A New Sufficient Condition for Graphs to Have (g, f)-Factors *†

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Abstract

Let a and b be integers such that $1 \leq a < b$, and let G be a graph of order n with $n > \frac{(a+b)(2a+2b-3)}{a+1}$, and the minimum degree $\delta(G) \geq \frac{(b-1)^2-(a+1)(b-a-2)}{a+1}$, and let $g(x)$ and $f(x)$ be two nonnegative integer-valued functions defined on $V(G)$ such that $a \leq g(x) < f(x) \leq b$ for each $x \in V(G)$. We prove that if $|N_G(x) \cup N_G(y)| \geq \frac{(b-1)n}{a+b}$ for any two nonadjacent vertices x and y in G, then G has a (g, f)-factor. Furthermore, it is showed that the result in this paper is best possible in some sense.

Keywords: graph, factor, (g, f)-factor, neighborhood

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1 Introduction

The graphs considered in this paper will be finite and undirected simple graphs. Let G be a graph. We denote by $V(G)$ and $E(G)$ the set of vertices

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and the set of edges, respectively. For any $x \in V(G)$, we denote by $d_G(x)$ the degree of x in G and by $N_G(x)$ the set of vertices adjacent to x in G. We write $N_G[x]$ for $N_G(x) \cup \{x\}$. The minimum degree of vertices in G is denoted by $\delta(G)$. For $S \subseteq V(G)$, we define $N_G(S) = \cup_{x \in S} N_G(x)$, and $G[S]$ is the subgraph of G induced by S. We write $G - S$ for $G[V(G) \setminus S]$. Let S and T be disjoint subsets of $V(G)$. We denote by $e_G(S,T)$ the number of edges joining S and T. A matching in a graph G is a set of edges of G with the property that no two edges are adjacent. A k-matching is a matching of size k.

Let $g(x)$ and $f(x)$ be two nonnegative integer-valued functions defined on $V(G)$ such that $g(x) \leq f(x)$ for each $x \in V(G)$. A (g,f)-factor of graph G is a spanning subgraph F of G such that $g(x) \leq d_F(x) \leq f(x)$ for each $x \in V(G)$ (Where of course d_F denotes the degree in F). If $g(x) = a$ and $f(x) = b$ for each $x \in V(G)$, then a (g,f)-factor of G is called an $[a,b]$-factor of G. If $g(x) = f(x) = k$ for each $x \in V(G)$, then a (g,f)-factor of G is called a k-factor of G. The other terminologies and notations not given in this paper can be found in [1].

The following results on k-factors and $[a,b]$-factors and (g,f)-factors are known.

Theorem 1 [2] *Let G be a graph, and let g and f be two non-negative integer-valued functions defined on $V(G)$ such that $g(x) < f(x)$ for each $x \in V(G)$. If $g(x) \leq d_G(x)$ and $(f(x)-1)d_G(y) \geq (d_G(x)-1)g(y)$ for each $x, y \in V(G)$, then G has a (g,f)-factor containing any edge e of G.*

Theorem 2 [3] *Let G be a graph, and let g and f be two non-negative integer-valued functions defined on $V(G)$ such that $g(x) < f(x)$ for each $x \in V(G)$. If $g(x) \leq d_G(x)$ and $(f(x)-k)d_G(y) \geq (d_G(x)-k)g(y)$ for each $x, y \in V(G)$, then G has a (g,f)-factor containing any k edges of G. Where k is one non-nagetive integer.*

Theorem 3 [3] *Let G be a graph, and let g and f be two non-negative integer-valued functions defined on $V(G)$ such that $g(x) < f(x)$ for each $x \in V(G)$, M is an $(rk-r+1)$-matching of G. If $g(x) \leq d_G(x)$ and $(f(x)-k)d_G(y) \geq (d_G(x)-k)g(y)$ for each $x, y \in V(G)$, then G has a (g,f)-factor containing M. Where r and k are two positive integers.*

Theorem 4 [4] *Let k be an integer such that $k \geq 2$, and let G be a connected graph of order n such that $n \geq 9k-1-4\sqrt{2(k-1)^2+2}$, kn is even, and the minimum degree is at least k. If G satisfies $|N_G(x) \cup N_G(y)| \geq \frac{1}{2}(n+k-2)$ for each pair of nonadjacent vertices $x, y \in V(G)$, then G has a k-factor.*

Theorem 5 [5] *Let aand b be integers such that $1 \le a < b$, and let G be a graph of order n with $n \ge \frac{2(a+b)(a+b-1)}{b}$, and $\delta(G) \ge a$. If*

$$|N_G(x) \cup N_G(y)| \ge \frac{an}{a+b}$$

for any two nonadjacent vertices x and y of G, then G has an $[a,b]$-factor.

Theorem 6 [6] *Let aand b be integers such that $2 \le a < b$, and let G be a graph of order n with $n \ge 6a+b$. Put $\lambda = \frac{a-1}{b}$. For any subset $X \subset V(G)$, we suppose*

$$N_G(X) = V(G) \qquad if\ |X| \ge \lfloor \frac{n}{1+\lambda} \rfloor;$$

or

$$|N_G(X)| \ge (1+\lambda)|X| \qquad if\ |X| < \lfloor \frac{n}{1+\lambda} \rfloor.$$

Then G has an $[a,b]$-factor.

Theorem 7 [7] *Let G be a graph, and let t, a and b be integers such that $0 \le a < b$ and $t \ge 3$. If G is a $K_{1,t}$-free graph and its minimum degree is at least*

$$(\frac{(t-1)(a+1)+b}{b})\lceil \frac{b+a(t-1)}{2(t-1)} \rceil - \frac{t-1}{b}(\lceil \frac{b+a(t-1)}{2(t-1)} \rceil)^2 - 1,$$

then G has an $[a,b]$-factor.

2 The Proof of Main Theorem

In this paper, we mainly prove the following theorem about the existence of a (g,f)-factor, which is an extension of Theorem 4 and Theorem 5. We extend Theorem 4 and Theorem 5 to (g,f)-factors.

Theorem 8 *Let a and b be integers such that $1 \le a < b$, and let G be a graph of order n with $n > \frac{(a+b)(2a+2b-3)}{a+1}$, and $\delta(G) \ge \frac{(b-1)^2-(a+1)(b-a-2)}{a+1}$, and let $g(x)$ and $f(x)$ be two nonnegative integer-valued functions defined on $V(G)$ such that $a \le g(x) < f(x) \le b$ for each $x \in V(G)$. If $|N_G(x) \cup N_G(y)| \ge \frac{(b-1)n}{a+b}$ for any two nonadjacent vertices x and y in G, then G has a (g,f)-factor.*

In order to prove our main theorem, we depend heavily on the following theorem, which is a special case of *Lovász's* (g,f)-factor theorem.

Theorem 9 [8] *Let G be a graph, and let $g(x)$ and $f(x)$ be two nonnegative integer-valued functions defined on $V(G)$ such that $g(x) < f(x)$ for each $x \in V(G)$. Then G has a (g, f)-factor if and only if*

$$\delta_G(S,T) = f(S) + d_{G-S}(T) - g(T) \geq 0$$

for all disjoint subsets S and T of $V(G)$.

The Proof of Theorem 8. Suppose that G satisfies the conditions of Theorem 8, but it has no (g, f)-factor. Then, by Theorem 9, there exist disjoint subsets S and T of $V(G)$ such that

$$\delta_G(S,T) = f(S) + d_{G-S}(T) - g(T) \leq -1. \tag{1}$$

We choose subsets S and T such that $|T|$ is minimum and S and T satisfy (1).

We first prove the following claims.

Claim 1. $d_{G-S}(x) < g(x) \leq b-1$ for each $x \in T$.

Proof. Suppose that there exists a vertex $x \in T$ such that $d_{G-S}(x) \geq g(x)$. Then the subsets S and $T - \{x\}$ satisfy (1), which contradicts the choice of T. Therefore,

$$d_{G-S}(x) < g(x) \leq b-1 \tag{2}$$

for each $x \in T$.

Claim 2. $|T| \geq a+2$.

Proof. If $|T| \leq a+1$, then by (1) and since $|S| + d_{G-S}(x) \geq d_G(x) \geq \delta(G) \geq \frac{(b-1)^2-(a+1)(b-a-2)}{a+1} \geq b-1$ for each $x \in T$ we obtain

$$\begin{aligned}
-1 &\geq \delta_G(S,T) = f(S) + d_{G-S}(T) - g(T) \\
&\geq (a+1)|S| + d_{G-S}(T) - (b-1)|T| \\
&\geq |T||S| + d_{G-S}(T) - (b-1)|T| \\
&= \sum_{x \in T} (|S| + d_{G-S}(x) - (b-1)) \geq 0,
\end{aligned}$$

which is a contradiction. So $|T| \geq a+2$.

Since $T \neq \varnothing$, let $h_1 = \min\{d_{G-S}(x) | x \in T\}$, and let $x_1 \in T$ be a vertex such that $d_{G-S}(x_1) = h_1$. According to (2), we get

$$0 \leq h_1 \leq b-2.$$

In the following, We shall consider two cases and derive a contradiction in each case.

Case 1. $T = N_T[x_1]$.

In view of Claim 2 and $|T| = |N_T[x_1]| \le d_{G-S}(x_1)+1 = h_1+1 \le b-1$, we have

$$h_1 \ge a+1. \tag{3}$$

and

$$b \ge a+3. \tag{4}$$

According to (1), (3), (4), $|T| \le b-1$, $|S|+h_1 = |S|+d_{G-S}(x_1) \ge d_G(x_1) \ge \delta(G) \ge \frac{(b-1)^2-(a+1)(b-a-2)}{a+1}$, and the definition of h_1, we obtain

$$\begin{aligned} -1 &\ge \delta_G(S,T) = f(S) + d_{G-S}(T) - g(T) \\ &\ge (a+1)|S| + d_{G-S}(T) - (b-1)|T| \\ &\ge (a+1)|S| + h_1|T| - (b-1)|T| \\ &= (a+1)|S| - (b-h_1-1)|T| \\ &\ge (a+1)(\frac{(b-1)^2-(a+1)(b-a-2)}{a+1} - h_1) \\ &\quad -(b-h_1-1)(b-1) \\ &\ge (b-a-2)h_1 - (a+1)(b-a-2) \ge 0. \end{aligned}$$

This is a contradiction.

Case 2. $T \ne N_T[x_1]$.

It is clear that $T \setminus N_T[x_1] \ne \varnothing$. Then we define

$$h_2 = \min\{d_{G-S}(x) | x \in T \setminus N_T[x_1]\},$$

and let $x_2 \in T \setminus N_T[x_1]$ be a vertex such that $d_{G-S}(x_2) = h_2$. Note that $0 \le h_1 \le h_2 \le b-2$ hold.

Obviously, two vertex x_1 and x_2 are not adjacent. In view of the condition of the theorem, we get that

$$\frac{(b-1)n}{a+b} \le |N_G(x_1) \cup N_G(x_2)| \le |S| + h_1 + h_2,$$

which implies

$$|S| \ge \frac{(b-1)n}{a+b} - h_1 - h_2. \tag{5}$$

By (1), (5), and $|S|+|T| \le n$, and $|N_T[x_1]| \le h_1+1$, and $n > \frac{(a+b)(2a+2b-3)}{a+1}$, we obtain

$$-1 \ge \delta_G(S,T) = f(S) + d_{G-S}(T) - g(T)$$

$$
\begin{aligned}
&\geq (a+1)|S| + d_{G-S}(T) - (b-1)|T| \\
&\geq (a+1)|S| + h_1|N_T[x_1]| + h_2(|T| - |N_T[x_1]|) - (b-1)|T| \\
&= (a+1)|S| + (h_1 - h_2)|N_T[x_1]| - (b-h_2-1)|T| \\
&\geq (a+1)|S| + (h_1 - h_2)|N_T[x_1]| - (b-h_2-1)(n-|S|) \\
&= (a+b-h_2)|S| + (h_1 - h_2)|N_T[x_1]| - (b-h_2-1)n \\
&\geq (a+b-h_2)(\frac{(b-1)n}{a+b} - h_1 - h_2) + (h_1-h_2)(h_1+1) \\
&\quad -(b-h_2-1)n \\
&= h_1^2 - (a+b-1)h_1 + h_2^2 + (n - \frac{(b-1)n}{a+b} - (a+b) - 1)h_2 \\
&= h_1^2 - (a+b-1)h_1 + h_2^2 + (\frac{(a+1)n}{a+b} - (a+b) - 1)h_2 \\
&> h_1^2 - (a+b-1)h_1 + h_2^2 + (a+b-4)h_2,
\end{aligned}
$$

i.e.

$$-1 > h_1^2 - (a+b-1)h_1 + h_2^2 + (a+b-4)h_2. \tag{6}$$

If $h_2 = 0$, then according to $0 \leq h_1 \leq h_2 \leq b-2$, we have $h_1 = 0$. By (6), we get that

$$-1 > 0,$$

a contradiction.

If $1 \leq h_2 \leq b-2$, then by (6) we get

$$
\begin{aligned}
-1 &> h_1^2 - (a+b-1)h_1 + h_2^2 + (a+b-4)h_2 \\
&\geq h_1^2 - (a+b-1)h_1 + h_1^2 + (a+b-4)h_1 \\
&= 2h_1^2 - 3h_1 \geq -1 \qquad (\textit{Since } h_1 \geq 0 \textit{ is an integer})
\end{aligned}
$$

which is a contradiction.

From the argument above, we deduce the contradictions. Hence, G has a (g, f)-factor.
Completing the proof of Theorem 8.

Remark. Let us show that the condition $|N_G(x) \cup N_G(y)| \geq \frac{(b-1)n}{a+b}$ in Theorem 8 can not be replaced by $|N_G(x) \cup N_G(y)| \geq \frac{(b-1)n}{a+b} - 1$. Suppose that $b = a+1$, and define $g(x) = a$ and $f(x) = b$ for each $x \in V(G)$. Let $G = (A, B)$ be a complete bipartite graph such that $|A| = at$ and $|B| = bt+1$, where t is any positive integer. Then it follows that $n = |A| + |B| = (a+b)t + 1$ and

$$\frac{(b-1)n}{a+b} > |N_G(x) \cup N_G(y)| = at = (b-1)t > \frac{(b-1)n}{a+b} - 1$$

for any subset $\{x, y\}$ of B. However, G has no $[a, b]$-factor since $b|A| < a|B|$, that is, G has no (g, f)-factor. In this sense, the condition $|N_G(x) \cup N_G(y)| \geq \frac{(b-1)n}{a+b}$ is the best possible.

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