Strongly Regular Graphs Admitting an Automorphism Group with Two Orbits

Nick C. Fiala

Department of Mathematics
St. Cloud State University
St. Cloud, MN 56301
ncfiala@stcloudstate.edu

Abstract

In this note, motivated by the non-existence of a vertex-transitive strongly regular graph with parameters $(3250, 57, 0, 1)$, we obtain a feasibility condition concerning strongly regular graphs admitting an automorphism group with exactly two orbits on vertices. We also establish a result on the possible orbit sizes of a potential strongly regular graph with parameters $(3250, 57, 0, 1)$. We use our results to obtain a list of only 11 possible orbit size combinations for a potential strongly regular graph with parameters $(3250, 57, 0, 1)$ admitting an automorphism group with exactly two orbits.

1 Introduction

A *strongly regular graph* with *parameters* v, k, λ, and μ, or an $srg(v, k, \lambda, \mu)$, is a finite simple k-regular graph with v vertices, not complete or edgeless, such that any two adjacent vertices have exactly λ common neighbors and any two distinct non-adjacent vertices have exactly μ common neighbors.

This note is motivated by the following two results concerning strongly regular graphs.

Theorem 1.1. ([2]) *If an* $srg(v, k, 0, 1)$ *exists, then* (v, k) *is* $(5, 2)$, $(10, 3)$, $(50, 7)$, *or* $(3250, 57)$.

It is well-known that there exists a unique $srg(5, 2, 0, 1)$ (the *pentagon*), a unique $srg(10, 3, 0, 1)$ (the *Petersen graph*), and a unique $srg(50, 7, 0, 1)$ (the *Hoffman-Singleton graph*) [2]. These three graphs are are all known to be *vertex-transitive* (they admit an automorphism group that is transitive on the set of vertices).

It is unknown is there exists an $srg(3250, 57, 0, 1)$. However, the following is known concerning a potential $srg(3250, 57, 0, 1)$ [1].

Theorem 1.2. (G. Higman, unpublished) *There does not exist a vertex-transitive $srg(3250, 57, 0, 1)$.*

Stated in a different way, Theorem 1.2 says that there does not exist an $srg(3250, 57, 0, 1)$ admitting an automorphism group with exactly one orbit (on vertices). Does there exist an $srg(3250, 57, 0, 1)$ admitting an automorphism group with exactly two orbits?

Motivated by this question, in this note we obtain a feasibility condition concerning strongly regular graphs admitting an automorphism group with exactly two orbits. We also establish a result on the possible orbit sizes of an $srg(3250, 57, 0, 1)$. We use our results to obtain a list of only 11 possible orbit size combinations for a potential $srg(3250, 57, 0, 1)$ admitting an automorphism group with exactly two orbits.

2 Results

We now establish our results and apply them to a potential $srg(3250, 57, 0, 1)$ admitting an automorphism group with exactly two orbits.

First, we need the following two well-known results.

Theorem 2.1. *If there exists an $srg(v, k, \lambda, \mu)$, then*

$$k(k - \lambda - 1) = (v - k - 1)\mu. \tag{1}$$

Theorem 2.2. *The eigenvalues of an $srg(v, k, \lambda, \mu)$ are $k > r > s$ where*

$$r = \frac{\lambda - \mu + \sqrt{(\lambda - \mu)^2 + 4(k - \mu)}}{2} \tag{2}$$

and

$$s = \frac{\lambda - \mu - \sqrt{(\lambda - \mu)^2 + 4(k - \mu)}}{2}. \tag{3}$$

Theorem 2.3. *Let G be an $srg(v, k, \lambda, \mu)$ admitting an automorphism group Γ with exactly two orbits, say O_1 and O_2. Then either*

$$\frac{(k - r)|O_1|}{v} \text{ and } \frac{(k - r)|O_2|}{v}$$

are both positive integers less than k or

$$\frac{(k - s)|O_1|}{v} \text{ and } \frac{(k - s)|O_2|}{v}$$

are both positive integers less than k.

Proof. Since Γ acts transitively on each of $\mathcal{O}_1$ and $\mathcal{O}_2$, the subgraph of G induced by $\mathcal{O}_i$, $i = 1, 2$, is regular. Therefore, since G is regular, given a vertex in $\mathcal{O}_i$, $i = 1, 2$, the number of vertices it is adjacent to in $\mathcal{O}_j$, $j = 1, 2$, depends only on i and j. Thus, we define k_{ij}, $i, j = 1, 2$, to be the number of vertices in $\mathcal{O}_j$ that a vertex in $\mathcal{O}_i$ is adjacent to.

Clearly, we have the three equations

$$|\mathcal{O}_1| + |\mathcal{O}_2| = v, \tag{4}$$

$$k_{11} + k_{12} = k, \tag{5}$$

and

$$k_{21} + k_{22} = k. \tag{6}$$

Counting in three different ways the number of ordered triples (x, y, z) of vertices of G such that x is adjacent to y, y is adjacent to z, $x \in \mathcal{O}_1$, and $z \in \mathcal{O}_2$ yields the two equations

$$|\mathcal{O}_1|k_{11}k_{12} + |\mathcal{O}_2|k_{21}k_{22} = |\mathcal{O}_1|k_{12}\lambda + |\mathcal{O}_1|(|\mathcal{O}_2| - k_{12})\mu \tag{7}$$

and

$$|\mathcal{O}_1|k_{11}k_{12} + |\mathcal{O}_2|k_{21}k_{22} = |\mathcal{O}_2|k_{21}\lambda + |\mathcal{O}_2|(|\mathcal{O}_1| - k_{21})\mu. \tag{8}$$

Solving equations (5), (6), (7), and (8) for k_{11}, k_{12}, k_{21}, and k_{22} and using (1), (2), (3), and (4) gives the two solutions

$$k_{11} = k - \frac{(k-r)|\mathcal{O}_2|}{v}, \quad k_{12} = \frac{(k-r)|\mathcal{O}_2|}{v},$$

$$k_{21} = \frac{(k-r)|\mathcal{O}_1|}{v}, \text{ and } k_{22} = k - \frac{(k-r)|\mathcal{O}_1|}{v}$$

and

$$k_{11} = k - \frac{(k-s)|\mathcal{O}_2|}{v}, \quad k_{12} = \frac{(k-s)|\mathcal{O}_2|}{v},$$

$$k_{21} = \frac{(k-s)|\mathcal{O}_1|}{v}, \text{ and } k_{22} = k - \frac{(k-s)|\mathcal{O}_1|}{v}.$$

The result now follows. □

The following result is essentially contained in [1].

Theorem 2.4. *There does not exist an* $srg(3250, 57, 0, 1)$ *admitting an automorphism group with all orbits of even size.*

Proof. Let G be an $srg(3250, 57, 0, 1)$ and let Γ be an automorphism group of G with all orbits of even size. Then $|\Gamma|$ is even and, therefore, Γ contains an involution γ.

It is known that any involutory automorphism of G must have exactly 56 fixed points. Thus, any involution in Γ contains exactly 1597 transpositions and is, therefore, an odd permutation.

Since γ is not fixed-point-free, γ is in some one-point stabilizer Γ_x. Therefore, $|\Gamma| = |x^\Gamma||\Gamma_x|$ is divisible by four since $|x^\Gamma|$ and $|\Gamma_x|$ are both even. Thus, $|\Gamma \cap A_{3250}| = |\Gamma||A_{3250}|/|\Gamma A_{3250}| = \Gamma/2$ since $\Gamma A_{3250} = S_{3250}$ because Γ contains an odd permutation.

Therefore, $|\Gamma \cap A_{3250}|$ is even and $\Gamma \cap A_{3250}$ contains an involution. However, this is a contradiction since any involution in Γ must be an odd permutation. □

Remark 2.5. It is clear from the proof of Theorem 2.4 that we do not actually need all orbits to be of even size, just an orbit containing a fixed point of an involution.

Corollary 2.6. *If there exists an* $srg(3250, 57, 0, 1)$ *admitting an automorphism group with exactly two orbits, say* $\mathcal{O}_1$ *and* $\mathcal{O}_2$ $(|\mathcal{O}_1| \leq |\mathcal{O}_2|)$*, then* $|\mathcal{O}_1| = 195 + 130i$, $|\mathcal{O}_2| = 3055 - 130i$, $k_{11} = 10 + 2i$, $k_{12} = 47 - 2i$, $k_{21} = 3 + 2i$, *and* $k_{22} = 54 - 2i$ *for some integer* i, $0 \leq i \leq 10$.

Proof. An $srg(3250, 57, 0, 1)$ has eigenvalues $k = 57$, $r = 7$, and $s = -8$. Therefore, $(k - r)/v = 1/65$ and $(k - s)/v = 1/50$.

Since $v = 3250$ is even, by Theorem 2.4, $|\mathcal{O}_1|$ and $|\mathcal{O}_2|$ are both odd and, therefore, cannot be divisible by 50. Thus, by Theorem 2.3, $|\mathcal{O}_1|$ and $|\mathcal{O}_2|$ are both divisible by 65.

The only integer values of $|\mathcal{O}_1|$ and $|\mathcal{O}_2|$ such that $|\mathcal{O}_1| + |\mathcal{O}_2| = 3250$, $|\mathcal{O}_1|$ and $|\mathcal{O}_2|$ are both odd, $|\mathcal{O}_1|$ and $|\mathcal{O}_2|$ are both divisible by 65, and $0 < |\mathcal{O}_1|/65 < 57$ and $0 < |\mathcal{O}_2|/65 < 57$ are $|\mathcal{O}_1| = 65 + 130i$ and $|\mathcal{O}_2| = 3185 - 130i$ for some integer i, $0 \leq i \leq 11$.

However, if $|\mathcal{O}_1| = 65$ and $|\mathcal{O}_2| = 3185$, then $k_{11} = 8$ and $k_{21} = 1$, implying that the subgraph induced by $\mathcal{O}_1$ is an $srg(65, 8, 0, 1)$. An $srg(65, 8, 0, 1)$ does not exist by Theorem 1.1. The result now follows. □

References

[1] P. J. Cameron, *Permutation Groups*, Cambridge Univ. Press, Cambridge, 1999.

[2] A. J. Hoffman and R. R. Singleton, *On Moore graphs with diameters two and three*, IBM J. Res. Develop. 4 (1960), 497-504.