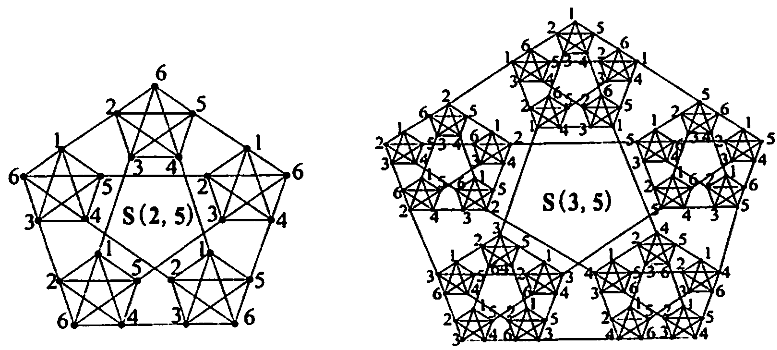

Fig. 6. $\{P_3\}$-free Colorings of $S(2,5)$ and $S(3,5)$

Theorem 4.2. *For any* $n \geq 2$ *and any* $k \geq 2$, $\chi_{P_4}(S(n,k)) = k+1$.

Proof. By Lemma 2.1 and 4.1, $\chi_{P_4}(S(n,k)) \leq \chi_{P_3}(S(n,k)) = k+1$. In the rest of the proof we only need to prove $\chi_{P_4}(S(n,k)) \geq k+1$. Since $S(2,k)$ is an isometric subgraph of $S(n,k)$ for any $n \geq 2$, $\chi_{P_4}(S(n,k)) \geq \chi_{P_4}(S(2,k))$. As the graph $S(2,k)$ consists of k complete subgraphs K_k and $\chi(K_k) = k$, it is sufficent to show that $\chi_{P_4}(S(2,k)) \geq k+1$ for any $k \geq 2$.

Suppose that $\chi_{P_4}(S(2,k)) = k$ for $k \geq 2$. Let $k = 2$. $S(2,2)$ is a path on 4 vertices and it is easy to see that a 2-coloring of $S(2,2)$ does not satisfy the definition of a $\{P_4\}$-free coloring. Suppose that $k \geq 3$. The graph $S(2,k)$ consists of k complete subgraphs on k vertices induced by the vertex sets $V_i = \{(ij) |\ j = 1, \ldots, k\}$, $i \in \{1, \ldots, k\}$. Note that the vertex $(ij) \in V_i$ is adjacent to the vertex $(ji) \in V_j$ for $i \neq j$. Since $\chi_{P_4}(S(2,k)) = k$, by the structure of $S(2,k)$, we get a path on 4 vertices colored by two colors, which violates the definition of $\{P_4\}$-free coloring. Thus $\chi_{P_4}(S(n,k)) \geq \chi_{P_4}(S(2,k)) \geq k+1$, and so $\chi_{P_4}(S(n,k)) = k+1$. The theorem is proved. □

Remark 4.3. (i)*For any* $n \geq 1$, $S(n,1)$ *is* K_1 *and* $\chi_{P_4}(S(n,1)) = 1$. (ii) *For any* $k \geq 2$, $\chi_{P_4}(S(1,k)) = \chi_s(K_k) = \chi(K_k) = k$.

Corollary 4.4. *For any* $n \geq 1$, $k \geq 2$ *and* $5 \leq r \leq |V(S(n,k))|$, $k \leq \chi_{P_r}(S(n,k)) \leq k+1$.

5 $\{P_r\}$-free colorings of $S^+(n,k)$ and $S^{++}(n,k)$

In this section we consider the $\{P_r\}$-free colorings on the extended Sierpiński graphs $S^+(n,k)$ and $S^{++}(n,k)$. Let φ_n be a $\{P_3\}$-free coloring of $S(n,k)$, as given in Lemma 4.1.

Theorem 5.1. *For any odd $n > 2$ and any $k \geq 2$, $\chi_{P_3}(S^+(n,k)) = k+1$.*
Proof. Suppose that $k = 2$. Note that $S^+(n,2)$ is an odd cycle on 2^n+1 vertices. Since $2^n + 1 \equiv 0 \pmod 3$ for any odd $n > 2$, it is clear that $\chi_{P_3}(S^+(n,2)) = 3$. Recall that $V(S^+(n,k)) = V(S(n,k)) \bigcup \{w\}$. By Lemma 4.1, $\chi_{P_3}(S^+(n,k)) \geq \chi_{P_3}(S(n,k)) = k+1$. We now only need to prove that $\chi_{P_3}(S^+(n,k)) \leq k+1$. Let f_n^+ be a coloring of $S^+(n,k)$ as follows:

$$f_n^+(u) = \begin{cases} k+1 & \text{if } u = w, \\ \varphi_n(u) & \text{if } u \in V(S(n,k). \end{cases}$$

By the definition of φ_n, $f_n^+(\overbrace{i\ldots i}^{n-1} j) = j$ for $i, j \in \{1, \ldots k\}$. Since $f_n^+(w) = k+1$, by the structure of $S^+(n,k)$, it is straightforward to verify that f_n^+ is a $\{P_3\}$-free coloring of $S^+(n,k)$ with $k+1$ colors. The theorem is proved. □

Note that $S^+(n,2)$ is an odd cycle on 2^n+1 vertices. By the definition of $\{P_4\}$-free coloring, $\chi_{P_4}(S^+(n,2)) = \begin{cases} 4 & n = 2, \\ 3 & n > 2. \end{cases}$

Theorem 5.2. *For any $n \geq 2$ and any $k \geq 3$, if n is odd, then $\chi_{P_4}(S^+(n,k)) = k+1$; if n is even, then $k+1 \leq \chi_{P_4}(S^+(n,k)) \leq k+2$.*
Proof. For any odd $n > 2$ and any $k \geq 3$, it is easy to see that $\chi_{P_4}(S^+(n,k)) = k+1$ by Theorem 5.1 and Lemma 2.1 and 4.1. For any even $n \geq 2$ and any $k \geq 3$, let f_n^+ be a coloring of $S^+(n,k)$ as follows:

$$f_n^+(u) = \begin{cases} k+2 & \text{if } u = w, \\ \varphi_n(u) & \text{if } u \in V(S(n,k). \end{cases}$$

By the definition of φ_n, $f_n^+(\overbrace{i\ldots i}^{n-1} j) = j$ for $i \neq j$, $f_n^+(\overbrace{i\ldots i}^{n-1} j) = k+1$ for $i = j$, $i, j \in \{1, \ldots k\}$ and $f_n^+(w) = k+2$, it is clear that $\chi_{P_4}(S^+(n,k)) \leq k+2$ and the theorem is proved. □

Remark 5.3. (i) *For any $n > 2$, $S^+(n,1) = K_2$, and so $\chi_{P_3}(S^+(n,1)) = \chi_{P_4}(S^+(n,1)) = 2$.* (ii) *For any $k \geq 2$, $S^+(1,k) = K_{k+1}$, and so $\chi_{P_3}(S^+(1,k)) = \chi_{P_4}(K_{k+1}) = \chi(K_{k+1}) = k+1$.*

Corollary 5.4. *For any $n \geq 2$, $k \geq 2$ and $5 \leq r \leq |V(S^+(n,k))|$, $k+1 \leq \chi_{P_r}(S^+(n,k)) \leq k+2$.*

Theorem 5.5. *For any $n \geq 2$ and $k \geq 2$, $\chi_{P_3}(S^{++}(n,k)) = k+1$.*
Proof. Suppose that $k = 2$. Note that $S^{++}(n,2)$ is an even cycle on $3 \cdot 2^{n-1}$ vertices. By the definition of $\{P_3\}$-free coloring, it is clear that $\chi_{P_3}(S^{++}(n,2)) = 3$.

For any $n \geq 2$ and any $k \geq 3$, since $S^{++}(n,k)$ consists of $k+1$ copies of $S(n-1,k)$, $V(S^{++}(n,k)) = V(S(n,k)) \bigcup V(S(n-1,k))$. Let f_n^{++} be a coloring of $S^{++}(n,k)$ as follows:

$$f_n^{++}(u) = \begin{cases} \varphi_{n-1}(u) & \text{if } u \in V(S(n-1,k), \\ \varphi_n(u) & \text{if } u \in V(S(n,k). \end{cases}$$

By the definition of φ_n, for $i,j \in \{1,\ldots,k\}$, if n is odd, then $f_n^{++}(\overbrace{i\ldots i}^{n-1}j) = f_n(\overbrace{i\ldots i}^{n-1}j) = j$, $f_n^{++}(\overbrace{i\ldots i}^{n-1}) = f_{n-1}(\overbrace{i\ldots i}^{n-1}) = k+1$, $f_n^{++}(\overbrace{i\ldots i}^{n-2}j) = f_{n-1}(\overbrace{i\ldots i}^{n-2}j) = j$; if n is even, then $f_n^{++}(\overbrace{i\ldots i}^{n}) = f_n(\overbrace{i\ldots i}^{n}) = k+1$, $f_n^{++}(\overbrace{i\ldots i}^{n-1}j) = f_n(\overbrace{i\ldots i}^{n-1}j) = j$, $f_n^{++}(\overbrace{i\ldots i}^{n-2}j) = f_{n-1}(\overbrace{i\ldots i}^{n-2}j) = j$. It is straightforward to verify that f_n^{++} is a $\{P_3\}$-free coloring of $S^{++}(n,k)$ with $k+1$ colors. The theorem is proved. □

Theorem 5.6. *For any $n \geq 2$ and any $k \geq 2$, $\chi_{P_4}(S^{++}(n,k)) = k+1$.*
Proof. Recall that $S^{++}(n,2)$ is an even cycle on $3 \cdot 2^{n-1}$ vertices. By the definition of $\{P_4\}$-free coloring, it is easy to see that $\chi_{P_4}(S^{++}(n,2)) = \chi_s(S^+(n,2)) = 3$. Since $S(n,k)$ is a subgraph of $S^{++}(n,k)$, by Theorem 4.1, $\chi_{P_4}(S^{++}(n,k)) \geq k+1$ for any $n \geq 1$ and $k \geq 2$. Since $\chi_{P_3}(S^{++}(n,k)) = k+1$ by Lemma 2.1, $\chi_{P_4}(S^{++}(n,k)) \leq k+1$ for any $n \geq 2$ and any $k \geq 1$. The theorem is proved. □

Remark 5.7. (i) *For any $n > 2$, $S^{++}(n,1) = K_2$, $\chi_{P_3}(S^{++}(n,1)) = \chi_{P_4}(S^{++}(n,1))$* $= 2$. (ii)*For any $k > 2$, $S^{++}(1,k) = K_{k+1}$, $\chi_{P_3}(S^{++}(1,k)) = \chi_{P_4}(S^{++}(1,k)) = \chi(K_{k+1}) = k+1$.*

Corollary 5.8. *For any $n,k \geq 2$, and $5 \leq r \leq |V(S^{++}(n,k))|$, $k \leq \chi_{P_r}(S^{++}(n,k)) \leq k+1$.*

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References

[1] N. Alon, C. McDiarmid, B. Reed, Acyclic coloring of graphs, Random Structures Algorithms 2 (1991) 277–288.

[2] N. Alon, B. Mohar, The chromatic number of graph powers, Combinatorics, Probability and Computing 11 (1) (2002) 1–10.

[3] M.O. Albertson, G.G. Chappell, H.A. Kierstead, A. Kündgen, R. Ramamurthi, Coloring with no 2–colored P_4's, Electronic J. Combin. 11 (1) (2004) ♯R26.

[4] G. Fertin, E. Godard, A. Raspaud, Acyclic and k–distance coloring of the grid, Inform. Process. Lett. 87 (2003) 51–58.

[5] G. Fertin, A. Raspaud, B. Reed, Star coloring of graphs, J. Graph Theory 47 (3) (2004) 163–182.

[6] G. Fertin, A. Raspaud, B. Reed, On star coloring of graphs, Graph-Theoretic Concepts in Computer Science, 27th International Workshop, WG 2001,

Lecture Notes in Computer Science, vol. 2204, Springer, Berlin, 2001, pp. 140–153.

[7] H.Y. Fu, D. Xie, Equitable $L(2,1)$–labelings of Sierpiński graphs, Australas. J. Combin. 46 (2010) 147–156.

[8] S. Gravier, S. Klavžar, M. Mollard, Codes and $L(2,1)$–labelings in Sierpiński graphs. Taiwanese J. Math. 9 (2005) 671–681.

[9] A.M. Hinz, A. Schief, The average distance on the Sierpiński gasket, Probab. Theory Related Fields 87 (1990) 129–138.

[10] M. Jakovac, S. Klavžar, Vertex-, edge-, and total-colorings of Sierpiński-like graphs, Discrete Math. 309 (2009) 1548–1556.

[11] M. Jakovac, A 2-parametric generalization of Sierpiński gasket graphs, Ars. Combin. to appear.

[12] V.A. Kaimanovich, Random walks on Sierpiński graphs: Hyperbolicity and stochastic homogenization, in: Fractal in Graz 2002, in : Trends Math., Birkhäuser, Basel, 2003, pp. 145–183.

[13] S. Klavžar, Coloring Sierpiński graphs and Sierpiński gasket graphs, Taiwanese J.Math. 12 (2008) 513–522.

[14] S. Klavžar, B. Mohar, Crossing numbers of Sierpiński–like graphs, J. Graph Theory 50 (2005) 186–198.

[15] S. Klavžar, U. Milutinović, Graphs $S(n,k)$ and a variant of the Tower of Hanoi problem, Czechoslovak Math. J. 47 (122) (1997) 95–104.

[16] S. Klavžar, U. Milutinović, C. Petr, 1–perfect codes in Sierpiński graphs, Bull. Austral. Math. Soc. 66 (2002) 369–384.

[17] F. Klix, K. Rautenstrauch-Goede, Struktur and Komponentenanalyse von Problemlösungsprozessen, Z. Psychol. 174 (1967) 167–193.

[18] S. L. Lipscomb, J.C. Perry, Lipscomb's $L(A)$ space fractalized in Hilbert's $l^2(A)$ space, Proc. Amer. Math. Soc. 115 (1992) 1157–1165.

[19] K.W. Lih, W.F. Wang, Coloring the square of an outerplanar graph, Taiwanese J. Math. 10 (2006) 1015–1023.

[20] U. Milutinović, Completeness of the Lipscomb space, Glas. Mat. Ser. III 27 (47) (1992) 343–364.

[21] M. Molloy, M.R. Salavatipour, A bound on the chromatic number of the square of a planar graph, Journal of Combinatorial Theory Ser. B 94 (2005) 189–213.

[22] D. Parisse, On some metric properties of the Sierpiński graphs $S(n,k)$, Ars Combin. 90 (2009) 145–160.

[23] R.S. Scorer, P.M. Grundy, C.A.B. Smith, Some binary games, Math. Gaz. 28 (1994) 96–103.