On regular $(2, q)$-extendable graphs

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Abstract

Let G be a graph with a maximum matching of size q, and let $p \leq q$ be a positive integer. Then G is called (p, q)-*extendable* if every set of p independent edges can be extended to a matching of size q. If G is a graph of even order n and $n = 2q$, then (p, q)-extendable graphs are exactly the p-extendable graphs defined by Plummer [11] in 1980.

Let $d \geq 3$ be an integer, and let G be a d-regular graph of order n with a maximum matching of size $q = \frac{n-t}{2} \geq 3$ for an integer $t \geq 1$ such that $n - t$ is even. In this work we prove that if

(i) $n \leq (t + 1)(d + 1) - 5$ or

(ii) $n \leq (t + 1)(d + 2) - 1$ when d is odd,

then G is $(2, q)$-extendable.

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We shall assume that the reader is familiar with standard terminology on graphs (see, e.g., Chartrand and Lesniak [3]). In this paper, all graphs are finite and simple. The vertex set of a graph G is denoted by $V(G)$, and $n = n(G) = |V(G)|$ is its *order*. The *neighborhood* $N_G(x)$ of a vertex x is the set of vertices adjacent with x, and the number $d_G(x) = |N_G(x)|$ is the *degree* of x in the graph G. By $\delta(G) = \delta$ we denote the *minimum degree* of the graph G. If A is a subset of the vertex set of a graph G, then $G[A]$ is the subgraph induced by A, and $N_G(A) = \bigcup_{x \in A} N_G(x)$. We denote by

K_n the *complete graph* of order n and by $K_{r,s}$ the *complete bipartite graph* with partite sets A and B, where $|A| = r$ and $|B| = s$. If G is a graph and $A \subseteq V(G)$, then $o(G - A)$ is the number of odd components in the subgraph $G - A$. The *closure* $C(G)$ of a graph G of order n is the graph obtained from G by recursively joining pairs of nonadjacent vertices whose degree sum is at least n until no such pair remains.

A graph G is *p-extendable* if it contains a set of p independent edges and every set of p independent edges can be extended to a perfect matching. In 1980, Plummer [11] studied the properties of p-extendable graphs. As an extension of p-extendable graphs, Liu and Yu [10] defined (p, q)-extendable graphs as follows. Let G be a graph with a maximum matching of size q, and let $p \leq q$ be a positive integer. Then G is called *(p, q)-extendable* if every set of p independent edges can be extended to a matching of size q. If G is a graph of even order n and $2q = n$, then (p, q)-extendable graphs are exactly the p-extendable graphs defined by Plummer [11]. Examples of (p, q)-extendable graphs are complete bipartite graphs $K_{q,r}$ with $r \geq q$.

In 2001, Liu and Yu [10] have given a characterization of (p, q)-extendable graphs, which generalize those given by Little, Grant and Holton [8] and Yu [14] for p-extendable graphs. The proof of this characterization is based on an extension of Tutte's famous 1-factor Theorem [13] by Berge [1]. For the proof of our main theoren, we use the following special case for $p = 2$.

Theorem 1 (Liu and Yu [10] 2001) Let q and n be positive integers such that $2 < q \leq \frac{n}{2}$. A graph G of order n with a maximum matching of size q is $(2, q)$-extendable if and only if for any subset $A \subseteq V(G)$

(1) $o(G - A) \leq |A| + n - 2q$ and

(2) $o(G - A) = |A| + n - 2q - 2k$ for $0 \leq k \leq 1$ implies that $G[A]$ contains a matching of size at most k.

In addition, we also use the following results.

Theorem 2 (König [7] 1931, Hall [5] 1935) Let G be a bipartite graph with bipartition X, Y. Then G contains a matching that saturates every vertex in X if and only if $|S| \leq |N_G(S)|$ for all $S \subseteq X$.

Theorem 3 (Zhao [15] 1991) Let $d \geq 2$ be an integer, and let G be graph without odd components such that $d \leq d_G(x) \leq d + 1$ for every vertex $x \in V(G)$. If $|V(G)| \leq 3d + 3$, then G has a perfect matching.

Theorem 4 (Ore [12] 1960) Let G be a graph of order $n \geq 3$. If

$$d_G(x) + d_G(y) \geq n$$

for all distinct nonadjacent vertices x and y of G, then G is Hamiltonian.

Theorem 5 (Ore [12] 1960, Bondy, Chvátal [2] 1976) Let G be a graph of order $n \geq 3$. If the closure $C(G)$ is complete, then G is Hamiltonian.

Now we present our main result.

Theorem 6 Let $d \geq 3$ be an integer, and let G be a d-regular graph of order n with a maximum matching of size $q = \frac{n-t}{2} \geq 3$ for an integer $t \geq 1$ such that $n - t$ is even. If

(i) $n \leq (t+1)(d+1) - 5$ or

(ii) $n \leq (t+1)(d+2) - 1$ when d is odd,

then G is $(2, q)$-extendable.

Proof. Suppose to the contrary that G is not $(2, q)$-extendable. Then it follows from the hypothesis and Theorem 1 that there exists a set $A \subseteq V(G)$ such that $o(G - A) \geq |A| + t + 1$ or $o(G - A) = |A| + t - 2k$ for $0 \leq k \leq 1$ and $G[A]$ contains a matching of size $k + 1$.

We call an odd component of $G - A$ large if it has more than d vertices and small otherwise. We denote by α and β the number of large and small components of $G - A$, respectively. Since G is a d-regular graph, it is easy to see that there are at least d edges in G joining each small component of $G - A$ with A. The d-regularity of G therefore implies

$$d\beta \leq d|A|. \tag{1}$$

Case 1. Assume that $o(G - A) \geq |A| + t + 1$. If n is even, then the numbers $o(G - A)$ and $|A|$ are of the same parity, and if n is odd, then the numbers $o(G - A)$ and $|A|$ are of different parity. Since n and t are of the same parity, we therefore deduce that

$$\alpha + \beta = o(G - A) \geq |A| + t + 2. \tag{2}$$

Inequality (1) shows that $\beta \leq |A|$ and thus (2) yields $\alpha \geq t + 2$. Applying the hypothesis (i), we arrive at the contradiction

$$(t+1)(d+1) - 5 \geq n \geq |A| + \alpha(d+1) + \beta \geq (t+2)(d+1).$$

If d is odd, then each large component contains at least $d+2$ vertices. Now the hypothesis (ii) leads to the contradiction.

$$(t+1)(d+2)-1 \geq n \geq |A| + \alpha(d+2) + \beta \geq (t+2)(d+2).$$

Case 2: Assume that $\alpha + \beta = o(G-A) = |A| + t$ and $G[A]$ contains an edge. This implies $|A| \geq 2$ and

$$d\beta \leq d|A| - 2.$$

This leads to $\beta \leq |A| - 1$ and thus $\alpha \geq t+1$. Now the hypothesis (i) or (ii) yields the contradiction

$$(t+1)(d+1) - 5 \geq n \geq |A| + \alpha(d+1) + \beta \geq (t+1)(d+1)$$

or

$$(t+1)(d+2) - 1 \geq n \geq |A| + \alpha(d+2) + \beta \geq (t+1)(d+2).$$

Case 3: Assume that $\alpha+\beta = o(G-A) = |A|+t-2$ and $G[A]$ contains a matching of size 2. This implies $|A| \geq 4$, $\beta \leq |A|-1$ und thus $\alpha \geq t-1$. If $\alpha \geq t+1$, then we obtain a contradiction as in Case 2.

If U is a small component of minimum order in $G-A$, then we observe that

$$|V(U)| \geq d - |A| + 1. \tag{3}$$

Subcase 3.1: Assume that $\alpha = t$. It follows that $\beta = |A| - 2 \geq 2$.

If $|A| \geq d$, then the hypothesis (i) or (ii) yields the contradiction

$$\begin{aligned}(t+1)(d+1)-5 \geq n &\geq |A| + t(d+1) + \beta \\ &= 2|A| - 2 + t(d+1) \\ &\geq 2d - 2 + t(d+1) \\ &= (t+1)(d+1) + d - 3\end{aligned}$$

or

$$\begin{aligned}(t+1)(d+2)-1 \geq n &\geq |A| + t(d+2) + \beta \\ &\geq d + t(d+2) + 2 \\ &= (t+1)(d+2).\end{aligned}$$

If $4 \leq |A| \leq d-1$, then the hypothesis (i) or (ii) and the bound (3) lead to the contradiction

$$\begin{aligned}(t+1)(d+1)-5 \geq n &\geq |A| + t(d+1) + (|A|-2)|V(U)| \\ &\geq |A| + t(d+1) + 2(d - |A| + 1) \\ &\geq |A| + t(d+1) + (d - |A| + 1) + 2 \\ &= (t+1)(d+1) + 2\end{aligned}$$

or

$$\begin{aligned}(t+1)(d+2)-1 \geq n &\geq |A| + t(d+2) + (|A|-2)|V(U)| \\ &\geq |A| + t(d+2) + 2(d-|A|+1) \\ &\geq |A| + t(d+2) + (d-|A|+1) + 2 \\ &= (t+1)(d+2)+1.\end{aligned}$$

Subcase 3.2: Assume that $\alpha = t-1$. It follows that $\beta = |A|-1 \geq 3$.

Subcase 3.2.1: Assume that $n \leq (t+1)(d+1)-5$. If $|A| \geq d$, then we obtain the contradiction

$$\begin{aligned}(t+1)(d+1)-5 \geq n &\geq |A| + (t-1)(d+1) + \beta \\ &= 2|A| - 1 + (t-1)(d+1) \\ &\geq 2d - 1 + (t-1)(d+1) \\ &= (t+1)(d+1) - 3.\end{aligned}$$

If $4 \leq |A| \leq d-1$, then it follows from (3) that

$$(t+1)(d+1)-5 \geq n \geq |A| + (t-1)(d+1) + (|A|-1)(d-|A|+1). \quad (4)$$

If we define $|A| = x$ and $g(x) = x + (x-1)(d-x+1)$, then, because of $4 \leq |A| \leq d-1$, we like to determine the minimum of the function g in the interval $I : 4 \leq x \leq d-1$. It is straightforward to verify that

$$\min_{x \in I}\{g(x)\} = g(4) = g(d-1) = 3d-5.$$

Combining this with the inequality (4), we arrive at the contradiction

$$\begin{aligned}(t+1)(d+1)-5 \geq n &\geq |A| + (t-1)(d+1) + (|A|-1)(d-|A|+1) \\ &\geq (t-1)(d+1) + 3d - 5 \\ &= (t+1)(d+1) + d - 7.\end{aligned}$$

Subcase 3.2.2: Assume that d is odd and $n \leq (t+1)(d+2)-1$. Since d is odd, n is even, $t \geq 2$ is even, and there exists at least one edge in G joining each large component of $G-A$ with A. This implies $\alpha + d\beta \leq d|A| - 4$. Since $\beta = |A|-1$ and $\alpha = t-1$, we deduce that $t+3 \leq d$ and thus $d \geq 5$.

Subcase 3.2.2.1: Assume that $|A| \geq d+3$. This assumption yields the contradiction

$$\begin{aligned}(t+1)(d+2)-1 \geq n &\geq |A| + (t-1)(d+2) + \beta \\ &= 2|A| - 1 + (t-1)(d+2) \\ &\geq 2d + 5 + (t-1)(d+2) \\ &= (t+1)(d+2) + 1.\end{aligned}$$

Subcase 3.2.2.2: Assume that $|A| = d+2$. This implies

$$\begin{aligned}(t+1)(d+2)-1 \geq n &\geq |A| + (t-1)(d+2) + \beta \\ &= 2|A| - 1 + (t-1)(d+2) \\ &= 2d + 3 + (t-1)(d+2) \\ &= (t+1)(d+2) - 1.\end{aligned}$$

Consequently, all large components of $G - A$ are of order $d+2$ and all small components of order one.

Next we will show that G contains a matching of size at least $q+1$. Since G is d-regular, there are at most $d-t-2$ edges in G joining each large component of $G - A$ with A. If Q is a large component, and hence of order $d+2$, and x and y are two nonadjacent vertices of Q, then we conclude that

$$d_Q(x) + d_Q(y) \geq 2d - (d-t-2) \geq d+2.$$

Therefore, by Theorem 4, the component Q is Hamiltonian.

Let zw be an edge joining a large component of $G - A$ with a vertex $w \in A$, and let B consist of the vertices of the small components of $G - A$. In addition, let H be the bipartite graph with the partite sets $A - w$ and B together with all edges of G between $A - w$ and B. Then $d_H(x) \geq d-1$ for all $x \in B$ and there are at least two vertices of degree d in B. Applying the theorem of König-Hall (Theorem 2), we deduce that H has a perfect matching. Altogether, we observe that G has a matching of size at least $q+1 = \frac{n-(t-2)}{2}$, a contradiction to our hypothesis.

Subcase 3.2.2.3: Assume that $|A| = d+1$. If there exists a small component with at least 5 vertices, then we arrive at the contradiction

$$\begin{aligned}(t+1)(d+2)-1 \geq n &\geq |A| + (t-1)(d+2) + 5 + \beta - 1 \\ &= 2|A| + (t-1)(d+2) + 3 \\ &= 2(d+2) + (t-1)(d+2) + 1 \\ &= (t+1)(d+2) + 1.\end{aligned}$$

If there is one small component of $G - A$ with exactly three vertices, then there are at least

$$(\beta - 1)d + 3(d-2) = d(d+2) - 6$$

edges from the small components to A. This is a contradiction to the fact that there are at most $d(d+1) - t - 3$ edges from A to the small components.

So there remains the case that all small components of $G-A$ are isolated vertices. If one large component has at least $d+6$ vertices, then we arrive at the contradiction

$$(t+1)(d+2)-1 \geq n \geq |A|+(t-1)(d+2)+4+\beta=(t+1)(d+2)+1.$$

Hence all large components of $G-A$ are of order at most $d+4$. As above, we will show that G contains a matching of size at least $q+1$. Since G is d-regular, there are at most $d-t-2$ edges in G joining each large component of $G-A$ with A. If Q is a large component, and hence of order at most $d+4$, and x and y are two nonadjacent vertices of Q, then we conclude that

$$d_Q(x)+d_Q(y) \geq 2d-(d-t-2) \geq d+4,$$

since $t \geq 2$. Therefore, by Theorem 4, the component Q is Hamiltonian.

Let zw be an edge joining a large component of $G-A$ with a vertex $w \in A$, and let B consist of the vertices of the small components of $G-A$. In addition, let H be the bipartite graph with the partite sets $A-w$ and B together with all edges of G between $A-w$ and B. Then $d_H(x) \geq d-1$ for all $x \in B$ and there is at least one vertex of degree d in B. Applying Theorem 2, we deduce that H has a perfect matching. Altogether, we observe that G has a matching of size at least $q+1=\frac{n-(t-2)}{2}$, a contradiction to our hypothesis.

Subcase 3.2.2.4: Assume that $|A|=d$. If there exists a small component of $G-A$ with at least 7 vertices, then we arrive at the contradiction

$$(t+1)(d+2)-1 \geq n \geq |A|+(t-1)(d+2)+7+\beta-1=(t+1)(d+2)+1.$$

If there exists a small component of $G-A$ with exactly 5 vertices, then there are at least

$$(\beta-1)d+5(d-4)=d^2+3d-20$$

edges from the small components to A, and there are at most d^2-t-3 edges from A to the small components of $G-A$. This leads to $3d+t \leq 17$, a contradiction when $d \geq 7$. In the case that $d=5$, we observe that $d=|A|=5$, $\beta=4$, $t=2$, the small component with exactly five vertices is the complete graph K_5, the remaining small components are of order one, the large component is of order $d+2=7$ and $n=20$. Now it is a simple matter to verify that G has a perfect matching, a contradiction to the hypothesis.

If there exists a small component of $G-A$ with exactly 3 vertices, then there are at least

$$(\beta-1)d+3(d-2)=d^2+d-6$$

edges from the small components to A, and there are at most d^2-t-3 edges from A to the small components of $G-A$. This leads to the contradiction $d+t \leq 3$.

So there remains the case that all small components of $G-A$ are isolated vertices. If one large component has at least $d+8$ vertices, then we arrive at the contradiction

$$(t+1)(d+2)-1 \geq n \geq |A|+(t-1)(d+2)+6+\beta = (t+1)(d+2)+1.$$

Hence all large components of $G-A$ are of order at most $d+6$, and there are at most $d-t-2$ edges in G joining each large component of $G-A$ with A. If Q is a large component, and hence of order at most $d+6$, and x and y are two nonadjacent vertices of Q, then

$$d_Q(x)+d_Q(y) \geq 2d-(d-t-2) = d+t+2 \geq d+6$$

when $t \geq 4$, and hence, by Theorem 4, the component Q is Hamiltonian. It follows as before that G has a matching of size at least $q+1 = \frac{n-(t-2)}{2}$ when $t \geq 4$, a contradiction to our hypothesis.

Suppose now that $t=2$, and let Q be the only large component. If Q has order at most $d+4$, then we arrive at a contradiction as before. Assume next that $|V(Q)| = d+6$. This implies that there are at most $d-4$ edges from Q to A in $G-A$.

If $d=5$, then let zw be the edge joining Q with a vertex $w \in A$. Then $Q-z$ is a connected graph of order 10 such that the degrees of the vertices in $Q-z$ are either 4 or 5. Using Theorem 3, we deduce that $Q-z$ has a perfect matching. Consequently, G has a perfect matching, a contradiction.

Assume that $d \geq 7$. Since there are at most $d-4$ edges from Q to A, the minimum degree $\delta(Q) \geq 4$. If Q has at most three vertices of degree less than d, then it is easy to see that the closure $C(Q)$ of Q is complete, and thus, by Theorem 5, Q is Hamiltonian. If Q has at least four vertices of degree less than d, then

$$d_Q(x)+d_Q(y) \geq 2d-(d-6) = d+6$$

for each pair x and y of nonadjacent vertices, and Q is also Hamiltonian according to Theorem 4. This again shows in each case that G has a perfect matching, a contradiction.

Subcase 3.2.2.5: Assume that $4 \leq |A| \leq d-1$. Using inequality (3), and the function $g(x)$ from Subcase 3.2.1, we arrive at the following contradiction for $d \geq 9$.

$$\begin{aligned}(t+1)(d+2)-1 \geq n &\geq |A|+(t-1)(d+2)+(|A|-1)(d-|A|+1)\\ &\geq (t-1)(d+2)+3d-5\\ &= (t+1)(d+2)+d-9.\end{aligned}$$

Assume that $d = 7$. If $|A|$ is even, then we have instead of (3) the better bound $|V(U)| \geq d - |A| + 2$. This implies

$$(t+1)(d+2) - 1 \geq n \geq |A| + (t-1)(d+2) + (|A|-1)(d - |A| + 2).$$

This yields for $|A| = 6$ the contradiction

$$\begin{aligned}(t+1)(d+2) - 1 \geq n &\geq 6 + (t-1)(d+2) + 5(d-4)\\ &= (t+1)(d+2) + 3d - 18,\end{aligned}$$

and for $|A| = 4$ the contradiction

$$\begin{aligned}(t+1)(d+2) - 1 \geq n &\geq 4 + (t-1)(d+2) + 3(d-2)\\ &= (t+1)(d+2) + d - 6.\end{aligned}$$

In the remaining case $|A| = 5$, inequality (3) implies $|V(U)| \geq 3$. If $|V(U)| = 3$ or $|V(U)| = 5$, then there are at least $3(d-2) + 3d = 36$ or $5(d-4) + 3d = 36$ edges joining the small components of $G - A$ with A, a contradiction to the fact that there are at most $5d - 4 - 1 = 30$ edges joining A with the small components of $G - A$. In the case $|V(U)| = 7$, we deduce that $n \geq 42$ when $t = 2$ and $n \geq 60$ when $t = 4$, a contradiction to $n \leq 26$ when $t = 2$ and $n \leq 44$ when $t = 4$.

Finally, assume that $d = 5$ and thus $|A| = 4$ and $t = 2$. Inequality (3) implies $|V(U)| \geq 3$. If all small components of $G - A$ are of order three, then there are at least $9(d-2) = 27$ edges joining the small components of $G - A$ with A, a contradiction to the fact that there are at most $4d - 4 - 1 = 15$ edges joining A with the small components of $G - A$. In the case that there is a small component of order 5, we conclude that $n \geq 22$, a contradiction to $n \leq 20$. Since we have discussed all possible cases, the proof of Theorem 6 is complete. □

The following examples will demonstrate that the bound given in Theorem 6 (i) is best possible when $d \geq 4$ is even.

Example 7 Let $d \geq 4$ be an even integer. Let $K_{d,d-1}$ be the complete bipartite graph with the larger partite set $\{x_1, x_2, \ldots, x_d\}$, and let H be consists of $K_{d,d-1}$ together with the edge set $\{x_1x_2, x_3x_4, \ldots, x_{d-1}x_d\}$. In addition, let $H_1, H_2, \ldots, H_{t-1}$ be $t-1$ copies of the complete graph K_{d+1}. We define the graph G of order $n = (t+1)(d+1) - 3$ as the disjoint union of $H, H_1, H_2, \ldots, H_{t-1}$. The resulting graph G is d-regular, and its maximum matching is of size $q = \frac{n-t}{2}$. However, the pair of edges x_1x_2 and x_3x_4 is not contained in a matching of of size q. This example shows that Theorem 6 (i) is best possible when $d \geq 4$ is even.

Example 8 Let H be the graph with vertex set $\{u_1, u_2, u_3, u_4, v_1, v_2\}$ and edge set $\{v_1u_1, v_1u_2, v_1u_3, v_2u_1, v_2u_2, v_2u_3, u_1u_2, u_3u_4\}$. In addition, let H_1' and H_2' be two copies of the complete graph K_5 with vertex sets $\{x_1, x_2, x_3, x_4, x_5\}$ and $\{y_1, y_2, y_3, y_4, y_5\}$, respectively. If we delete the edges x_1x_2, x_3x_4, x_4x_5 in H_1' and y_1y_2, y_3y_4, y_4y_5 in H_2', then we denote the resulting graphs by H_1 and H_2. Now let G be the disjoint union of H, H_1 and H_2 together with the two edges x_4u_4 and y_4u_4. Then G is a 3-regular graph of order 16 with a maximum matching of size 7, however, the pair of edges u_1u_2 and u_3u_4 is not contained in a matching of size 7. This example shows that Theorem 6 (ii) is best possible for $d = 3$ and $t = 2$.

Example 9 Let H' be a bipartite graph with the partite sets

$$A = \{u_1, u_2, u_3, u_4, u_5\} \quad \text{and} \quad B = \{v_1, v_2, v_3\}$$

such that each vertex of B is connected with each vertex of A by an edge. Now let H consist of H' together with the edges u_1u_2, u_2u_3, u_3u_4 and u_4u_1. In addition, let H_1' and H_2' be two copies of the complete graph K_7 with vertex sets $\{x_1, x_2, x_3, x_4, x_5, x_6, x_7\}$ and $\{y_1, y_2, y_3, y_4, y_5, y_6, y_7\}$, respectively. If we delete the edges $x_1x_2, x_3x_4, x_5x_6, x_6x_7$ in H_1' and $y_1y_2, y_3y_4, y_5y_6, y_6y_7$ in H_2', then we denote the resulting graphs by H_1 and H_2. Now let G be the disjoint union of H, H_1 and H_2 together with the two edges x_6u_5 and y_6u_5. Then G is a 5-regular graph of order 22 with a maximum matching of size 10, however, the pair of edges u_1u_2 and u_3u_4 is not contained in a matching of size 10. This example shows that Theorem 6 (ii) is best possible for $d = 5$ and $t = 2$.

For odd integers $d \geq 7$, I think that the following better bound for n is valid.

Conjecture 10 Let $d \geq 7$ be an odd integer, and let G be a d-regular graph of order n with a maximum matching of size $q = \frac{n-t}{2} \geq 3$ for an even integer $t \geq 2$. If $n \leq (t+2)(d+2) - 8$, then G is $(2, q)$-extendable.

The next examples will show that Conjecture 10 would be best possible for $t = 2$.

Example 11 Let $d \geq 7$ be an odd integer, and let H' be a bipartite graph with the partite sets $A = \{u_1, u_2, \ldots, u_d\}$ and $B = \{v_1, v_2, \ldots, v_{d-2}\}$ such that each vertex of B is joined to each vertex of A by an edge. Now let H consist of H' together with the edges $u_1u_2, u_2u_3, \ldots, u_{d-2}u_{d-1}$ and $u_{d-1}u_1$. In addition, let H_1' and H_2' be two copies of the complete graph K_{d+2} with vertex sets $\{x_1, x_2, \ldots, x_{d+2}\}$ and $\{y_1, y_2, \ldots, y_{d+2}\}$, respectively. If we delete the edges $x_1x_2, x_3x_4, \ldots, x_dx_{d+1}$ and $x_{d+1}x_{d+2}$ in H_1'

and $y_1y_2, y_3y_4, \ldots, y_dy_{d+1}$ and $y_{d+1}y_{d+2}$ in H_2', then we denote the resulting graphs by H_1 and H_2. Now let G be the disjoint union of H, H_1 and H_2 together with the two edges $x_{d+1}u_d$ and $y_{d+1}u_d$. Then G is a d-regular graph of order $4d+2$ with a maximum matching of size $2d$, however, the pair of edges u_1u_2 and u_3u_4 is not contained in a matching of size $2d$.

In Theorem 6 we considered regular $(2,q)$-extendable graphs that did not have perfect matchings. We now consider $(2,q)$-extendable graphs with perfect matchings.

Observation 12 Let $d \geq 5$ be an integer, and let G be a d-regular of even order n. If $n \leq 2d-4$, then G is 2-extendable.

Proof. Let uv and xy be two arbitrary nonincident edges of G, and define the subgraph $G' = G - \{u,v,x,y\}$. Then G' is of even order such that $n(G') \leq 2d-8$ and $\delta(G') \geq d-4$. By the classical theorem of Dirac [4], G' has a Hamiltonian cycle. Consequently, the pair of edges uv and xy is contained in a perfect matching of G, and thus G is 2-extendable. □

Example 13 Let $d \geq 5$ be an integer. Let $K_{d,d-2}$ be the complete bipartite graph with the larger partite set $\{x_1, x_2, \ldots, x_d\}$, and let G consists of $K_{d,d-2}$ together with the edge set $\{x_1x_2, x_2x_3, \ldots, x_{d-1}x_d, x_dx_1\}$. The resulting graph G of order $2d-2$ is d-regular, and it has a perfect matching. However, the pair of edges x_1x_2 and x_3x_4 is not contained in a perfect matching. This example shows that Observation 12 is best possible.

Remark 14 If q is the size of a maximum matching in a d-regular graph of order n with $d \geq 3$, then Henning and Yeo [6] have proved recently that

$$q \geq \min\left\{\left(\frac{d^2+4}{d^2+d+2}\right) \times \frac{n}{2}, \frac{n-1}{2}\right\} \text{ when } d \text{ is even}$$

and

$$q \geq \frac{(d^3-d^2-2)n-2d+2}{2(d^3-3d)} \text{ when } d \text{ is odd.}$$

In the papers by Yu [14] and Liu and Yu [9] one can find other extensions of p-extendability, which are stronger and which are only defined for graphs with a perfect or almost perfect matching.

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