No 4-isosceles set with eight points on a circle *

Xianglin Wei
wxlhebtu@126.com
College of Science, Hebei University of Science and Technology, Shijiazhuang, 050018, China

Abstract

A finite planar set is k-isosceles for $k \geq 3$ if every k-point subset of the set contains a point equidistant from two others. There exists no convex 4-isosceles 8-point set with 8 points on a circle.

1 Introduction

For $k \geq 3$, a set in R^2 is k-isosceles if every k-point subset of the set includes a 3-point subset in which one point is equidistant from the other two. In [1] Fishburn discussed 3-isosceles and 4-isosceles planar point sets. At the end of his paper, he put forward several open questions about 4-isosceles planar sets. Now there exists one problem which has not been answered, that is: is there an 8-point 4-isosceles set whose points are the vertices of a convex octagon? A finite subset of R^2 is convex if its points are the vertices of a convex polygon. We know of no convex 4-isosceles 8-point set, but there are convex 4-isosceles n-point sets ($n \leq 7$) as shown in [1] [3] [4].

Let $d(x, y)$ denote the Euclidean distance between x and y in R^2. Fishburn [1] considers planar sets with four or more points on a line. In this note we consider the planar sets with eight points on a circle, and give the following answer: There does not exist a 4-isosceles 8-point set with 8 points on a circle.

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2 Proof of Main Results

We assume C_O be the unit circle. For two points $x, y \in C_O$, let $\widehat{xy}$ denote the arc which contained in a semicircle, $d(\widehat{xy}) \le \pi r$ denote the arc-length of arc $\widehat{xy} \subset C_O$. Clearly for three points $x, y, z \in C_O$, $d(x, y) = d(x, z) \iff d(\widehat{xy}) = d(\widehat{xz})$. For $k \ge 3$, a set in which all points lie on a circle is k-arcisosceles if every k-point subset of the set includes a 3-point subset in which one point has the same arc-length from the other two. In this note, let P be a 8-point set with points on a circle, for a successive 5-point subset $Q \subset P$, say $Q = \{a, b, c, d, e\}$, we denote point a and point e be the two end points of Q.

Lemma 1. *Let P be a set in which all points lie on a circle. Then P is 4-isosceles if and only if P is also 4-arcisosceles.*

Proof. Since $d(x, y) = d(x, z) \iff d(\widehat{xy}) = d(\widehat{xz})$, the result is clearly correct. □

Lemma 2. *[1] [2] A 5-point set on a line is 4-isosceles if and only if its linear successive distance pattern is* $2:1:1:2$ *or* $1:1:1:3$ *or* $1:1:2:3$ *or* $2:2:1:3$. *No set of 6 points on a line is 4-isosceles.*

From Lemma 1 and Lemma 2 we can conclude the following Lemma:

Lemma 3. *A 5-point subset with points on an arc whose length is at most of a semicircle is 4-arcisosceles if and only if its successive arc-length pattern of arcs is* $2:1:1:2$ *or* $1:1:1:3$ *or* $1:1:2:3$ *or* $2:2:1:3$. *No 6-point set with points on any arc whose length is at most of a semicircle is 4-arcisosceles.*

Proof. Since the linear successive distance pattern of n points lying on a line is same as the successive arc-length pattern of n points lying on a semicircle, the result is correct. □

Theorem 4. *There does not exist a 4-isosceles 8-point set P with 8 points on a circle.*

Proof. By Lemma 1, we should prove that there exists no 4-arcisosceles 8-point set. By Lemma 3, any semicircle contains at most 5 points of the set P, that is to say, the set P must contain a successive 5-point subset Q with points on an arc which contained in a semicircle. Without loss of generality, we assume $Q = \{1, 2, 3, 4, 5\}$, then points 1 and 5 are two end points of it. By Lemma 3, we know $d(\widehat{12}) : d(\widehat{23}) : d(\widehat{34}) : d(\widehat{45})$ should be $2:1:1:2$ or $1:1:1:3$ or $1:1:2:3$ or $2:2:1:3$. We consider two cases to prove the result.

(a) (b)

Figure 1: (a) $d(\widehat{12}) : d(\widehat{23}) : d(\widehat{34}) : d(\widehat{45})$ =2 : 1 : 1 : 2 and $d(\widehat{12}) = \frac{\pi}{3}$; (b) $d(\widehat{12}) : d(\widehat{23}) : d(\widehat{34}) : d(\widehat{45})$ =1 : 1 : 1 : 3 and $d(\widehat{12}) = \frac{\pi}{6}$

Case 1: Points 1 and 5 are two end points of a diameter of a circle.

(1) $d(\widehat{12}) : d(\widehat{23}) : d(\widehat{34}) : d(\widehat{45})$ =2 : 1 : 1 : 2.

As shown in figure 1 (a), $d(\widehat{12}) = d(\widehat{45}) = \frac{\pi}{3}, d(\widehat{23}) = d(\widehat{34}) = \frac{\pi}{6}$. Point 6 does not lie on the arc I containing the other end point, since otherwise $\{2,3,4,5,6\}$ is not 4-arcisosceles. Point 8 does not lie on the arc II containing the other end point, since otherwise $\{8,1,2,3,4\}$ is not 4-arcisosceles. Hence the remaining three points 6, 7, 8 should lie on the arc III not containing the two end points. But $\{6,7,8,1,2\}$ is not 4-arcisosceles.

(2) $d(\widehat{12}) : d(\widehat{23}) : d(\widehat{34}) : d(\widehat{45})$ =1 : 1 : 1 : 3.

As shown in figure 1(b), $d(\widehat{12}) = d(\widehat{23}) = d(\widehat{34}) = \frac{\pi}{6}, d(\widehat{45}) = \frac{\pi}{2}$. Point 6 does not lie on the arc I containing the other end point, since otherwise $\{2,3,4,5,6\}$ is not 4-arcisosceles. Point 8 does not lie on the arc II not containing the other end point, since otherwise $\{8,1,2,3,4\}$ is not 4-arcisosceles. Hence the remaining three points 6, 7, 8 should lie on the arc III containing one end point. But $\{4,5,6,7,8\}$ is not 4-arcisosceles.

(a) (b)

Figure 2: (a) $d(\widehat{12}) : d(\widehat{23}) : d(\widehat{34}) : d(\widehat{45})$ =1 : 1 : 2 : 3 and $d(\widehat{12}) = \frac{\pi}{7}$; (b) $d(\widehat{12}) : d(\widehat{23}) : d(\widehat{34}) : d(\widehat{45})$ =2 : 2 : 1 : 3 and $d(\widehat{12}) = \frac{\pi}{4}$

(3) $d(\widehat{12}) : d(\widehat{23}) : d(\widehat{34}) : d(\widehat{45})$ =1 : 1 : 2 : 3

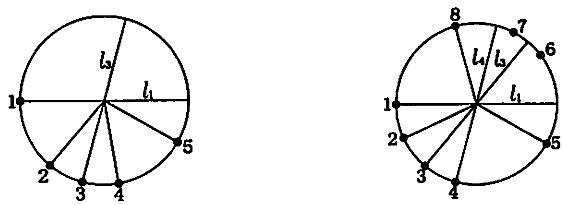
As shown in figure 2 (a), $d(\widehat{12}) = d(\widehat{23}) = \frac{\pi}{7}, d(\widehat{34}) = \frac{2\pi}{7}, d(\widehat{45}) = \frac{3\pi}{7}$. Point 6 does not lie on the arc I containing the other end point, since otherwise $\{2,3,4,5,6\}$ is not 4-arcisosceles. If point 8 lies on the arc II containing the other end point, then $d(\widehat{18}) = \frac{2\pi}{7}$, since $d(\widehat{81}) : d(\widehat{12}) : d(\widehat{23}) : d(\widehat{34}) = 2 : 1 : 1 : 2$. Hence $d(\widehat{58}) = \frac{5\pi}{7}$, and $\{5,6,7,8,1\}$ is not 4-arcisosceles. So we can assume point 8 lies on the arc III not containing the other end point. Then $\{4,5,6,7,8\}$ is not 4-arcisosceles.

(4) $d(\widehat{12}) : d(\widehat{23}) : d(\widehat{34}) : d(\widehat{45}) = 2 : 2 : 1 : 3$

As shown in figure 2(b), $d(\widehat{12}) = d(\widehat{23}) = \frac{\pi}{4}, d(\widehat{34}) = \frac{\pi}{8}, d(\widehat{45}) = \frac{3\pi}{8}$. Point 6 does not lie on the arc I containing the other end point, since otherwise $\{2,3,4,5,6\}$ is not 4-arcisosceles. Point 8 does not lie on the arc II containing the other end point, since otherwise $\{8,1,2,3,4\}$ is not 4-arcisosceles. Hence the remaining three points 6, 7, 8 should lie on the arc III. Then $\{4,5,6,7,8\}$ is not 4-arcisosceles.

Case 2: Points 1 and 5 are not two end points of a diameter of a circle.

Let l_i be the diameter through point i. By lemma 3, we know that four points of P lie on one side of the diameter l_i and the remaining three points of P lie on the other side of it. Without loss of generality, we may assume points 2, 3, 4, 5 lie on one side of l_1 and points 6, 7, 8 lie on the other side of l_1.



(a) (b)

Figure 3: (a) $d(\widehat{12}) : d(\widehat{23}) : d(\widehat{34}) : d(\widehat{45}) = 2 : 1 : 1 : 2$ and $d(\widehat{12}) < \frac{\pi}{3}$; (b) $d(\widehat{12}) : d(\widehat{23}) : d(\widehat{34}) : d(\widehat{45}) = 1 : 1 : 1 : 3$ and $d(\widehat{12}) < \frac{\pi}{6}$

(1) $d(\widehat{12}) : d(\widehat{23}) : d(\widehat{34}) : d(\widehat{45}) = 2 : 1 : 1 : 2$.

As shown in figure 3 (a), $d(\widehat{12}) < \frac{\pi}{3}$. If points 4, 5, 6, 7 lie on the same side of l_3, then the subset $\{3,4,5,6,7\}$ is not 4-arcisosceles. Otherwise points 7, 8, 1, 2 lie on the same side of diameter l_3, and $\{7,8,1,2,3\}$ is not 4-arcisosceles.

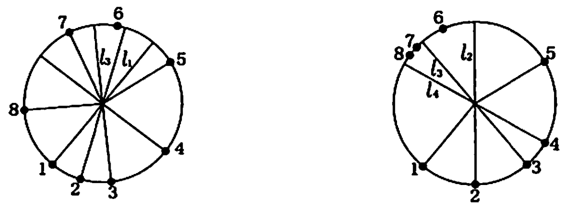
(2) $d(\widehat{12}) : d(\widehat{23}) : d(\widehat{34}) : d(\widehat{45}) = 1 : 1 : 1 : 3$.

As shown in figure 3 (b), $d(\widehat{12}) < \frac{\pi}{6}$. If points 4, 5, 6, 7 lie on the same side of l_3, then $\{3,4,5,6,7\}$ is not 4-arcisosceles. So we may assume points 7, 8, 1, 2 lie on the same side of l_3. If points 5, 6, 7, 8 lie on the same side of l_4, then $\{7,8,1,2,3\}$ is not 4-arcisosceles. If points 8, 1, 2, 3 lie on the same side of l_4, then clearly $d(\widehat{81}) : d(\widehat{12}) : d(\widehat{23}) : d(\widehat{34}) = 3 : 1 : 1 : 1$, and

$\{7,8,1,2,3\}$ is not 4-arcisosceles.

(3) $d(\widehat{12}) : d(\widehat{23}) : d(\widehat{34}) : d(\widehat{45}) = 1 : 1 : 2 : 3$

As shown in figure 4 (a), $d(\widehat{12}) < \frac{\pi}{7}$. If points 3, 4, 5, 6 lie on the same side of l_2, then $\{2,3,4,5,6\}$ is not 4-arcisosceles. So we may assume points 6, 7, 8, 1 lie on the same side of l_2. If points 5, 6, 7, 8 lie on the same side of l_4, then $\{6,7,8,1,2\}$ is not 4-arcisosceles. If points 8, 1, 2, 3 lie on the same side of l_4, then clearly $d(\widehat{81}) : d(\widehat{12}) : d(\widehat{23}) : d(\widehat{34}) = 2 : 1 : 1 : 2$, and $\{6,7,8,1,2\}$ is not 4-arcisosceles.



(a) (b)

Figure 4: (a) $d(\widehat{12}) : d(\widehat{23}) : d(\widehat{34}) : d(\widehat{45}) = 1 : 1 : 2 : 3$ and $d(\widehat{12}) < \frac{\pi}{7}$; (b) $d(\widehat{12}) : d(\widehat{23}) : d(\widehat{34}) : d(\widehat{45}) = 2 : 2 : 1 : 3$ and $d(\widehat{12}) < \frac{2\pi}{8}$

(4) $d(\widehat{12}) : d(\widehat{23}) : d(\widehat{34}) : d(\widehat{45}) = 2 : 2 : 1 : 3$

As shown in figure 4 (b), $d(\widehat{12}) < \frac{2\pi}{8}$. We may assume points 5, 6, 7, 8 lie on the same side of l_4, since otherwise $\{8,1,2,3,4\}$ is not 4-arcisosceles. And we may assume points 6, 7, 8, 1 lie on the same side of l_2, since otherwise $\{2,3,4,5,6\}$ is not 4-arcisosceles. If points 4, 5, 6, 7 lie on the same side of l_3, then $\{3,4,5,6,7\}$ is not 4-arcisosceles; if points 7, 8, 1, 2 lie on the same side of l_3, then $\{7,8,1,2,3\}$ is not 4-arcisosceles.

The proof is complete. □

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